Analytical Solution of Linear Fractional Partial Differential Equation of Order $0 < \alpha \leq 1$ by Improved Adomian Decomposition Method

Abstract

The paper aims to obtain exact analytical solution of linear nonhomogeneous space-time fractional order partial differential equation by improved Adomian decomposition method coupled with fractional Taylor expansion series. The solution of these equations are in series form may have rapid convergence to a closed-form solution. The effectiveness and sharpness of this method is shown by obtaining the exact solution of these equations with suitable initial conditions (ICs). With the help of this method, it is possible to investigate nature of solutions when we vary order of the fractional derivative. Behaviour of the solution of these equations are represented by graphs using Matlab software.

Keywords : Improved Adomian decomposition method; Fractional Taylor expansion series; Mittage-Leffler function; Caputo fractional derivative

2010 Mathematics Subject Classifications : Primary: 26A33, 33E12 34A08, 35R11

1   Introduction

The study of nonlinear partial differential equations (PDEs) is a well developed research area. There are physical models governed by nonlinear fractional partial differential equations (FPDEs) in various sciences such as sciences of Mathematics, sciences of Physics, sciences of Chemistry, and sciences of Biology as well as in technologies [21, 31, 32, 35, 38, 39]. Several researchers have focused on the study of physical models directed by FPDEs. The difficulty of getting the exact solution of equations in such models is an important and attractive area of research. Not all nonlinear equations in physical models have an exact solution, therefore, many researchers have developed various methods of solving nonlinear FPDEs. The Adomian decomposition method (ADM) [6, 7] is a powerful tool to determine solution of fractional differential equations. In a present years, a numerous attention has been devoted to the study of the Adomian's Decomposition Method (ADM) [1, 6, 7].

Cherrualt et al. [16], Adomian and Rach [8], Duan et al. [22], which permit us to survey the properties and solutions of a huge variety of ordinary and PDEs, as well as of FPDEs, which express several mathematical problems, or can be used to mathematically model diverse physical processes. From a historic view, the ADM was first introduced, and enormously used in the 1980's [2, 3, 4, 5], and afterward many mathematicians and scientists have constantly modified the ADM in an attempt to upgrade its accuracy and/or to widen the applications of the initial method. Duan, Rach and Wazwaz [26, 27, 28, 41, 42, 43, 45, 49] put in a lot of effort to improve ADM. The ADM procedure to solve linear or nonlinear boundary value problems using the Duan-Rach recursion [23, 24, 25], which is intrinsic to the ADM, does not itself require Green's functions, Dirac delta functions, and discretization techniques such as a finite-difference method or a finite element method. Also, it does not invoke the shooting method, special basis functions, guessing the starting term, linearization, perturbation, and so on. Importantly, fast, efficient, cost-effective and accurate solutions can be found without the need to resort to high performance computing. As the nonlinear terms are not ignored or crudely linearized, a much better appreciation of the physics of a particular problem ensues. This aspect of simulation is often lost in numerical methods. A key concept is that the Adomian decomposition series is any rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function that permits solution by recursion, in which the aforesaid rearrangement is accomplished through the choice of the recursion scheme. The ADM yields a rapidly convergent sequence of analytic functions as the approximate solutions of the original mathematical model. The most important work about convergence has been carried out by Cherruault [15]. Further remarks about the convergence of the decomposition method are in [16]. Historical view of ADM given by Rach in [40].Thus the ADM subsumes even the classic power series method while extending the class of amenable nonlinearities to include any analytic nonlinearity.

While Shawagfeh [46] has employed ADM for solving nonlinear fractional partial differential equations, Daftardar-Gejji and Jafri have obtained solution of numerous problems [33] by using ADM. Also Dhaigude and Birajdar [19, 20] extended the discrete ADM for obtaining the numerical solution of system of FPDEs. Chitalkar-Dhaigude and Bhadgaonkar in [17] have shown that the ADM is more convenient than the Charpit's method to solve first-order nonlinear PDEs. Sontakke and Pandit [47, 48] investigates the iterative solution of linear and NFPDEs using fractional ADM. Bhadgaonkar and Sontakke [14] obtained exact Solution of Space-Time FPDEs by ADM. El-Wakil and Abdou have discussed a new application of ADM on nonlinear physical equations in [30]. Az-Zo'bi improved the Laplace decomposition method to obtain approximate analytical solutions of linear and nonlinear differential equations and systems in [9] and also apply the modified decomposition method for the solution of isentropic flow of an inviscid gas model (IFIG) in [11]. Az-Zo'bi and Al-Khaled proposed a new convergence proof of the ADM for a mixed hyperbolic elliptic system of conservation laws in [12] and also Az-Zo'bi apply ADM to develop a fast and accurate algorithm for systems of conservation laws of mixed hyperbolic elliptic type

in [10]. Az-Zo'bi et al. [13] have modify the reduced differential transform method to be applicable for a wide range of nonlinear PDEs using Adomian polynomials, and new generalizations of transformed formulas are established.

Rach et al. [44] created a new modification of the ADM for solving ordinary differential equations (ODEs) using the Taylor expansion series for a nonhomogeneous term. N. Khodabakshi et al. [34], discussed the basic ADM method and extended the proposed method in [44] to solve time-fractional ODEs. As a result of these ideas, Dhaigude and Bhadgaonkar in [18] combined the ADM with a fractional Taylor expansion series and obtain an almost analytical solution of physical models such as Gas dynamics model, Advection model, Wave model, and Klein-Gordon model in nonlinear nonhomogeneous space-time fractional PDEs. The fractional Taylor expansion series used in this method is represented as a differential transform in the differential transform method [37]. This method is developed specifically for nonhomogeneous differential equations. The main aim of this paper is to implement improved ADM to solve the linear nonhomogeneous space-time fractional partial differential equation so we write proposed method in [18] for linear fractional PDE. The solution of this equation is calculated in the form of convergent series with easily computable components. The space-time fractional derivatives are described in the Caputo sense.

The paper is structured in this way: in section 2 few basic results about fractional calculus and related properties are given which are used in this paper, while in section 3 we clarify the steps of the improved ADM for solving nonlinear nonhomogeneous space and time fractional order PDEs in section 4. Section 5 is conclusions.

2 Basic Definitions

In this section, basic definitions on fractional calculus are discussed which are useful for further discussion.

Definition 2.1 *[39] Let $f \in \mathcal{C}_\alpha$ and $\alpha \geq -1$, then Riemann-Liouville fractional integral operator (RLFIO) of $w(x,t)$ with respect to t of order α is indicated by $I_t^\alpha w(x,t)$ and is explained as*

$$I_t^\alpha w(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} w(x,\tau) d\tau, \quad t > 0, \alpha > 0. \qquad (2.1)$$

Definition 2.2 *[39] Let $m - 1 < \alpha < m$, $t \in R$ and $t > 0$. The Caputo fractional derivative operator (CFDO) for the function $f \in H^1([a,b], \mathbb{R}_+)$ with order $\alpha \geq 0$ is explained as*

$$D_t^\alpha w(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \dfrac{\partial^m w}{\partial \tau^m} d\tau, \\ \dfrac{\partial^m w}{\partial t^m}, \qquad \alpha = m \in N. \end{cases} \qquad (2.2)$$

We have following properties of RLFIO and CFDO

$$D_t^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{(\mu-\alpha)}, \tag{2.3}$$

$$I_t^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{(\mu+\alpha)}, \quad \alpha > 0, \ \mu > -1. \tag{2.4}$$

Note that the relation between RLFIO and CFDO is given by:

$$I_t^\alpha D_t^\alpha w(x,t) = w(x,0) - \sum_{k=0}^{m-1} w^{(k)}(x,0)\frac{t^k}{k!}, \quad m-1 < \alpha \le m. \tag{2.5}$$

Mittage-Leffler function (MLF): The MLF for one parameter and two parameter is explained as follows

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbf{C}, Re(\alpha) > 0),$$

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta \in \mathbf{C}, Re(\alpha, \beta) > 0).$$

When we apply CFDO on MLF we get

$$D_t^\alpha E_\alpha(at^\alpha) = aE_\alpha(at^\alpha), \tag{2.6}$$

where a is constant.

3   Analysis of method

Consider the initial value problem (IVP) for space-time FPDE of order $0 < \alpha \le 1$,

$$D_t^\alpha w(x,t) + D_x^\alpha w(x,t) = g(x,t), \tag{3.1}$$
$$w(x,0) = h(x), \tag{3.2}$$

or equivalently

$$L(w(x,t)) = g(x,t), \tag{3.3}$$
$$w(x,0) = h(x), \tag{3.4}$$

where $w(x,t)$ is unrecognized function which we want to determined, t is time variable, x is the space coordinate, $L(w(x,t))$ is fractional differential operator and $g(x,t)$ is nonhomogeneous function.
Now, applying the RLFIO I_t^α on both side of equation(3.1) and use the IC (3.2), we attain:

$$w(x,t) = w(x,0) + I_t^\alpha\big[g(x,t) - D_x^\alpha w(x,t))\big]. \tag{3.5}$$

The unrecognized function $w(x,t)$ can be expressed as an infinite series of the form

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) \tag{3.6}$$

Suppose that $g(x,t)$ is analytic. Its fractional Taylor expansion series [29, 36, 37] is:

$$g(x,t) = \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h)x^{k\alpha}t^{h\alpha}, \tag{3.7}$$

where

$$G_{\alpha,\alpha}(k,h) = \frac{1}{\Gamma(k\alpha+1)\Gamma(h\alpha+1)}(D^{\alpha})_x^k(D^{\alpha})_t^h g(x,t)\big|_{x=t=0}$$
$$\text{and} \quad (D^{\alpha})_x^k = D_x^{\alpha}D_x^{\alpha}...D_x^{\alpha}, \quad k \quad times.$$

By using (3.6) and (3.7) in (3.5) we attain

$$\sum_{n=0}^{\infty} w(x,t) = h(x) + I_t^{\alpha}\left[\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h)x^{k\alpha}t^{h\alpha} - D_x^{\alpha}\sum_{n=0}^{\infty} w_n(x,t)\right], \tag{3.8}$$

$$\sum_{n=0}^{\infty} w(x,t) = h(x) + I_t^{\alpha}\left[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} + \cdots\right]$$
$$- I_t^{\alpha}\left[D_x^{\alpha}\sum_{n=0}^{\infty} w_n(x,t)\right]. \tag{3.9}$$

Taking term by term comparison on both side of equation (3.9), we set recursion scheme like:

$$w_0(x,t) = h(x),$$
$$w_1(x,t) = I_t^{\alpha}\left[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} - D_x^{\alpha}w_0\right],$$
$$w_2(x,t) = I_t^{\alpha}\left[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} - D_x^{\alpha}w_1\right],$$
$$w_3(x,t) = I_t^{\alpha}\left[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha} - D_x^{\alpha}w_2\right],$$

and so forth. Then the solution $w(x,t)$ of IVP $(3.1) - (3.2)$ is

$$\phi_{m+1} = \sum_{n=0}^{m} w_n(x,t) \tag{3.10}$$

which gives

$$\lim_{m\to\infty} \phi_{m+1} = w(x,t). \tag{3.11}$$

4   Numerical Applications

The effectiveness and sharpness of the improved ADM can be demonstrated by applying it to space-time fractional nonhomogeneous linear partial differential equation.

Example 4.1 *Consider the linear nonhomogeneous space-time fractional PDE,*

$$\frac{x^\alpha}{\Gamma(\alpha+1)} D_x^\alpha w(x,t) + D_t^\alpha w(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)} t^\alpha E_{2\alpha,1+\alpha}(t^{2\alpha}) + w, \qquad 0 < \alpha \le 1 \quad (4.1)$$

with initial condition

$$w(x,0) = \frac{x^\alpha}{\Gamma(\alpha+1)}. \tag{4.2}$$

Applying I_t^α on both sides of equation (4.1) and use initial condition (4.2), we have

$$w(x,t) = w(x,0) + I_t^\alpha \left[-\frac{x^\alpha}{\Gamma(\alpha+1)} D_x^\alpha w(x,t) + \frac{x^\alpha}{\Gamma(\alpha+1)} t^\alpha E_{2\alpha,1+\alpha}(t^{2\alpha}) + w \right],$$

$$w(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha \left[-\frac{x^\alpha}{\Gamma(\alpha+1)} D_x^\alpha w(x,t) + \frac{x^\alpha}{\Gamma(\alpha+1)} t^\alpha E_{2\alpha,1+\alpha}(t^{2\alpha}) + w \right]. \tag{4.3}$$

Here

$$g(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)} t^\alpha E_{2\alpha,1+\alpha}(t^{2\alpha}).$$

By using (3.6) and (3.7) in (4.3) we have

$$\sum_{n=0}^\infty w_n(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha \left[-\frac{x^\alpha}{\Gamma(\alpha+1)} D_x^\alpha \sum_{n=0}^\infty w_n + \sum_{k=0}^\infty \sum_{h=0}^\infty G_{\alpha,\alpha}(k,h) x^{k\alpha} t^{h\alpha} + \sum_{n=0}^\infty w_n \right],$$

$$\sum_{n=0}^\infty w_n(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha \left[-\frac{x^\alpha}{\Gamma(\alpha+1)} D_x^\alpha \sum_{n=0}^\infty w_n + \sum_{n=0}^\infty w_n + \sum_{k=0}^\infty G_{\alpha,\alpha}(k,0) x^{k\alpha} t^{0\alpha} \right.$$

$$\left. + \sum_{k=0}^\infty G_{\alpha,\alpha}(k,1) x^{k\alpha} t^\alpha + \sum_{k=0}^\infty G_{\alpha,\alpha}(k,2) x^{k\alpha} t^{2\alpha} + \cdots \right]. \tag{4.4}$$

Here first few coefficients of $G_{\alpha,\alpha}(k,h)$ are given in Table 1.

Taking term by term comparison on both sides of equation (4.4), we set recursion scheme as follows:

Table 1: The components of $G_{\alpha,\alpha}(k,h)$

$G_{\alpha,\alpha}(k,h)$ $k-varies\,vertically$ $h-varies\,horizontally$	$G_{\alpha,\alpha}(k,0)$	$G_{\alpha,\alpha}(k,1)$	$G_{\alpha,\alpha}(k,2)$	$G_{\alpha,\alpha}(k,3)$	$G_{\alpha,\alpha}(k,4)$	$\cdots$
$G_{\alpha,\alpha}(0,h)$	0	0	0	0	0	$\cdots$
$G_{\alpha,\alpha}(1,h)$	0	$\frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+1)}$	0	$\frac{1}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}$	0	$\cdots$
$G_{\alpha,\alpha}(2,h)$	0	0	0	0	0	$\cdots$
$G_{\alpha,\alpha}(3,h)$	0	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

$$w_0(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)},$$

$$w_1(x,t) = I_t^\alpha\left[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha w_0 + w_0 + \sum_{k=0}^{\infty}G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha}\right],$$

$$= I_t^\alpha\left[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)} + 0\right],$$

$$= I_t^\alpha\left[-\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)}\right] = 0,$$

$$w_2(x,t) = I_t^\alpha\left[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha w_1 + w_1 + \sum_{k=0}^{\infty}G_{\alpha,\alpha}(k,1)x^{k\alpha}t^\alpha\right],$$

$$= I_t^\alpha\left[-0 + 0 + \frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^\alpha}{\Gamma(\alpha+1)}\right],$$

$$= \frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$w_3(x,t) = I_t^\alpha\left[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha w_2 + w_2 + \sum_{k=0}^{\infty}G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha}\right],$$

$$= I_t^\alpha\left[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha\left[\frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] + \frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 0\right],$$

$$= I_t^\alpha\left[-\frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right] = 0,$$

$$w_4(x,t) = I_t^\alpha\left[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha w_3 + w_3 + \sum_{k=0}^{\infty}G_{\alpha,\alpha}(k,3)x^{k\alpha}t^{3\alpha}\right],$$

$$= I_t^\alpha\left[-0 + 0 + \frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right],$$

$$= \frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)},$$

and so on. Then the exact solution of IVP (4.1)-(4.2) is

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)}\left[1 + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots\right],$$

$$= \frac{x^\alpha}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{t^{2\alpha k}}{\Gamma(2\alpha k+1)},$$

$$w(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)}E_{2\alpha}(t^{2\alpha}). \tag{4.5}$$

If $\alpha = 1$ then IVP (4.1)-(4.2) is

$$xw_x + w_t = x\sinh(t) + w, \tag{4.6}$$

with IC

$$w(x,0) = x. \tag{4.7}$$

the exact solution of given IVP is

$$w(x,t) = x\cosh(t). \tag{4.8}$$

If $\alpha = \frac{1}{2}$ then IVP (4.1)-(4.2) is

$$2\sqrt{\frac{x}{\Pi}}D_x^{\frac{1}{2}}w(x,t) + D_t^{\frac{1}{2}}w(x,t) = 2\sqrt{\frac{xt}{\Pi}}E_{1,\frac{3}{2}}(t) + w, \tag{4.9}$$

with IC

$$w(x,0) = 2\sqrt{\frac{x}{\Pi}}, \tag{4.10}$$

the solution of given IVP is

$$w(x,t) = 2\sqrt{\frac{x}{\Pi}}e^t. \tag{4.11}$$

Remark 4.1 : *Figure 1 is the graphical behaviour of improved ADM solution (4.5) for different values of α such as $\alpha = 1, 0.8, 0.6, 0.4$ and exact solution (4.8) when $x = 0.25$. 3D graphical representation of solution (4.5) of IVP (4.1)-(4.2) for different values of α such as $\alpha = 1, 0.8, 0.6, 0.4$ are given in Figure2 and Figure3. It is clear from Figure 1, Figure 2 and Figure 3 that, when the limit $\alpha \to 1$, the solution (4.5) approaches to the exact solution (4.8).*
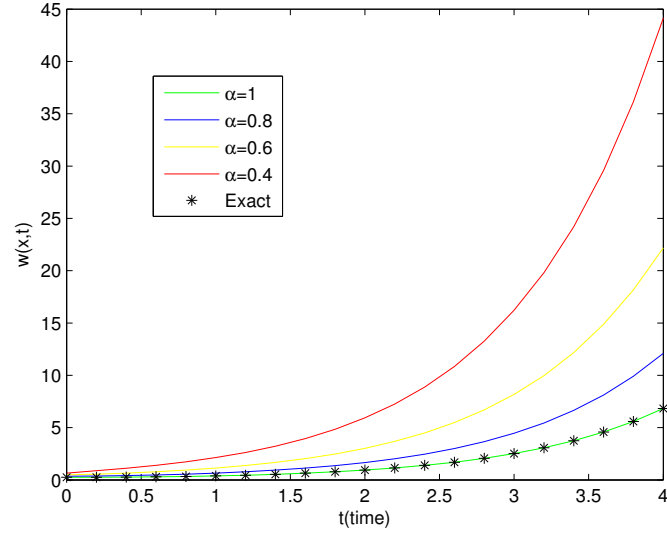
Figure 1: 2D Graphical representation of solution (4.5) of IVP (4.1)-(4.2) for different values of α such as $\alpha = 1, 0.8, 0.6, 0.4$ and exact when $x = 0.25$.
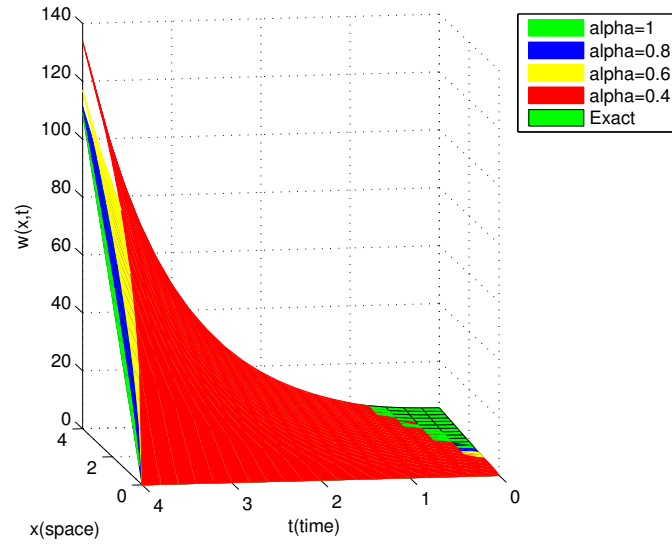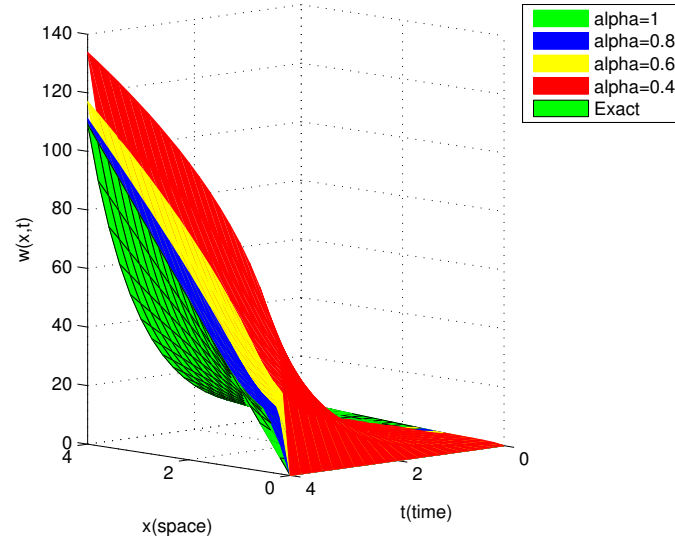

Figure 2

Figure 3

Figure 2 and Figure 3 shows the 3D Graphical representation of solution (4.5) of IVP (4.1)-(4.2) for different values of α such as $\alpha = 1, 0.8, 0.6, 0.4$ and exact solution (4.8).

5   Conclusions

The applicability of improved ADM is demonstrated by some physically significant linear nonhomogeneous fractional partial differential equation of order $0 < \alpha \leq 1$. It returns either a fast convergent series or an exact solution. Another advantage of this method is that we can see where we want to stop the recursion by looking at the coefficient table that is created during the process. The solution of these models are in series form may have rapid convergence to a closed-form solution. It is a more convenient way to solve such types of partial differential equation with the help of improved ADM than general ADM.

Acknowledgement

References

[1] G. Adomian and R. Rach,*Inversion of nonlinear stochastic operators*, J. Math. Anal. Appl. **91**, No. 1(1983)39-46.

[2] G. Adomian, *Convergent series solution of nonlinear equations*, J. Comput. Appl. Math. 1984 Oct **11**, No.2(1984a)225-30.

[3] G. Adomian, *On the convergence region for decomposition solutions*, J. Comput.Appl. Math. **11**, No.3(1984b)379-80.

[4] G. Adomian, *Nonlinear stochastic dynamical systems in physical problems*, J. Math. Anal. Appl. **111**, No.1(1985)105-13.

[5] G. Adomian, *Decomposition solution for Duffing and Van der Pol oscillators*, International Journal of Mathematics and Mathematical Sciences **9**, No.4(1986)731-732.

[6] G. Adomian,*A Review of Decomposition Method in Applied Mathematics*, J. Math. Anal. Appl. **135**(1988)501–544.

[7] G. Adomain, *Solving Frontier Problems of Physics: the decomposition method, With a preface by Yves Cherruault*, Academic Publishers Group, Dordrecht 1 (1994).

[8] G. Adomian and R. Rach, *Modified adomian polynomials*, Mathematical and computer modelling **24**, No. 11(1996) 39-46.

[9] E. A. Az-Zo'bi,*Modified Laplace decomposition method*, World Appl. Sci. J. **18**, No.4(2012), 1481-1486.

[10] E. A. Az-Zo'bi, *Construction of solutions for mixed hyperbolic elliptic Riemann initial value system of conservation laws*, Applied Mathematical Modelling **37**, No.8 (2013), 6018-6024.

[11] E. A. Az-Zo'bi,*An approximate analytic solution for isentropic flow by an inviscid gas model*, Archives of Mechanics **66**, No.3 (2014), 203-212.

[12] E. A. Az-Zo'bi and K. Al-Khaled, *A new convergence proof of the Adomian decomposition method for a mixed hyperbolic elliptic system of conservation laws*, Applied Mathematics and Computation **217**, No.8 (2010), 4248-4256.

[13] E. A. Az-Zo'bi, M. O. Al-Amb, A. Yildirim and W. A. Alzoubi, *Revised reduced differential transform method using Adomian's polynomials with convergence analysis*, Mathematics in Engineering, Science and Aerospace (MESA) **11**, No.4(2020)827-840.

[14] V. N. Bhadgaonkar and B. R. Sontakke, *Exact Solution of Space-Time Fractional Partial Differential Equations by Adomian Decomposition Method*, J. Adv. Math. Comput. Sci. **36**, No.6(2021)75-87. https://doi.org/10.9734/jamcs/2021/v36i630373.

[15] Y. Cherruault,*Convergence of Adomian's decomposition method*, Math. Comput. Modelling **14**, (1990) 83–86.

[16] Y. Cherruault, G. Adomian, K. Abbaoui, R. Rach,*Further remarks on convergence of decomposition method*, Int. J. Bio-med. Comput. **38**, No.1(1995)89–93.

[17] C. Chitalkar-Dhaigude and V. Bhadgaonkar, *Adomian Decomposition Method Over Charpit's Method for Solving Nonlinear First Order Partial Differential Equations*, Bull. Marathwada Math. Soc. **18**, No.1(2017)1–18.

[18] D. B. Dhaigude and V. N. Bhadgaonkar, Analytical solution of nonlinear non-homogeneous space and time fractional physical models by improved Adomian decomposition method. Punjab Univ. J. Math. **54**, No.2(2022)127-147.

[19] D. B. Dhaigude and G. A. Birajdar, *Numerical Solution of System of Fractional Partial Differential Equations by Discrete Adomian Decomposition Method*, J. Frac. Cal.Appl. **3**, No.12(2012)1–11.

[20] D. B. Dhaigude, G. A. Birajdar and V.R. Nikam, *Adomian Decomposition Method for Fractional Benjamin-Bona-Mahony-Burger's Equations*, Int. J. Appl. Math. Mech. **8**, No.12(2012)42–51.

[21] K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Springer Science and Business Media, (2010).

[22] J. S. Duan, R. Rach, D. Baleanu and A. M. Wazwaz, *A review of the Adomian decomposition method and its applications to fractional differential equations*, Commun. Frac. Calc. **3**, No.2(2012)73-99.

[23] J. S. Duan and R. Rach, *New higher-order numerical one-step methods based on the Adomian and the modified decomposition methods*, Appl. Math. Comput. **218**, No.6(2011)2810-2828.

[24] J. S. Duan and R. Rach, *A new modification of the Adomian decomposition method for solving boundary value problems for higher order differential equations*, Appl. Math. Comput. **218**, No. 8 (2011)4090-4118.

[25] J. S. Duan and R. Rach, *Higher-order numeric Wazwaz–El-Sayed modified Adomian decomposition algorithms*, Computers and Mathematics with Applications **63**, No. 11(2012) 1557-1568.

[26] J. S. Duan, R. Rach and A.M. Wazwaz, *Solution of the model of beam-type micro- and nano-scale electrostatic actuators by a new modified Adomian decomposition method for nonlinear boundary value problems*, International Journal of Non-Linear Mechanics **49**, (2013) 159-169, https://doi.org/10.1016/j.ijnonlinmec.2012.10.003.

[27] J. S. Duan, R. Rach and A. M. Wazwaz, *A reliable algorithm for positive solutions of nonlinear boundary value problems by the multistage Adomian decomposition method*, Open Engineering **5**, No. 1(2014)59-74.

[28] J. S. Duan, R. Rach and A. M. Wazwaz, *Simulation of the eigenvalue problem for tapered rotating beams by the modified decomposition method*, International

Journal for Computational Methods in Engineering Science and Mechanics **23**, No.1 (2021)20-28, https://doi.org/10.1080/15502287.2021.1904461.

[29] M. Edelman, Elbert E.N.Macau and Miguel A.F.Sanjuan, *Chaotic, Fractional and Complex Dynamics: New Insights and Perspectives*, Springer Int. Publ.,(2018).

[30] S. A. El-Wakil and M. A. Abdou, *New applications of Adomian decomposition method*, Chaos, Solitons & Fractals **33**, No.2(2007)513-522.

[31] B. Guo, Xueke Pu and Fenghui Huang, *Fractional partial differential equations and their numerical solutions*, New Jersey:World Scientific,(2015).

[32] R. Herrmann, *Fractional Calculus: An Introduction for Physicists*, World Scientific, River Edge, New Jerzey, 2nd edition, (2014).

[33] H. Jafari and V. Daftardar-Gejji, *Adomian Decomposition: A Tool for Solving a System of Fractional Differential Equations*, J. Math. Anal. Appl. **301**, No.2(2005)508-518.

[34] N. Khodabakhshi, S. Mansour Vaezpour and Dumitru Baleanu, *Numarical Solution of the Initial Value Problem for Fractional Differential Equation By Modification of Adomian Decomposition Method*, Frac. Cal. Appl. Anal. **17**, No.2(2014)382–400.

[35] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo,*Theory and Applications of Fractional Differential Equations*,Elsevier: Amsterdam,The Netherlands, (2006).

[36] Z. Odibat and N. Shawagfeh, *Generalized Taylor's formula*, Appl. Math. Comput. **186**, No.1(2007) 286–293.

[37] Z. Odibat and S. Momani, *A generalized differential transform method for linear partial differential equations of fractional order*, Appl. Math. Lett. **21**, No.2(2008)194–199.

[38] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, (1974).

[39] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, (1999).

[40] R. Rach, *A bibliography of the theory and applications of the Adomian decomposition method, 1961-2011*, Kybernetes **41**, No.7/8 (2012)1087-1148.

[41] R. Rach, J. S. Duan and A. M. Wazwaz, *On the solution ofnon-isothermal reaction-diffusion model equations in a spherical catalyst by the modified Adomian method*, Chemical Engineering Communications **202**, No. 8(2015)1081-1088, https://doi.org/10.1080/00986445.2014.900054.

[42] R. Rach, A. M. Wazwaz and J. S. Duan, *The Volterra integral form of the Lane–Emden equation: New derivations and solution by the Adomian decomposition method*, J. Appl. Math. Comput. **47**, No.1(2015)365-379.

[43] R. Rach, J. S. Duan and A. M. Wazwaz, *Simulation of large deflections of a flexible cantilever beam fabricated from functionally graded materials by the Adomian decomposition method*, Int. J. Dyn. Syst. Differ. Equ. **10**, No.4(2020)287-298.

[44] R. Rach, A. M. Wazwaz and J. S. Duan, *A Reliable Modification of the Adomian Decomposition Method for Higher-Order Nonlinear Differential Equations*, Kybernetes **42**, No.2(2013)282–308.

[45] R. Rach, A. M. Wazwaz, J. S. Duan, *A reliable modification of the Adomian decomposition method for higher-order nonlinear differential equations*, Kybernetes **42**, No. 1/2 (2013)282-308.

[46] N. T. Shawagfeh, *Analytical Approximate Solution for Nonlinear Fractional Differential Equations*, Appl. Math. Comput. **131**, No.2-3(2002)517–529.

[47] B. R. Sontakke and R. Pandit, *Convergence analysis and approximate solution of fractional differential equations*, Malaya J. Mat. **7**, No.2(2019)338-344.

[48] B. R. Sontakke, Adomian decomposition method for solving highly nonlinear fractional partial diffeential equations, IOSR j. Eng. **9**, No.3(2019)39-44.

[49] A. M. Wazwaz, R. Rach and J.S. Duan, *Adomian decomposition method for solving the Volterra integral form of the Lane–Emden equations with initial values and boundary conditions*, Appl. Math. Comput. **219**, No.10(2013)5004-5019.