
Safe Sets in Some Graph Families

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Original Article

Abstract

For a connected simple graph G , a non-empty set $S \subseteq V(G)$ of vertices is a safe set if, for every component A of $\langle S \rangle_G$ and every component B of $\langle V(G) - S \rangle_G$ adjacent to A , it holds that $|A| \geq |B|$. The safe number denoted by $s(G)$ of G is the minimum cardinality of a safe set G . In this paper, it examines the characterization of a safe set in complete bipartite graph. It also discusses the minimum cardinality of a safe sets of path graph and cycle graph via modulus. Moreover, this study generates the possible exact values of the safe number of the complete graph, complete bipartite graph, and star graph.

Keywords: Safe Sets, Safe Number.

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1 Introduction

A facility location problem (FLP) refers to the placement and management of a facility in order to obtain or achieve the maximum goal with minimizing costs. The literature on combinatorial optimization can be used to learn more about FLPs. Fujita et al., [7] studied the FLP and introduced the concept of safe set and connected safe set. Their ideas come from the type of facility location problem in which the goal is to determine a "safe" subset of nodes in a network where facilities can be placed. They established that obtaining a minimal safe set and a minimum connected safe set are both NP-hard tasks in general. They also demonstrated that in linear time, a minimum connected safe set in a tree can be discovered. It was shown in [7] that $s(P_n) = cs(P_n) = \lceil \frac{n}{3} \rceil$ and $s(C_n) = cs(C_n) = \lceil \frac{n}{2} \rceil$, where P_n and C_n are the path and cycle graph.

The study of Fujita et al., [7] has motivated this study. In this paper, we extended the study of safe sets in some graph families. We characterize the safe sets in graphs resulting from complete bipartite graph. We also present a new method in computing the minimum cardinality of path graph and cycle graph, this method is via modulus. Lastly, we generate the exact values of the safe number of complete graph, complete bipartite graph, and star graph.

All graphs under considered here are undirected and nontrivial connected simple graph.

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2 Preliminary Notes

Some definitions of the concepts covered in this study are included below. You may refer on the remaining terms and definition in [1,2,3,4,5,6].

Definition 2.1. [5] Let G and H be graphs. The *complete product* or the *join* of G and H , denoted by $G \vee H$, is a graph having a vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{v_1 v_2 : v_1 \in E(G), v_2 \in V(H)\}$.

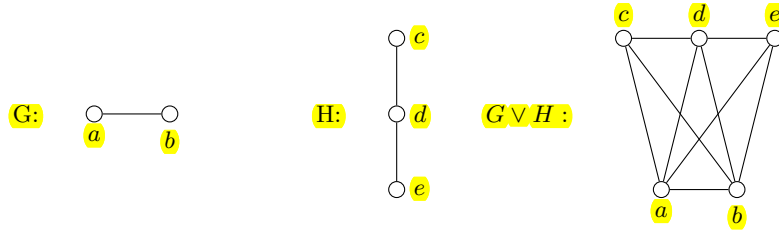


Figure 1: The join of graph P_2 and graph P_3 .

Definition 2.2. [6] The subgraph of a graph G induced by $S \subseteq V(G)$ is denoted by $\langle S \rangle_G$. A *component* of G is a connected induced subgraph of G with an inclusionwise maximal vertex set. For vertex-disjoint subgraphs A and B of G , if there is an edge between A and B , then A and B are *adjacent*. A non-empty set $S \subseteq V(G)$ of vertices is a *safe set* if, for every component A of $\langle S \rangle_G$ and every component B of $\langle V(G) - S \rangle_G$ adjacent to A , it holds that $|A| \geq |B|$. The safe number denoted by $s(G)$ of G is the minimum cardinality of a safe set of G .

To understand Definition 2.2, consider the path P_6 in the given figure below. It can be seen that set $S = \{c, d\}$ is a safe set and a minimum safe set.



Figure 2: The path graph P_6 .

Observe that $\langle V(P_6) - S \rangle_{P_6} = \{B_1, B_2\}$, where $B_1 = \{a, b\}$ and $B_2 = \{e, f\}$, each of which with $|B_1| = 2$ and $|B_2| = 2$. It shows that $|S| \geq |B_1|$ and $|S| \geq |B_2|$. Hence, $s(P_6) = 2$.

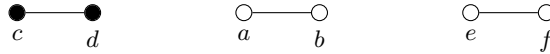


Figure 3: The path P_6 with a safe set $S = \{c, d\}$ and $s(P_6) = 2$.

3 Main Results

In this section, the safe number of path, cycle, complete graph, complete bipartite graph, and star graph are shown. As well as, the characteristics of a safe set of complete bipartite graph.

3.1 Safe Number of the Path Graph, P_n

For convenience, we consider the path graph G of order $n \geq 2$ with vertex set $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$. The following shows the parameter of the safe number of path graph via modulus.

Theorem 3.1. Let $G = (V, E)$ be a nontrivial connected graph. If $G = P_n$, $n \geq 2$, then

$$s(P_n) = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 \pmod{3} \\ \frac{n+2}{3} & \text{if } n \equiv 1 \pmod{3} \\ \frac{n+1}{3} & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

Proof. Let $P_n = \{v_0, v_1, \dots, v_{n-1}\}$, $n \geq 2$, and S be a non-empty subset of $V(P_n)$. Consider the following cases:

Case 1: $n \equiv 0 \pmod{3}$

Choose $S = \{v_k, \dots, v_{2k-1}\}$ where $k = \frac{n}{3}$. Then the component of $\langle S \rangle_{P_n}$ is S itself with $|S| = \frac{n}{3}$. Now, $V(P_n) - S = \{v_0, \dots, v_{k-1}\} \cup \{v_{2k}, \dots, v_{n-1}\}$. Clearly, $\langle V(P_n) - S \rangle_{P_n}$ has two components, $B_1 = \{v_0, \dots, v_{k-1}\}$ and $B_2 = \{v_{2k}, \dots, v_{n-1}\}$, each of which with $|B_1| = \frac{n}{3}$ and $|B_2| = \frac{n}{3}$. Hence, $|S| \geq |B_1|$ and $|S| \geq |B_2|$. Thus, S is a safe set. Now, we want to show that S is a minimum safe set of P_n . Suppose S is not a minimum safe set of P_n . Then there exist a safe set $S_0 \subseteq V(P_n)$ such that $|S_0| < |S|$. Thus, $|S_0| < \frac{n}{3}$ and $|V(P_n) - S_0| > 2(\frac{n}{3})$. Then for each component A in $\langle S_0 \rangle_{P_n}$, $|A| < |S_0| < \frac{n}{3}$ and there exist a component B in $\langle V(P_n) - S_0 \rangle_{P_n}$ such that $|B| > \frac{n}{3}$. Thus, $|A| < |B|$. A contradiction. Thus, S is a minimum safe set. Hence, $s(P_n) = \frac{n}{3}$.

Case 2: $n \equiv 1 \pmod{3}$

Choose $S = \{v_k, \dots, v_{2k}\}$, where $k = \frac{n-1}{3}$. Then the component of $\langle S \rangle_{P_n}$ is S itself with $|S| = \frac{n+2}{3}$. Now, $V(P_n) - S = \{v_0, \dots, v_{k-1}\} \cup \{v_{2k+1}, \dots, v_{n-1}\}$. Clearly, $\langle V(P_n) - S \rangle_{P_n}$ has two components, $B_1 = \{v_0, \dots, v_{k-1}\}$ and $B_2 = \{v_{2k+1}, \dots, v_{n-1}\}$, each of which with $|B_1| = \frac{n-1}{3}$ and $|B_2| = \frac{n-1}{3}$. Hence, $|S| \geq |B_1|$ and $|S| \geq |B_2|$. Thus, S is a safe set. Now, we want to show that S is a minimum safe set of P_n . Suppose S is not a minimum safe set of P_n . Then there exist a safe set $S_0 \subseteq V(P_n)$ such that $|S_0| < |S|$. Thus, $|S_0| < \frac{n+2}{3}$ and $|V(P_n) - S_0| > 2(\frac{n-1}{3})$. Now for each component A in $\langle S_0 \rangle_{P_n}$, $|A| < |S_0| < \frac{n+2}{3}$ and there exist a component B in $\langle V(P_n) - S_0 \rangle_{P_n}$ such that $|B| > \frac{n+2}{3}$. Thus, $|A| < |B|$. A contradiction. Thus, S is a minimum safe set. Hence, $s(P_n) = \frac{n+2}{3}$.

Case 3: $n \equiv 2 \pmod{3}$

Choose $S = \{v_k, \dots, v_{2k}\}$, where $k = \frac{n-2}{3}$. Then the component of $\langle S \rangle_{P_n}$ is S itself with $|S| = \frac{n+1}{3}$. Now, $V(P_n) - S = \{v_0, \dots, v_{k-1}\} \cup \{v_{2k+1}, \dots, v_{n-1}\}$. Clearly, $\langle V(P_n) - S \rangle_{P_n}$ has two components, $B_1 = \{v_0, \dots, v_{k-1}\}$ and $B_2 = \{v_{2k+1}, \dots, v_{n-1}\}$, each of which with $|B_1| = \frac{n-2}{3}$ and $|B_2| = \frac{n-2}{3}$. Hence, $|S| \geq |B_1|$ and $|S| \geq |B_2|$. Thus, S is a safe set. Now, we want to show that S is a minimum safe set of P_n . Suppose S is not a minimum safe set of P_n . Then there exist a safe set $S_0 \subseteq V(P_n)$ such that $|S_0| < |S|$. Thus, $|S_0| < \frac{n+1}{3}$ and $|V(P_n) - S_0| > \frac{n-2}{3}$, $|V(P_n) - S_0| > \frac{n+1}{3}$. Now for each component A in $\langle S_0 \rangle_{P_n}$, $|A| < |S_0| < \frac{n+1}{3}$ and there exist a component B in $\langle V(P_n) - S_0 \rangle_{P_n}$ such that $|B| > \frac{n+1}{3}$. Thus, $|A| < |B|$. A contradiction. Thus, S is a minimum safe set. Hence, $s(P_n) = \frac{n+1}{3}$. \square

3.2 Safe Number of the Cycle Graph, C_n

For convenience, we consider the cycle graph G of order $n \geq 3$ with vertex set $V(G) = \{v_0, v_1, \dots, v_{n-1}\}$. The following shows the parameter of the safe number of cycle graph via modulus.

Theorem 3.2. For a cycle graph G of order $n \geq 3$, the following holds:

$$s(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0(\text{mod } 2) \\ \frac{n+1}{2} & \text{if } n \equiv 1(\text{mod } 2) \end{cases}$$

Proof. Let $C_n = \{v_0, \dots, v_{n-1}\}$, $n \geq 3$, S be a non-empty subset of $V(C_n)$. Consider the following cases:

Case 1: $n \equiv 0(\text{mod } 2)$

Choose $S = \{v_k, \dots, v_{2k}\}$, where $k = \frac{n-2}{2}$. Then the component of $\langle S \rangle_{C_n}$ is S itself with $|S| = \frac{n}{2}$. Now, $V(C_n) - S = \{v_0, \dots, v_{k-1}, v_{2k+1}\}$, the component of $\langle V(C_n) - S \rangle_{C_n}$ is $V(C_n) - S$ itself with $|V(C_n) - S| = \frac{n}{2}$. Hence, $|S| \geq |V(C_n) - S|$. Thus, S is a safe set. Now, we want to show that S is a minimum safe set of C_n . Suppose S is not a minimum safe set of C_n . Then there exist a safe set $S_0 \subseteq V(C_n)$ such that $|S_0| < |S|$. Thus, $|S_0| < \frac{n}{2}$ and $|V(C_n) - S_0| > \frac{n}{2}$. Then the component A in $\langle S_0 \rangle_{C_n}$, $|A| < |S_0| < \frac{n}{2}$ and the component B in $\langle V(C_n) - S_0 \rangle_{C_n}$ such that $|B| > \frac{n}{2}$. Thus, $|A| < |B|$. A contradiction. Thus, S is a minimum safe set. Hence, $s(C_n) = \frac{n}{2}$.

Case 2: $n \equiv 1(\text{mod } 2)$

Choose $S = \{v_k, \dots, v_{2k}\}$, where $k = \frac{n-1}{2}$. Then the component of $\langle S \rangle_{C_n}$ is S itself with $|S| = \frac{n+1}{2}$. Now, $V(C_n) - S = \{v_0, \dots, v_{k-1}\}$, the component of $\langle V(C_n) - S \rangle_{C_n}$ is $V(C_n) - S$ itself with $|V(C_n) - S| = \frac{n-1}{2}$. Hence, $|S| \geq |V(C_n) - S|$. Thus, S is a safe set. Now, we want to show that S is a minimum safe set of C_n . Suppose S is not a minimum safe set of C_n . Then there exist a safe set $S_0 \subseteq V(C_n)$ such that $|S_0| < |S|$. Thus, $|S_0| < \frac{n+1}{2}$ and $|V(C_n) - S_0| > \frac{n-1}{2}$. Then the component A in $\langle S_0 \rangle_{C_n}$, $|A| < |S_0| < \frac{n+1}{2}$ and the component B in $\langle V(C_n) - S_0 \rangle_{C_n}$ such that $|B| > \frac{n-1}{2}$. Thus, $|A| < |B|$. A contradiction. Thus, S is a minimum safe set. Hence, $s(C_n) = \frac{n+1}{2}$. \square

Let G be a complete graph K_n with $n \geq 3$. Since the component of $\langle S \rangle_{K_n}$ is S itself and the component of $\langle V(G) - S \rangle_{K_n}$ is $V(K_n) - S$ itself, such that $|S| \geq |V(G) - S|$. Hence, S is a safe set. Thus the following remark holds.

Remark 1. Let G be a complete graph K_n , $n \geq 3$. Then,

$$s(G) = s(K_n) \begin{cases} \frac{n}{2} & \text{if } n \equiv 0(\text{mod } 2) \\ \frac{n+1}{2} & \text{if } n \equiv 1(\text{mod } 2) \end{cases}$$

3.3 Safe Set of the Complete Bipartite Graph, K_n

Before we introduce the safe number of the complete bipartite graph, $K_{m,n}$, we first initiate the characterization of safe sets in the complete product of two empty graphs, since this will be important in proving the parameters of complete bipartite graph, $K_{m,n}$.

Theorem 3.3. Let G and H be empty graphs with $|V(G)| = m$ and $|V(H)| = n$. Then a non-empty subset $S \subseteq V(G \vee H)$ is a safe set of $G \vee H$ if and only if one of the following holds:

- (i) $S \subseteq V(G)$, $S = V(G)$.
- (ii) $S \subseteq V(H)$, $S = V(H)$.

(iii) $S = S_1 \cup S_2$, where S_1 is a non-empty subset of $V(G)$ and S_2 is a non-empty subset of $V(H)$, such that $|S| \geq |V(G \vee H) - S|$ where $1 \leq |S_1| \leq m$ and $1 \leq |S_2| \leq n$.

Proof. Let S be a non-empty subset of $V(G \vee H)$ be a safe set of $G \vee H$. For the case (i), suppose $S \subseteq V(G)$, clearly $S = V(G)$. Similarly, for the case (ii), if $S \subseteq V(H)$. Now for the case (iii). Suppose $S = S_1 \cup S_2$ such that $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$. If either of S_1 or S_2 are empty, then S is in case (i) or (ii). Since $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$, then $|S_1| \geq 1$ and $|S_2| \geq 1$. Suppose further that $|S| < |V(G \vee H) - S|$. Note that, $\langle S \rangle_{G \vee H}$ is connected and $\langle V(G \vee H) - S \rangle_{G \vee H}$ is also connected. Thus, there can only be one component of S and one component of $V(G \vee H) - S$. A contradiction, since S is a safe set. Thus, $|S| \geq |V(G \vee H) - S|$. Obviously, $1 \leq |S_1| \leq m$ and $1 \leq |S_2| \leq n$.

For the converse. Let S be a nonempty subset of $V(G \vee H)$. Suppose S satisfies (i). Then, $\langle S \rangle_{G \vee H} = G$, and for each component A in $\langle S \rangle_{G \vee H} = G$ and each component B in $\langle V(G \vee H) - S \rangle_{G \vee H} = H$, $|A| = 1 \geq 1 = |B|$. Thus, S is a safe set. Similarly for S satisfying (ii). Now, suppose S satisfies (iii). Note that in this case, $\langle S \rangle_{G \vee H}$ and $\langle V(G \vee H) - S \rangle_{G \vee H}$ has only one component in each induced subgraphs, that is the $\langle S \rangle_{G \vee H}$ itself and the $\langle V(G \vee H) - S \rangle_{G \vee H}$ itself. Since, $|S| \geq |V(G \vee H) - S|$, then $|\langle S \rangle_{G \vee H}| \geq |\langle V(G \vee H) - S \rangle_{G \vee H}|$. Thus S is a safe set of $G \vee H$. \square

From Theorem 3.3, the following parameter is an immediate consequence.

Corollary 3.4. Let G be a complete bipartite graph, $K_{m,n}$, then $s(K_{m,n}) = \min\{m, n\}$.

Proof. Let X and Y be the partite set of $K_{m,n}$ where $|X| = m$ and $|Y| = n$. Let S be a non-empty subset of $V(K_{m,n})$. By Theorem 3.3 (i), if $S \subseteq X$ then $|S| = |X| = m$ and S is a safe set of $K_{m,n}$. Now if $S \subseteq Y$, then by Theorem 3.3 (ii) $|S| = |Y| = n$ and S is a safe set of $K_{m,n}$. Thus, $s(K_{m,n}) = \min\{m, n\}$. \square

A special type of graph resulting from the complete bipartite graph $K_{m,n}$ is the star graph $K_{1,n}$. Thus we have the following corollary.

Corollary 3.5. For any star graph, $K_{1,n}$, $n \geq 1$, $s(K_{1,n}) = 1$.

Proof. This immediately follows from Corollary 3.4. \square

4 Conclusion

In this article, the safe set resulting from complete bipartite graph and safe number of path, cycle, complete graph, complete bipartite graph, and star graph are studied. As future line of research, we plan to investigate the safe set and safe number for some other graph families that has not been studied.

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Competing Interests

The authors declare that they have no competing interests.

References

- [1] Acosta H.R., Eballe R.G., Cabahug I.S., *Downhill Domination in the Tensor product of Graphs*, *International Journal of Mathematical Analysis*, Vol. 13, no. 12, 555-564
- [2] Cabahug, I.S., Canoy, S.R.,(2016) *Independent dr-Power Dominating Sets in Graphs*, *Applied Mathematical Sciences*, Vol.10, no. 8, 377-387
- [3] Cabahug, I.S., Canoy, S.R.,(2016) *dr-Power Dominating Sets in Graphs* *International Journal of Mathematical Analysis*, Vol. 10, n0. 3, 139-149
- [4] Cabahug, I.S., Canoy, S.R.,(2016) *Connected dr-Power Dominating Sets in Graph*, *Advances and Applications in Discrete mathematics*, Vol. 19, no. 3, 171-182
- [5] Chartrand, G., Lesniak, L., Zhang, P., (2015). *Graphs & Digraphs (Discrete Mathematics and Its Applications)* (6th ed.), Chapman and Hall/CRC.
- [6] Cohen, N., Agneda, R., Fujita, S., Legay, S., Manoussakis, Y., et al.. *Safe Sets in Graphs: Graph Classes and Structural Parameters*. COCOA 2016 - 10th International Conference Combinatorial Optimization and Applications, Dec 2016, Hong Kong, China. pp.241-253. hal-0162455
- [7] Fujita, S., MacGillivray, G., Sakuma, T., 2016. *Safe Set Problem on Graphs*. *Discrete Applied Mathematics* 215, 106-111. doi:<http://dx.doi.org/10.1016/j.dam.2016.07.020>.
- [8] Fujita, S., Park, B., Jensen, T., and Sakuma, T., *On the Weighted Safe Set Problem on Paths and Cycles*, *Journal of Combinatorial Optimization*, vol. 37, no. 2, pp. 685-701, 2019.
- [9] Kang, B., Kim, S., and Park, B., *On Safe Sets of the Cartesian Product of Two Complete Graphs*, *Ars Combinatoria*, vol. 141, pp. 243-257, 2018.
- [10] Korte, B., Vygen, J., *Combinatorial Optimization*, fifth ed., Springer-Verlag, Berlin Heidelberg, 2011.
- [11] Malaguti, E., & Pedrotti, V. (2021). *A New Formulation for the Weighted Safe Set Problem*. *Procedia Computer Science*, 195, 508-515. <https://doi.org/10.1016/j.procs.2021.11.061>