Spectral Properties Of Compact Operators

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Original Article

Abstract

The spectral properties of a compact operator $T : X \longrightarrow Y$ on a normed linear space resemble those of square matrices. For a compact operator, the spectral properties can be treated fairly completely in the sense that Fredholm's famous theory of integral equations may be extended to linear functional equations $Tx - \lambda x = y$ with a complex parameter λ. This paper has studied and investigated the spectral properties of compact operators in Hilbert spaces. The spectral properties of compact linear operators are relatively simple generalization of the eigenvalues of finite matrices. As a result, the paper has given a number of corresponding propositions and interesting facts which are used to prove basic properties of compact operators. The Fredholm theory has been introduced to investigate the solvability of linear integral equations involving compact operators.

Keywords: spectral properties, normed linear space, Fredholm theory, compact operators
Mathematics Subject Classification:

1 Introduction

Let X be a normed linear space and $T : X \to Y$ be a linear transformation. A point $\lambda \in \mathbb{C}$ is said to belong to the **resolvent set of** T (denoted by $\rho(T)$) if $T - \lambda I$ is O- invertible. Thus

$$\rho(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)\} \text{ is 0-invertible}$$

The complement of $\rho(T)$, i.e. the subset $\mathbb{C} - \rho(T)$ is called the **spectrum of** T and represented by the symbol $\sigma(T)$. Thus $\sigma(T) = \{\lambda \in \mathbb{C} : (T - \lambda I) \text{ or } (\lambda I - T)\}$ is not invertible. If λ is an eigenvalue of T, then there exists an $x \in X$ such that $x \neq \bar{0}$ and $Tx = \lambda x$ (x is called on eigenvector corresponding to the eigenvalue λ) [5].

Hence $(T - \lambda I)x = \bar{0}$ for an $x \neq \bar{0}$. Thus $T - \lambda I$ is not one to one. Hence $T - \lambda I$ is not O-invertible. Therefore, λ is in the spectrum of T i.e. $\lambda \in \sigma(T)$. Thus λ is an eigenvalue of T implies that $\lambda \in \sigma(T)$. Thus the spectrum of T includes all the eigenvalues of T [2].

In general, $\sigma(T)$ includes some points of $\mathbb{C}$ which are not even eigenvalues of T. For instance even

if $T-\lambda I$ is a bijection for some $\lambda \in \mathbb{C}$, if the set inverse $(T-\lambda I)^{-1}$ (which exists) is not bounded, then $\lambda \in \sigma(T)$.
The null space $\eta_{T-\lambda I}$ of the operator $T-\lambda I$ is called the eigenspace of T corresponding to the eigenvalue λ [8].

Illustration: We have seen that if X is a Banach space, $T \in B(x)$ and $|\beta| > \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ the $\beta I - T$ or $T - \beta I$ is O-invertible [10]. Thus if $|\beta| > \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ then $\beta \in \rho(T)$ $\left(\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r_\sigma(T)\right.$ is the spectral radius of T). Thus if $|\beta| > r_\sigma(T) \Rightarrow \beta \in \rho(T)$

Therefore, the complement of the spherical ball of radius $r_\sigma(T)$ consists of points belonging to the resolvent set of T. Hence the spectrum $\sigma(T)$ of T must be contained inside the neighborhood $\bar{N}(O, r_\sigma(T))$. This paper has studied and investigated the spectral properties of compact operators on Hilbert spaces. It has introduced the famous Fredholm theory to investigate the solvability of the linear integral equations involving compact operators. The definitions in this paper are all standard and can be found in [1], [3], [4] , [6], [7], [9], [10].

2 Spectral Properties

The spectral theory for compact linear operators represents the most natural introduction to the general spectral theory of linear operators in a Hilbert space.

Proposition 2.1. *Let H be a Hilbert space and $T \in B(H)$. Then the following conditions are equivalent dim $(H) = \infty$:*

(i) T is compact.

(ii) $\|T - F_n\| \to 0$ as $n \to \infty$.

(iii) There is a sequence (F_n) of operators of finite rank on (H) such that $\|T - F_n\| \to 0$ as $n \to \infty$.

Proof. (i) $\Rightarrow$ (ii) : Suppose (i) is satisfied. We know that $e_n \xrightarrow{w} \bar{0}$. Since T is compact T$e_n \xrightarrow{s} \bar{0}$. Using the Cauchy–Bunyakovsky–Schwarz Inequality we have

$$\lim_{n\to\infty} |\langle Te_n, c_n\rangle| \leq \lim_{n\to\infty} \|Te_n\| = 0.$$

(ii) $\Rightarrow$ (iii) : Given a positive integer, consider the class H of all orthonormal sets $E \in (H)$ for which

$$|\langle Te, e\rangle| \geq \frac{1}{4n}(e \in E)$$

(We allow the void set as one possible choice of E). By the hypothesis, each $E \in H$ is a finite set. Since the union of a strictly increasing sequence of sets in H is again a member of H (and therefore finite) each such sequence terminates. It follows that H has a maximal element E_0. If M is the (finite-dimensional) linear subspace of H generated by E_0, then $|\langle Tx, x\rangle| < \frac{1}{4n}$ whenever $x \in M^\perp$ and $\|x\| = 1$; for otherwise H contains $E_0 \cup \{x\}$, contradicting the maximality of E_0. Then $|\langle Tx, x\rangle| \leq \frac{1}{n}$ whenever $x \in M^\perp$ and $\|x\| \leq 1$. From the relation

$$\langle Tu, v\rangle = \frac{1}{4}[\langle T(u+v), u+v\rangle - \langle (u-v, u-v)\rangle] + i[\langle T(u+iv), u+iv\rangle - \langle T(u-iv), u-iv\rangle]$$

it follows that

$$|\langle Tu, v\rangle| \leq \frac{1}{n} \text{ for } u, v \in M^\perp, \|u\| \leq 1, \|v\| \leq 1 \tag{2.1}$$

By taking $u = (I-P)x$ and $v = (I-P)y$, where p is the orthoprojector on H onto M, we deduce from (2.1) that

$$|\langle (I-P)T(I-P)x, y\rangle| \leq \frac{1}{n}$$

whenever $x, y \in H, \|x\| \leq 1$ and $\|y\| \leq 1$. Then $\|(I-P)T(I-P)\| \leq \frac{1}{n}$. The operator

$$F_n = PT + TP - PTP$$

has finite rank and $\|T - F_n\| \leq \frac{1}{n}$. Since F_n is compact and $B_\infty(H)$ is closed in $B(H)$, it follows that T is compact. □

Remark 2.1. *If* f *is a mapping from a set* A *of infinite cardinality into a normed linear space* X*, we shall say that* f *vanishes at infinity and write $f(a) \to 0$ as $n \to \infty$, if the following condition is satisfied: Given any positive ε, the set*

$$\{a \in A : \|f(a)\| \geq \varepsilon\}$$

is finite. When this is so, the set $\{a \in A : f(a) \neq \bar{0}\}$ is at most countable since it is the union of all the finite sets $\{a \in A : \|f(a)\| \geq \frac{1}{n}\}$ $(n \in \mathbb{N})$. Clearly, if $\{\int(a) : a \in A\}$ is summable, then $f(a) \to \infty$ as $n \to \infty$. With this interpretation, we can rewrite in place of condition (ii) in Proposition 2.1 in the form:
(ii)′ For every orthonormal system $(\{e_\alpha : \alpha \in \Lambda\} \in H)$ $\langle Te_\alpha, e_\alpha\rangle \to \infty$ as $n \to \infty$ (in the sense described in the Remark above).

Proposition 2.2. *Let $T \in B(H)$ be compact and $\lambda \in \mathbb{K}$ be not 0 . Then $\mathcal{R}(\lambda I - T)$ is closed.*

Proof. Let $y \in \overline{\mathcal{R}(\lambda I - T)}$. So there is a sequence (x'_n) of elements in H such that

$$y_n = (\lambda I - T)x'_n \xrightarrow{s} y$$

writing $H = \eta^{\perp}_{\lambda 1 - T} \oplus \eta_{\lambda I - T}$, we have the decomposition $x'_n = x_n + x''_n$ where $x_n \in \eta^{\perp}_{\lambda I - T}$ and $x''_n \in \eta_{\lambda I - T}$ and hence

$$y_n = (\lambda I - T)(x_n + x''_n) \xrightarrow{s} y.$$

But $(\lambda I - T)x''_n = \bar{0}$ $(x''_n \in \eta_{\lambda I - T})$ i.e. $y_n = (\lambda I - T)x_n \xrightarrow{s} y$.
We shall show that (x_n) is bounded. Assume the contrary. Then we can choose a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| \to 0$ as $k \to \infty$. Without loss of generality, we may assume that $\|x_n\| \to 0$ as $n \longrightarrow \infty$. Define $\hat{x}_n = \|x_n\|^{-1} x_n \quad \forall n \in \mathbb{N}$. (So $\hat{x}_n \in \eta^{\perp}_{\lambda I - T}$) so $\|\hat{x}_n\| = 1 \quad \forall n \in \mathbb{N}$. Since $(\hat{x}_n)$ is bounded (for $\|\hat{x}_n\| = 1 \quad \forall n \in \mathbb{N}$) and T is compact; there exists a subsequence $(\hat{x}_{n_k})$ of $(\hat{x}_n)$ such that $T\hat{x_n}$ converges strongly to some element of H. Now

$$\hat{x}_{nk} = \frac{1}{\lambda}\{(\lambda I - T)\hat{x}_{n_k} - T\hat{x}_{n_k}\}$$

Since

$$(\lambda I - T)\hat{x}_n = (\lambda I - T)\frac{x_n}{\|x_n\|} = \frac{y_n}{\|x_n\|} \longrightarrow \bar{0} \tag{2.2}$$

So $(\lambda I - T)\hat{x}_{n_k} \xrightarrow{s} \bar{0}$ by (2.2)
Therefore, $\hat{x}_{n_k} = \frac{1}{\lambda}\{(\lambda I - T)\hat{x}_{n_k} - T\hat{x}_{n_k}\} \xrightarrow{s} x$ for some $x \in H$
Now $\|\hat{x}_{n_k}\| = 1 \quad \forall n \in N$.
Therefore, $\|\hat{x}_{n_k}\| \longrightarrow \|x\|$ as $k \to \infty$. Therefore $\|x\| = 1$. Since each $\hat{x}_{n_k} \in \eta^{\perp}_{\lambda I - T}$ and $\eta^{\perp}_{\lambda I - T}$ is closed, so $\lambda \in \eta^{\perp}_{\lambda I - T}$.

On the other hand, $(\lambda I - T)\hat{x}_n \xrightarrow{s} 0$. We get

$$(\lambda I - T)\hat{x}_{n_k} \xrightarrow{s} 0 \tag{2.3}$$

Since $\hat{x}_{n_k} \xrightarrow{s} x$ and $\lambda I - T$ is bounded. Therefore, $(\lambda I - T)\hat{x}_{n_k} \to (\lambda I - T)x = \bar{0}$ by 2.3
i.e. $x \in \eta_{\lambda I - T}$

But (2.2) contradicts (2.3) since $\|x_n\| = 1$. This contradiction shows that the supposition (x_n) is

unbounded is unacceptable.
Therefore, (x_n) must be bounded. Hence since T is compact there must be a sequence (x_{n_k}) of (x_n) such that Tx'_{n_k} converges strongly. Hence there is a subsequence (x'_{n_k}) of (x_n) such that $x'_{n_k} \xrightarrow{\omega} x$ say (known result).
Since T is compact, $Tx'_{n_k} \xrightarrow{s} Tx$.
Now $y'_{n_k} = (\lambda I - T)x'_{n_k}$. Since $y_n \xrightarrow{s} y$ and (y'_{n_k}) is a subsequence of (y_n) so $y'_{n_k} \xrightarrow{s} y$ Now

$$x'_{n_k} = \frac{1}{\lambda}\left(y'_{n_k} + Tx'_{n_k}\right) \tag{2.4}$$

Since $y_n \xrightarrow{s} y$ so (y'_{n_k}) being a subsequence of (y_n) also converges to y, i.e. $y'_{n_k} \longrightarrow y$. We have seen above that (Tx'_{n_k}) converges strongly. Hence $x_{n_k} \xrightarrow{s} \hat{x} \in H$. Using this in (2.4)

$$\hat{x} = \frac{1}{\lambda}(y + T\hat{x}) \text{ i.e. } \lambda\hat{x} = y + T\hat{x} \text{ or}$$

$(\lambda I - T)\hat{x} = y$ which implies $y \in \mathcal{R}_{\lambda I - T}$

$$\text{Therefore } \bar{\mathcal{R}}_{\lambda I - T} \subseteq \mathcal{R}_{\lambda I - T}$$

Thus, $\mathcal{R}_{\lambda I - T}$ is closed □

Proposition 2.3. *Let $T \in B(H)$ be compact. Then $P\sigma(T) - \{0\} = \sigma(T) - \{0\}$.*

Proof. Since $P\sigma(T) \subseteq \sigma(T)$, so $P\sigma(T) \subseteq \sigma(T) - \{0\}$. We need to prove the reverse inclusion i.e.

$$P\sigma(T) - \{0\} \subseteq \sigma(T) - \{0\}$$

i.e. for a $\lambda \neq 0$,

$$\lambda \notin P\delta(T) \Rightarrow \lambda \notin \sigma(T)$$

i.e. $\lambda \in \rho(T)$. Let $\lambda \notin P\sigma(T)$. Hence $\lambda I - T$ is one to one. Hence the inverse linear map $(\lambda I - T)^{-1} : \mathcal{R}_{\lambda I - T} \to H$ exists.

By proposition 2.2, $\mathcal{R}_{\lambda I - T}$ is closed in H and so $\mathcal{R}_{\lambda I - T}$ is a Hilbert space (A closed linear subspace of a Banach space is a Banach space (with induced norm)).
Now $\lambda I - T : H \to \mathcal{R}_{\lambda I - T}$ is a bounded bijection and hence by the Banach inverse theorem, $\lambda I - T$ is invertible i.e. $(\lambda I - T)^{-1} \in B(R_{\lambda I - T}, H)$. We will show that $R_{\lambda I - T} = H$; it would then follow that $\lambda \in \rho(T)$ ie $\lambda \notin \sigma(T)$. Suppose H is not true, that $R_{\lambda I - T} = H$ in which case $R_{\lambda I - T}$ is a proper closed subspace of H.

Define $H_0 = H$ and let $H_n = (\lambda I - T)^n H = \mathcal{R}_{(\lambda I - T)^n}$ $\forall n \in \mathbb{N}$. Thus we get a decreasing nested sequence of linear subspaces (all of which are closed)

$$H = H_0 \xrightarrow{\lambda I - T} H_1 \underset{R_{(\lambda I - T)}}{\downarrow} \xrightarrow{\lambda I - T} H_2 \underset{R_{(\lambda I - T)^2}}{\downarrow} \xrightarrow{\lambda I - T} H_3 \underset{R_{(\lambda I - T)^3}}{\downarrow} \dots H_{n-1} \underset{R_{(\lambda I - T)^{n-1}}}{\downarrow} \xrightarrow{\lambda I - T} H_n \underset{R_{(\lambda I - T)^n}}{\downarrow}$$

Indeed, $H_n = \mathcal{R}(\lambda I - T)^n$. Now,

$$\begin{aligned}(\lambda I - T)^n &= \lambda^n I - \lambda^{n-1}T + \lambda^{n-2}T^2 + \cdots + (-1)^n T^n \\ &= \lambda^n I - T\left[\lambda^{n-1}I - \lambda^{n-2}T + \cdots + (-1)^{n-1}T^{n-1}\right] \\ &= \lambda^n I - T_s\end{aligned}$$

Where $s = \lambda^{n-1}I - \lambda^{n-2}T + \cdots + (-1)^{n-1}T^{n-1}$. Since $T \in B(H)$, so $s \in B(H)$. Since T is compact, so TS is compact. Thus $H_n = \mathcal{R}_{(\lambda I-T)^n} = \mathcal{R}(\lambda^n I - Ts)$. We claim that the sequence (H_n) is strictly nested (decreasing). Suppose the contrary, in which case there is $n_0 \in \mathbb{N}$ such that

$$H_{n_0} = H_{n_0+1} \text{ i.e } \mathcal{R}_{(\lambda I-T)^n = \mathcal{R}_{(\lambda I-T)^{n_0+1}}}$$

$$H = H_0 \overset{\lambda I-T}{\longrightarrow} H_1 \underset{R_{(\lambda I-T)}}{\downarrow} \overset{\lambda I-T}{\longrightarrow} H_2 \underset{R_{(\lambda I-T)^2}}{\downarrow} \overset{\lambda I-T}{\longrightarrow} \ldots H_{n-1} \underset{R_{(\lambda I-T)^{n-1}}}{\downarrow} \ldots H_n \underset{R_{(\lambda I-T)^n}}{\downarrow} \overset{\lambda I-T}{\longrightarrow} H_{n+1} \underset{R_{(\lambda I-T)^{n+1}}}{\downarrow}$$

In this case, we get $\left((\lambda I-T)^{-1}\right)^{n_0} H_n = H$ i.e. $(\lambda I-T)^{-n_0} H_{n_0} = H(\lambda I-T)^{-n_0} H_{n_0+1} = H_1$. Since $H_{n_0} = H_{n_0+1}$, so $H = H_1$ which is a contradiction to our assumption. Hence the sequence $(H_n)_{n=1}^{\infty}$ is strictly nested i.e. $\mathrm{H} = \mathrm{H}_0 \subset \mathrm{H}_1 \subset \mathrm{H}_2 \subset \mathrm{H}_3 \subset \ldots$
Since $H_{n-1} \subset H_n \forall n \in \mathbb{N}$, we can from the subspaces $H_{n-1} \ominus H_n = H_{n-1} \cap H_n^{\perp}$ which are all non-empty. (Note $H_{n-1} \ominus H_n = \{x \in H_{n-1} : x \perp H_n\}$). Choose an $x_n \in H_{n-1}$ such that $\|x_n\| = 1$ and $x_n \perp H_n$ $\forall n \in \mathbb{N}$. Since $x_n \in H_{n-1}$, so $(\lambda I-T)x_n \in H_n$. Now $Tx_n = \underbrace{\lambda x_n}_{\in H_{n-1} \ominus H_n} - \underbrace{(\lambda I-T)x_n}_{\in H_n}$

Therefore $\lambda x_n \perp (\lambda I-T)x_n$.
Hence by Pythagorean theorem,

$$\|Tx_n\|^2 = \|\lambda x_n\|^2 + \|(\lambda I-T)x_n\|^2 \geq |\lambda|^2 \|x_n\|^2 = |\lambda|^2 \; \forall n \in \mathbb{N} \text{ i.e. } \|Tx_n\| \geq \lambda \; \forall n \in \mathbb{N} \tag{2.5}$$

Now (x_n) is an orthonormal sequence in H. {Indeed: $x_n \in H_{n-1} \ominus H_n : x_n \in H_{n-1}$ and $x_n \in H_n x_{n+1} \in H_n$ and $x_{n+1} \perp H_{n+1}$
Therefore, $x_n \perp x_{n+1}$ $\forall n \in \mathbb{N}$ and$\|x_n\| = 1$ $\forall n\mathbb{N}$ }. Hence $(x_n) \overset{\omega}{\longrightarrow} \bar{0}$.
Since T is compact,

$$Tx_n \overset{s}{\longrightarrow} \bar{0}$$

i.e. by continuity of the norm $\|.\|$ in H we have $\|Tx_n\| \to \|\bar{0}\| = 0$ as $n \to \infty$.
But this contradicts (2.5) since $\lambda \neq 0$. This shows that the supposition that $\mathcal{R}_{\lambda I-T} \neq H$ is unaccepted. Therefore, $\mathcal{R}_{\lambda I-T} = H$ i.e. $\lambda \in \rho(T)$ i.e. $\lambda \notin \sigma(T)$ and the proof is complete. □

Corollary 2.1. *Let dim $H = \infty$ and T be compact.Then $\sigma(T) = P\sigma(T) \cup 0$.*

Proof. In this case, (by proposition 2.3). So

$$(\sigma(T) - \{0\}) \cup \{0\} = (P\sigma(T) - \{0\}) \cup \{0\}$$
$$\therefore \sigma(T) = P\rho(T) \cup \{0\}$$

It follows from proposition 2.3 and the corollary that when dim $H = +\infty$ any non-zero complex number must be either in $\rho(T)$ or be an eigenvalue of T i.e.

$$\begin{aligned} \lambda \neq 0 &\Rightarrow \lambda \in \sigma(T) \text{ or } \lambda \in \sigma(T) \\ &\Rightarrow \lambda \in \rho(T) \text{ or } \lambda \in P\sigma(T) \cup \{0\} \\ &\Rightarrow \lambda \in \rho(T) \text{ or } \lambda \in P\sigma(T) \end{aligned}$$

□

Proposition 2.4. *Let $T \in B(H)$ be compact and $\rho > 0$. Then there can be at most a finite number of linearly independent eigenvectors of T corresponding to eigenvalues λ of T satisfying $|\lambda| \geq \rho$.*

Proof. Suppose there is an infinite set of linearly independent eigenvectors corresponding to all eigenvalues λ with $|\lambda| \geq \rho$.

{**Background knowledge**: If an eigenvalue λ is of geometric multiplicity n i.e. dim $\eta_{\lambda I-T}=n$ then there exists a linearly independent set $\{x_1,\ldots,x_n\}$ such that

$$[\{x_1,\ldots,x_n]]=\eta_{\lambda I-T}.$$

If λ,μ are distinct eigenvalues and x,y are eigenvectors corresponding to λ,μ respectively, then $\{x,y\}$ is linearly independent. So if $\{x_1,\ldots,x_n\},\{y_1,\ldots,y_n\}$ are linearly independent sets spanning the eigenspaces $\eta_{\lambda I-T},\eta_{\mu I-T}$ respectively then $\{x_1,\ldots,x_n\},\{y_1,\ldots,y_m$ is also linearly independent.

Choose a sequence (x_n) of distinct eigenvectors from this linearly independent set. So $\{x_1,x_2,\ldots,x_n,\ldots\}$ is linearly independent. Let the eigenvalue corresponding to x_n be λ_n.
By employing the Gram- Schmidt orthogonalization process, we can find an orthornomal sequence (e_n) such that

$$[\{x_1,\ldots,x_n\}]=[\{e_1,\ldots,e_n\}]\quad\forall n\in\mathbb{N}$$

and so for each e_n we can write
$e_n=\alpha_{n,1}x_1+\alpha_{n,2}x_2+\cdots+\alpha_{n,n}x_n$.
$\therefore Te_n=\alpha_{n,1}Tx_1+\alpha_{n,2}Tx_2+\cdots+\alpha_{n,n}Tx_n$.
Now $Tx_n=\lambda_nx_n$ $\forall n\in\mathbb{N}$.
$\therefore Te_n=\alpha_{n,1}\lambda_1x_1+\alpha_{n,2}\lambda_2x_2+\ldots+\alpha_{n,n}\lambda_nx_n$.
$\therefore Te_n-\lambda_ne_n=(\alpha_{n,1}\lambda_1x_1+\alpha_{n,2}\lambda_2x_2+\ldots+\alpha_{n,n}\lambda_nx_n)-\lambda_n(x_n\alpha_{n,1}+\alpha_{n,2}x_2+\cdots+\alpha_{n,n}x_n)$.
$=\alpha_{n,1}(\lambda_1-\lambda_n)x_1+\alpha_{n,2}(\lambda_2-\lambda_n)x_2+\cdots+\alpha_{n-1}(\lambda_{n-1}-\lambda_n)x_{n-1}$
$=\beta_{n-1,1}e_1+\beta_{n-2,2}e_2+\cdots+\beta_{n-1,n-1}e_{n-1}$
Since $[\{x_1,\ldots,x_{n-1}\}]=[\{e_1,\ldots,e_{n-1}]]$
$\therefore Te_n=\beta_{n-1,1}e_1+\beta_{n-1,2}e_2+\cdots+\beta_{n-1,n-1}e_{n-1}+\lambda_ne_n$.
Hence

$$\langle Te_n,e_n\rangle=\langle\lambda_ne_n,e_n\rangle=\lambda_2\|e_n\|^2=\lambda_n$$

since (e_n) is an orthornormal sequence. (But $e_n\xrightarrow{\omega}\bar{0}$ and T is compact. So $Te_n\xrightarrow{s}\bar{0}$ etc.)
Infact, we have already seen that, if T is compact

$$\lim_{n\to\infty}|\langle Te_n;\beta_n\rangle|=0$$

i.e. $\lim_{n\to\infty}|\lambda_n|=0$ (For $\langle Te_n,e_n\rangle=\lambda_n\forall n\in\mathbb{N}$) and this contradicts the hypothesis for we consider all eigenvalues λ with $|\lambda|\geq\rho$(so $|\lambda_n|\geq\rho>0$). Hence the supposition that there are infinitely many linearly independent eigenvectors corresponding to all eigenvalues λ with $|\lambda|\geq p$ is unacceptable and this proves the theorem. □

Corollary 2.2. *Let $T\in B(H)$ (dim $H=\infty$) be compact. Then for each $\lambda\neq0$ $\eta_{\lambda I-T}$ must be finite dimensional.*

Proof. Suppose $\eta_{\lambda I-T}$ is infinite dimensional. Let $\{x_n:x\in\mathbb{N}\}$ be a linearly independent subset of H spanning $\eta_{\lambda I-T}$.
Now $|\lambda|\neq0$. So we have an infinite set of linearly independent eigenvectors corresponding to the eigenvalue λ. $\{(T-\lambda I)x_n=\bar{0}\quad\forall n\in\mathbb{N}\}$.
Hence there are infinitely many linearly independent eigenvectors corresponding to eigenvalues μ with $|\mu|\geq|\lambda|$ which contradicts the result of the proposition 2.4.
Hence the supposition that $\eta_{\lambda I-T}$ is infinite dimensional is unacceptable. □

Proposition 2.5. *Let $T\in B(H)$ be compact. Then T has atmost Countably many eigenvalues. If the number of eigenvalue of T is infinite, then 0 is the only limit point of $P\sigma(T)$.*

Proof. Suppose a non-zero λ is a limit point of $P\sigma(T)$. Then we can find an infinite sequence (λ_n) of distinct eigenvalues such that $\lambda_n\to\lambda$ as $n\to\infty$.
Thus there will exist a sequence (x_n) of ements of H such that $x_n\neq\bar{0}$ and x_n is an eigenvector

corresponding to the eigenvalue λ_n. From a result of linear algebra we see that $\{x_n : n \in \mathbb{N}\}$ is linearly independent.

Let $H_n = [\{x_1, \cdots, x_n\}]\, \forall n \in \mathbb{N}, \{H_0 = \{\bar{o}\}\}$. So we have the strict containment

$$H_0 \subset H_1 \subset H_2 \subset \ldots \subset H_{n-1} \subset H_n \subset \ldots.$$

All the H_n are closed linear subspaces of H, (= Hilbert spaces). Consider $H_n \ominus H_{n-1}$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}, \exists y_n \in H_n$ such that $\|y_n\| = 1$ and $y_n \perp H_{n-1}$. Thus we obtain a sequence (y_n) of unit vectors. This sequence is orthonormal.
Indeed, let $m \neq n$ $(m, n \in \mathbb{N})$ and for definiteness, let $m < n$ consider the elements y_m, y_n. Now $y_m \in H_m \ominus H_{m-1}, y_n \in H_n \ominus H_{n-1}$. Therefore, $y_n \in H_n$ and $\perp H_{n-1}$ and $y_m \in H_m$ (and $\perp H_{m-1}$). Since $m < n$, so $m \leqslant n - 1$

$$\therefore H_n \subseteq H_{n-1} \subseteq H_n.$$
$$y_m \in H_n, y_n \in H_n.$$
$$\therefore y_n \in H_{n-1}, y_n \in H_n$$

Now $y_n \in H_n \ominus H_{n-1} \Rightarrow y_n \perp H_{n-1}$. From $y_m \in H_{m-1}$ and $y_n \perp H_{n-1}$ we get $y_m \perp y_n$ and this is valid $\forall m \neq n$.
Thus (y_n) is orthonormal. Hence $y_n \xrightarrow{\omega} \bar{0}$ and since T is compact, $Ty_n \xrightarrow{s} \bar{0} \in H$ say. Now

$$Ty_n = \lambda_n y_n - (\lambda_n I - T)\, y_n \;\forall n \in \mathbb{N} \tag{2.6}$$

the element $\lambda_n y_n \in H_n \ominus H_{n-1}$
On the other hand, since $y_n \in H_n = [(x_1, \ldots, x_n)]$ we can write

$$y_n = a_{n1}x_1 + a_{n2}x_2 + \ldots, a_{nn}x_n \text{ for scalars } a_{ni}\ (i = 1, \ldots n)$$

Consequently,

$$\begin{aligned}(\lambda_n I - T)\, y_n &= (\lambda_n I - T)\,(a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n)\\ &= \lambda_n\,(a_{n1}x_1 + \cdots + a_{nn}x_n) - (a_{n1}Tx_1 + \ldots + a_{nn}Tx_n)\,.\\ &= \lambda_n\,(a_n x_1 + \cdots + a_{nn}x_n) - (a_{n1}\lambda_1 x_1 + \cdots + a_{nn}\lambda_n x_n)\,.\end{aligned}$$

Since λ_i is an eigenvalue with eigenvector x_i $\forall i \in \mathbb{N}$,

$$\begin{aligned}&= (\lambda_n - \lambda_1)\, a_{n1}x_1 + \cdots + (\lambda_n - \lambda_{n-1})\, a_{n1(n-1)}x_{n-1}\\ &\in [\{x_1, \ldots, x_{n-1}\}] = H_{n-1}\end{aligned}$$

Since $\lambda_n y_n \in H_n \ominus H_{n-1}$, so

$$(\lambda_n I - T)\, y_n \perp \lambda_n y_n$$

Hence it follows from 2.6, using Pythagorean theorem

$$\|Ty_n\|^2 = \|\lambda_n y_n\|^2 + \|(\lambda_n I - T)\, y_n\|^2\, \forall n \in \mathbb{N}$$
$$\therefore \|Ty_n\| \geq \|\lambda_n y_n\| = \|\lambda_n\| \to |\lambda|$$

(Since $\lambda_n \to \lambda$ as $n \to \infty$). This contradicts the result $Ty_n \xrightarrow{s} \bar{0}$ (above). Hence the supposition that $\lambda \neq 0$ is a limit point of $P\sigma(T)$ is unacceptable. Thus if the number of eigenvalues is infinite, then 0 can be the only limit point. □

Proposition 2.6. *If $\lambda \neq 0$ is an eigenvalue of a compact $T \in B(H)$ then, $\bar{\lambda}$ is an eigenvalue of T^*.*

Proof. We saw earlier that T is compact implies T^* is compact. For any $T \in B(H)$, we have the result,

$$\sigma(T^*) = \overline{\sigma(T)}, \text{ where}$$
$$\overline{\sigma(T)} = \{\bar{\lambda} : \lambda \in \sigma(T)\}$$

Since T^* is compact,

$$\begin{aligned} P\sigma(T^*) - \{0\} &= \sigma(T^*) - \{0\} \\ &= \overline{\sigma(T)} - 0 \\ &= \overline{\sigma(T) - \{0\}} \\ &= \overline{P\sigma(T) - \{0\}} \\ &= \overline{P\sigma(T)} - \{0\} \end{aligned}$$

Therefore, if $\lambda \in P\sigma(T)$ and $\lambda \neq 0$, then $\bar{\lambda} \in P\sigma(T^*)$ and the proof is over. ☐

Proposition 2.7. *Let H be a Hilbert space and $T \in B(H)$ be compact. Let $\lambda \neq 0$. Then $\lambda \in \rho(T)$ if and only if $\mathcal{R}_{\lambda I - T} = H$.*

Proof. If $\lambda \in \rho(T)$, we have $\mathcal{R}_{\lambda I - T} = H$, as seen in proposition 2.2 (since T is closed). Conversely, let $R_{\lambda I - T} = H(\lambda \neq 0)$. Suppose $\lambda \in \sigma(T)$. Then by Proposition 2.2, $\lambda \in P\sigma(T)$. Let x_1 be a corresponding eigenvector. Since $\mathcal{R}_{\lambda I - T} = H$, we can inductively construct a sequence such that.

$$(T - \lambda I)x_n = x_{n-1} \forall n \geq 1 \left(x_0 = \bar{0}\right)$$

Again by induction we show that the vectors x_n $(n \geq 1)$ must be linearly independent. Clearly, $x_1 \neq \bar{0}$ for x_1 is an eigenvector. Suppose $\{x_1, \cdots, x_{n-1}\}$ is linearly independent and

$$\sum_{k=1}^{n} \alpha_k x_k = \bar{0} \tag{2.7}$$

Then we have

$$\bar{0} = (T - \lambda I)\left(\sum_{k=1}^{n} \alpha_k x_k\right) = \sum_{k=1}^{n} \alpha_k (T - \lambda I) x_k = \sum_{k=1}^{n} \alpha_k x_{k-1} \left(x_0 = \bar{0}\right)$$

We conclude that $\alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$ and by (7) since $x_1 \neq \bar{0}$, also $\alpha_1 = 0$. Therefore $\{x_1, \cdots, x_n\}$ is linearly independent. Let. $\{e_n\}_{n=1}^{\infty}$ be the orthonormal sequence obtained from $\{x_n\}_{n=1}^{\infty}$ by the Gram-Schmidt orthonormalization process. As in the proof of Proposition 2.4, we have (writing $e_n = \alpha_{n1}x_1 + \alpha_{n2}x_2 + \cdots + \alpha_{nn}x_n$)

$$\begin{aligned} (T - \lambda l)e_n = (T - \lambda I)\left(\sum_{k=1}^{n} n_{nk} x_k\right) &= \left(\sum_{k=1}^{n} \sigma_{nk} (T - \lambda l) x_k\right) \\ &= \sum_{k=1}^{n} \alpha_{n_k} x_{k-1} = g_n (say) \end{aligned}$$

Hence $g_n = V_{k=1}^{n-1}\{x_k\} = V_{k=1}^{n-1}\{c_k\}$. We can therefore write

$$g_n = \beta_{n1}e_1 + \cdots + \beta_{n,n-1}e_{n-1}.$$

Thus $g_n \perp e_n$. Now $Te_n = \lambda e_n + g_n$. Therefore

$$\langle Te_n, e_n \rangle = \langle \lambda e_n + g_n, e_n \rangle = \lambda \langle e_n, e_n \rangle = \lambda \neq 0 \forall n \in \mathbb{N}$$

and this contradicts Proposition 2.1((i) $\Rightarrow$ (ii)). Hence λ is not in $P\sigma(T)$, so λ is not in $\sigma(T)$ ☐

An immediate consequence of Proposition 2.7 is the next result.

Proposition 2.8. *Let T be a compact linear operator in a Hilbert space H and if for a fixed $\lambda \neq 0$ the equation*

$$Tx - \lambda x = y \tag{2.8}$$

has a solution for each $y \in H$, then the equation

$$Tx - \lambda x = \bar{0} \tag{2.9}$$

has the unique solution $x = \bar{0}$, i.e λ is not an eigenvalue of the operator T. The conclusion holds in pre-Hilbert spaces as well.

Proof. It is clear from (2.8) that $R_{T-\lambda I} = H$), since $\lambda \neq 0, \lambda \in P\sigma(T)$ or $\rho(T)$. Since $x = \bar{0}$ is the only solution of (2.9), it. follows that λ is not. in $P\sigma(T)$.

We now establish Proposition 2.8 when H is a pre-Hilbert space. Assume the contrary, namely, that equation (2.9) has a solution $x_1 \neq \bar{0}$. Thus vector $x_2 : Tx_2 - \lambda x_2 = x_1$. Then we find a vector x_3 such that $Tx_3 - \lambda x_3 = x_2$. Continuing this process we find an infinite sequence of vectors $(x_n)_{n=1}^{\infty}$ such that

$$Tx_k - \lambda x_k = x_{k-1} (k \in \mathbb{N}).$$

We now claim that the set of vectors $\{x_k : k \in \mathbb{N}\}$ is linearly independent. We do this as we did in the proof of Proposition 2.7. Orthogonalizing this sequence, we get an orthogonal sequence $(\tilde{x}_k)$, where

$$\tilde{x}_1 = \alpha_{11}x_1, \bar{x}_2 = \alpha_{21}x_1 + \alpha_{22}x_2, \cdots, \bar{x}_k = \alpha_{k1}x_1 + \alpha_{k2}x_2 + \cdots + \alpha_{kk}x_k$$

It follows that

$$\begin{aligned}
T\tilde{x}_k &= \alpha_{k1}Tx_1 + \alpha_{k2}Tx_2 + \cdots + \alpha_{kk}Tx_k \\
&= \alpha_{k1}\lambda x_1 + \alpha_{k1}\left(x_1 + \lambda x_2\right) + \cdots + \alpha_{kk}\left(x_{k-1} + \lambda x_k\right) \\
&= \left(\alpha_{k2}x_1 + \alpha_{k3}x_2 + \cdots + \alpha_{kk}x_{k-1}\right) + \lambda\left(\alpha_k x_1 + \alpha_k x_2 + \cdots + \alpha_{kk}x_k\right) \\
&= \left(\alpha_{k2}x_1 + \alpha_{k3}x_2 + \cdots + \alpha_{kk}x_{k-1}\right) + \lambda\tilde{x}_k \\
&= \beta_{k1}\tilde{x}_1 + \beta_{k2}\tilde{x}_2 + \cdots + \beta_{k,k-1}\tilde{x}_{k-1} + \lambda\tilde{x}_k (k \in \mathbb{N})
\end{aligned}$$

Since $\langle T\tilde{x}_k, \bar{x}_k\rangle = \lambda \neq 0 \forall k \in \mathbb{N}$ we have $\lim_{k\to\infty}\langle T\bar{x}_k, \bar{x}_k\rangle \neq 0$. On the other hand, since T is compact, we must have $\lim_{k\to\infty}\langle T\tilde{x}_k, \tilde{x}_k\rangle = \lim_{k\to\infty}\beta_{kk} = 0$. Thus we have a contradiction. Hence the assumption $x_1 \neq \bar{0}$ is inadmissible. Hence $x_1 = \bar{0}$. This proves that (2.9) has unique solution $x = \bar{0}$. □

Corollary 2.3. *If for a fixed $\lambda \neq 0$, equation (2.8) is solvable for each $y \in H$, then given $y \in H$, this equation has a unique solution and consequently the operator $(T - \lambda I)$ has an inverse on all of H.*

Proof. The main Proposition shows that if (2.8) is solvable for each $y \in H$ then (2.9) has the unique solution $x = \bar{0}$ (λ fixed). Now fix y at y_0 and assume that (2.8) is solvable. Suppose there are two vectors $x_1, x_2 \in H$ which solve (2.8) for the given λ and the given y_h. Then

$$Tx_1 + \lambda x_1 = Tx_2 + \lambda x_2 = y_0, \text{ i.e, } T\left(x_1 - x_2\right) - \lambda\left(x_1 - x_2\right) = \bar{0}$$

□

Proposition 2.9. *Let H be a Hilbert space and $T \in B_\infty(H)$. A complex number $\lambda \neq 0$ is an eigenvalue of T if and only if $\bar{\lambda}$ is an eigenvalue of T^*.*

Proof. This follows from $\sigma\left(T^*\right) = \overline{\sigma(T)} = \{\bar{\lambda} : \lambda \in \sigma(T)\}$ and $\sigma\left(T^*\right)/\{0\} = P\sigma\left(T^*\right)/\{0\}$. The last equality holds since T^* is compact. □

Remark 2.2. *The last Proposition cannot be extended to bounded normal operators on H.*

Proposition 2.10. *Let T be a compact linear operator in a pre-Hilbert space H and fix $\lambda \neq 0$. Then there exists a constant L depending on T and λ such that if the equation*

$$Tx - \lambda x = y \tag{2.10}$$

is solvable for a fixed right member y, at least one of its solutions x satisfies

$$\|x\| \leq L\|y\| \tag{2.11}$$

Remark 2.3. *Before providing the proof, we clarify as to what the Proposition conveys. In asking whether $\lambda \in \sigma(T)$, we could be interested in the invertibility of $(T-\lambda I)$. An inverse of $T-\lambda I$ would assign to every $y \in R_{T-\lambda I}$ a unique* x. *Moreover, if $(T-\lambda I)^{-1}$ is bounded, then we must have $\|x\| \leq L\|y\|$ for some constant L (which depends only on T and λ). The statement of Proposition 2.10 asserts that we can always reverse the action of $(T-\lambda I)$ in a bounded way. Ignoring the question of uniqueness of $(T-\lambda I)^{-1}(\{y\})$ (when (2.10) is solvable for a $y \in H$) there is always a candidate* x *associated with* y *by some sort of bounded inverse of $(T-\lambda I)$, the bound being* L.

Proof. Fix y and assume that (2.10) has a solution x^*. If λ is an eigenvalue of T, let $x_1, \cdots, x_k$ be a linearly independent set. of eigenvectors spanning $\eta_{T-\lambda l}$. In this case, the general solution of (2.10) has the form

$$x = x^* + \alpha_1 x_1 + \cdots + \alpha_k x_k$$

where $\alpha_1, \cdots, \alpha_k$ are arbitrary complex numbers. We select these numbers solution of (2.10) with minimum norm. If $\lambda \in \rho(T)$, then $\tilde{x} = x^*(k=0)$. Now let y vary over the: set M of all vectors for which (2.10) is solvable. To each vector $y \in M$, there corresponds a minimal solution $\bar{x}$. We now claim that.

$$\sup_{y \in M} \frac{\|i\|}{\|v\|} < \infty$$

Suppose the contrary. Then there exists a sequence (y_k) of vectors such that as $k \rightarrow\rightarrow \infty$, $\frac{\|\bar{r}_\lambda\|}{\|y_k\|} \rightarrow \infty$, where $\bar{x}_k$ is the minimal solution of (2.10) with right. member y_k. Dividing both sides of the equation

$$T\tilde{x}_k - \lambda\tilde{x}_k = y_k (k \in \mathbb{N})$$

by $\|\bar{x}_k\|$, we get

$$T\bar{x}_k' - \lambda\bar{x}_k' = y_k' (k \in \mathbb{N})$$

where $y_k' = \|\bar{x}_k\|^{-1} y_k, \|\bar{x}_k'\| = 1 \forall k \in \mathbb{N}$. Thus, the minimal solution $\bar{x}_k'$ of (2.10) has norm 1 if the right member is y_k'. Since T is compact, there exists a subsequence $(\bar{x}_{n_i}')$ of $(\tilde{x}_k')$ for which $s-\lim_{i\rightarrow\infty} T\tilde{x}_{n_i}'$ exists. Since $y_k' s_{\rightarrow}\bar{0}$ as $k \rightarrow \infty$, $s-\lim_{i\rightarrow\infty} T\tilde{x}_{n_i}$ also exists, say z, and consequently, $Tz - \lambda z = \bar{0}$ where $\|z\| = 1$. Thus z is an eigenvector of the operator T. Both the vectors $\bar{x}_{n_2}' - z$ and $\bar{x}_{n_i}'$ are solutions of (2.10) with right member y_{n_i}'. But because of the minimum norm of a solution of this equation being 1 , we have for each i

$$\left\|\tilde{x}_{n_1}' - z\right\| \geq 1$$

Since this is impossible, the proposition is proved. □

Example 1. *When H is a Hilbert. space, we can modify the proof of Proposition* 2.10 *as follows: Let P_λ be the orthoprojector on H onto $\eta_{T-\lambda l}$. Given any $y \in R_{T-\lambda I}$ and any x such that (2.10) holds, we observe that*

$$(T-\lambda I)\left(x - x'\right) = y \text{ if and only if } x' \in \eta_{T-\lambda I}.$$

For $x' = P_{\rm A} x \in \eta_{T-\lambda l}$ and $\bar{x} = x - P_\lambda x$ we obtain

$$(T-\lambda I)\tilde{r} = y \tag{2.12}$$

where $\|\dot{x}\| = \|x - P_X x\| = \min\{\|x - x'\| : x' \in \eta_T - \lambda\}$ *and therefore*

$$\|\bar{x}\| = \min\{\|x''\| : (T - \lambda I)x'' = y\} \tag{2.13}$$

Note that in this manner we have associated every $y \in R_{T-\lambda I}$ *with a unique vector* $\bar{x}$ *such that (2.12) holds. We now assert that there is a real constant* $L > 0$ *such that* $\|\hat{x}\| \leq L\|y\|\ \forall y \in R_{T-\lambda l}$. *Assuming the contrary we have*

$$\sup\left\{\frac{\|\vec{i}\|}{\|y\|} : y \neq \bar{0}, y \in R_{T-\lambda I}\right\} = \infty$$

We can therefore choose a sequence $(y_n)_{n=1}^{\infty}$ *of elements from* $R_{T-\lambda l}$ *such that* $y_n \neq \bar{0}y\ n \in \mathbb{N}$ *and* $\lim_{n\to\infty} \frac{\|\bar{I}_n\|}{\|y_n\|} = \infty$. *For* $\tilde{x}'_n = \frac{\bar{x}_n}{\|\bar{x}_n} \left\|_1 y'_n = \frac{y_n}{\|\bar{I}_n}\right\|$, *we obtain from (2.12) and (2.13)*

$$(T - \lambda I)\tilde{x}'_n = y'_n \tag{2.14}$$

$$\|\bar{x}'_n\| = \min\{\|x''\| : (T - \lambda I)x'' = y'_n\} \tag{2.15}$$

$$\lim_{n\to\infty} y'_n = 0$$

Let $\left(\tilde{x}'_{n_k}\right)$ *be a weakly converging subsequence of* $(\tilde{x}'_n)$ *(Every bounded sequence in a Hilbert space contains a weakly convergent subsequence). Then* $T\bar{x}'_{n_k}$ *converges strongly to some vector* z *and as a consequence*

$$\lim_{k\to\infty} \lambda\bar{x}'_{n_k} = \lim_{k\to\infty}\left(T\tilde{x}'_{n_k} - y'_{n_k}\right) = z$$

Since $\lambda \neq 0$, *the sequence* $\left(\tilde{x}'_{n_k}\right)$ *converges strongly and its limit is the vector* $z' = \frac{z}{\lambda}$. *From*

$$(T - \lambda I)z' = \lim_{k\to\infty}(T - \lambda I)\tilde{x}'_{n_k} = \lim_{k\to\infty} y'_{n_k} = \bar{0}$$

and (2.14) we conclude

$$(T - \lambda I)\left(\bar{x}'_n - z'\right) = y'_n$$

while $\lim_{k\to\infty}\|\tilde{x}_{n_k} - z'\| = 0$ *and therefore* $\|\tilde{x}'_n - z'\| < 1$ *for infinitely many* $n \in \mathbb{N}$. *This however contradicts (2.15).*

We now strengthen Proposition 2.8 which asserts that if $\lambda \neq 0$ *is an eigenvalue of a compact linear operator* T *in a pre-Hilbert space* H, *then the equation*

$$Tx - \lambda x = y \tag{2.16}$$

is not solvable for every $y \in H$. *If* H *is a Hilbert space, we shall determine the set of all vectors* y *for which (2.16) is solvable.*

Proposition 2.11. *Let* T *be a compact linear operator in a Hilbert space* H *and let* λ *be a nonzero element of* $\mathbb{K}$. *Then the equation (2.16) not an eigenvalue of* T^*, *then* $\eta_{T^*-\bar{\lambda}I} = \{\bar{0}\}$, *that is, in this case, equation (2.16) is solvable for every* $y \in H$.

Proof. Now $Tx - \lambda x = y$ is solvable for each $y \in R_{T-\lambda I}$. By Proposition 2.1, $R_{T-\lambda I}$ is closed. Now, $\bar{R}_{T-\lambda I} = \eta^{\perp}_{T^*-\bar{\lambda}I}$. Hence $R_{T-\lambda I} = \eta^{\perp}_{T^*-\bar{\lambda}I}$. Thus equation (2.16) is solvable if and only if $y \in \eta^{\perp}_{T^*-\bar{\lambda}I^*}$ If $\bar{\lambda}$ is not an eigenvalue of T^* (which is compact since T is compact) it follows (by the Corollary to Proposition 2.8) that (2.16) is uniquely solvable for all $y \in H$ □

Remark 2.4. *The reader can formulate an analogous proposition for the compact operator* T^*. *(see proposition 2.13).*

Proposition 2.12. *Suppose* H *is a Hilbert space and* $T \in B_\infty(H)$. *If* $\lambda \neq 0$ *is an eigenvalue of* T *(hence* $\bar{\lambda}$ *is an eigenvalue of* T^* *) then* $\eta_{\lambda I-T}$ *and* $\eta_{\bar{\lambda}I-T^*}$ *have the same dimension (which is finite) (This is called Fredholm's Third Theorem) (See proposition 2.13).*

Proof. We have $\dim \eta_{\lambda I-T} \leq \dim \eta_{\bar{\lambda} I-T}$. There exists an isometric operator V on $\eta_{\lambda I-T}$ into $\eta_{\lambda I-T^*} = R^{\perp}_{\bar{\lambda} I-T}$. Let P represent the orthoprojector on H onto $\eta_{\lambda I-T}$. The operator is of finite rank and hence is compact. Consider the operator $T_1 = T + VP$. Clearly T_1 is compact. We shall show that $\eta_{\lambda l-T_1} = \{\bar{0}\}$, i.e., λ is not an eigenvalue of T_1. Indeed, let $(\lambda I - T_1)\, r = \bar{0}_2$ that is, $\lambda x - Tx - VPx = \bar{0}$ or $\lambda x - Tx = VPx$. Now $\lambda x - Tx \in R_{\lambda I-T}$, whereas $\|Px \in R^{\perp}_{\bar{\lambda} I-T}$. Hence $\lambda x - Tx = \bar{0} = VPx$. Since V is an isometry, $0 = \|VPx\| = \|Px\|$. Hence $x \in \eta^{\perp}_{\lambda l-T}$. The relation $\lambda x - Tx = \bar{0}$ shows that $x \in \eta_{\lambda I-T}$. Thus $x = \bar{0}$.

By Proposition 2.11, $\bar{\lambda}$ is not an eigenvalue for T_1^*, that, is, $\eta_{\bar{\lambda} I-T^*} = \{\bar{0}\}$. Since $\eta_{\bar{\lambda} I-T_1} = R^{\perp}_{\lambda I-T_1}$. We have $R^{\perp}_{\lambda I-T_1} = \{\bar{0}\}$. We now show that.

$$R_{\lambda I-T_1} = R_{\lambda I-T} \oplus R_{VP}.$$

If $y \in R_{\lambda I-T}$, then $y = (\lambda I - T_1)\, x$ for some $x \in \mathbb{Z}$ and thus which shows that $y \in R_{\lambda I-T} \oplus R_{VP}$, that is

$$R_{\lambda I-T_1} \subseteq R_{\lambda I-T} \oplus R_{VP}$$

Conversely. let $y \in R_{\lambda l-T} \oplus R_{VP}$. Then

$$y = (\lambda I - T)x' + VPx'' \text{ for } x', x'' \in H) \,.$$

We can write $x' = x_1' + x_2', x'' = x_1'' + x_2''$ where $x_1', x_1'' \in \eta_{\lambda I-T}, x_2', x_2'' \in \eta^{\perp}_{\lambda I-T}$. Clearly

$$(\lambda I - T)x' = (\lambda I - T)\left(x' + x_2'\right) = (\lambda I - T)x_2' \text{ and}$$
$$VPx'' = VP\left(x_1'' + x_2''\right) = VPx_1''$$

Thus,

$$\begin{aligned} y = (\lambda I - T)x_2' + VPx_1'' =& (\lambda I - T - VP)\left(x_2' - x_1''\right) \\ =& (\lambda I - T_1)\left(x_2' - x_1''\right) \end{aligned}$$

which shows that $y \in R_{\lambda I-T}$. Next it is easily seen that $R_{VP} = R_V$. Thus $R_{\lambda I-T} = R_{\lambda I-T} \oplus R_V$. Consequently,

$$\{\bar{0}\} = \eta_{\bar{\lambda} I-T} = R^{\perp}_{\lambda I-T_1} = [R_{\lambda I-T} \oplus R_V]^{\perp}.$$

which yields

$$R_V = R^{1}_{\lambda l-T}$$

Since V is an isometry

$$\dim \eta_{\lambda I-T} = \dim R_V = \dim \eta_{\bar{\lambda} I-T}$$

The other case can he treated similarly.

Alternative proof:
Suppose $\eta_{T-\lambda I}$ (which is the same as $\eta_{\lambda I-T}$) has dimension p less that $\dim \eta_T \cdot -\vec{\lambda} I = q$. Let $\{e_j; j = 1, \cdots, p\}$ be an orthonormal basis for $\eta_{T-\lambda l}$ and $\{f_k; k = 1, \cdots, q\}$ be an orthonormal basis for $\eta_{T^*-\bar{\lambda} I}$. We define in H an operator T_1 by

$$T_1 x = Tx + \sum_{j=1}^{p} \langle x, e_j \rangle\, f_j \tag{2.17}$$

Clearly T_1 is compact and λ is not an eigenvalue of T_1. For suppose there was a vector $y \neq \bar{0}$ such that

$$T_1 y = \lambda y \tag{2.18}$$

Hence by (2.17),

$$(T - \lambda y) + \sum_{j=1}^{p} \langle y, e_j \rangle\, f_j = \bar{0}$$

Now, $\langle (T-\lambda I)y+\sum_{j=1}^{p} < y, e_j \int_j, f_i\rangle = 0, i=1,\cdots,p$.

$$\langle y, \left(T^4-\bar{\lambda} I\right) f_i\rangle 1\langle y, e_1\rangle = 0$$

But since the first term on the left is 0 , we have

$$\langle y, e_1\rangle = 0 (i=1,2,\cdots,p) \tag{2.19}$$

Hence, it follows from (2.18) that $T_1 y = Ty$ and from (2.18) that $Ty = \lambda y$. Thus y is an eigenvector of the operator T and by (2.19) it is not. a linear combination of the eigenvectors $e_j (j=1,\cdots,p)$. This is impossible. Hence (2.18) is impossible, that is, λ is not an eigenvalue of the operator T_1. Hence there is a vector x such that.

$$\left(T_1-\lambda I\right) x = f_{p+1}$$

But since, by (2.16)

$$\langle T_1 x, f_{p+1}\rangle = \langle Tx, f_{p+1}\rangle = \langle x, T^* f_{p+1}\rangle = \langle x, \bar{\lambda} f_{p+1}\rangle = \lambda\langle x_1 f_{p+1}\rangle$$

we have

$$1 = \langle f_p, f_{p+1}\rangle = \langle\left(T_1-\lambda I\right) x, f_{p+1}\rangle = 0$$

Thus the hypothesis that $q>p$ leads to an absurdity. Since $T=(T^*)^*$, we can reverse the roles of the operators T and T^*. Hence, by what has already been proved, it is also impossible that $p>q$. Hence the Proposition. □

Remark 2.5. *The reader should note that occasionally we write $\lambda I-T$ in place of $T-\lambda I$ and correspondingly $\bar{\lambda} I-T^*$ in place of $T^*-\bar{\lambda} I$. He can adopt uniformly in this matter for the proofs are the same for all the propositions (with minor alterations). We now summarize the assertions of propositions 2.11 and 2.12 and the remark following the former into a single proposition called the Fredholm alternative. This proposition is a generalization of the Fredholm alternative in the theory of linear integral equations.*

Proposition 2.13. *(FREDHOLM ALTERNATIVE).*
Let H be a Hilbert space and $T \in B(H)$ be a compact operator and λ be a nonzero element of $\mathbb{K}$. Then we have the Frodholm alternative. Either the inhomogeneous equations

$$(T-\lambda I)x = y \text{ and } \left(T^*-\bar{\lambda} I\right) \tilde{x} = \tilde{y}$$

are uniquely solvable $\forall y, \tilde{y} \in H$ or the homogeneous

$$(T-\lambda I)\cdot x = \bar{0} \text{ and } \left(T^*-\bar{\lambda} I\right) \bar{x} = \bar{0}$$

hate non-trivial solutions.
The spaces of the solutions of the two homogeneous equations have the same (finite) dimension and $(T-\lambda I)x = y$ is solvable if and only if y is orthogonal to every solution $\bar{x}$ of $\left(T^*-\bar{\lambda} I\right) \hat{x} = \bar{0}$. that is, if and only if $\bar{y} \perp \eta_{T^*-\bar{\lambda} I}$. Similarly, $\left(T^*-\bar{\lambda} I\right) \tilde{x} = \hat{y}$ is solvable if and only if $\tilde{y} \perp \eta_{T-\lambda I}$.

Proof. Either $\lambda \in \rho(T)$ and so $\bar{\lambda} \in \rho\left(T^*\right)$ or $\lambda \in P\sigma(T)$ and hence $\bar{\lambda} \in P\sigma\left(T^*\right)$ (propositions 2.2, 2.9). In the first case we have

$$x = (T-\lambda I)^{-1} y, \tilde{x} = \left(T^*-\bar{\lambda} I\right) \tilde{y}$$

where $(T-\lambda I)^{-1}, \left(T^*-\bar{\lambda} I\right)^{-1} \in B(H)$. In the second case, we have

$$\eta_{T-\lambda I}^{\frac{1}{T}} = R_{T^*-\bar{\lambda} I} \text{ (for } R_{T^*-\bar{\lambda} I} \text{ is a closed linear subspace).}$$

The equation $\left(T^*-\bar{\lambda} I\right) \bar{x} = \bar{y}$ has a solution $\tilde{x}$ if and only if $\tilde{y} \in R_{T^*-\bar{\lambda} I}$, that is, if and only if $\tilde{y}$ is orthogonal to $\eta_{T-\lambda I}$, which in turn consists of all solutions of $(T-\lambda I)x = \bar{0}$. Similarly from

$$R_{T-\lambda I} = \eta_{T^*-\bar{\lambda} I}^{\perp}$$

the remaining statements of the Proposition follow. □

The following example shows that a non-zero compact linear operator may not have any eigenvalues at all (the spectrum then reduces to its absolute minimum, consisting of one single point only. At the same time it illustrates that the conclusion of Proposition 2.9 breaks down for $\lambda = 0$).

Example 2. *Let T be the linear operator on $\ell^2(\mathrm{N})$ defined by*

$$T\left(n_k\right)_{k=1}^{\infty} = \left(\frac{1}{k}\alpha_{k-1}\right)_{k=1}^{\infty} \quad \alpha_0 = 0$$

that is, $T\left(\alpha_1, \alpha_2, \cdots\right) = \left(0, \frac{1}{2}\alpha_1, \frac{1}{3}\alpha_2\right) \forall \left(\alpha_n\right)_{n=1}^{\infty} \in \ell^2(\mathbb{N})$. Show that.

(a) T *is compact.*

(b) $P\sigma(T) = 0$.

(c) The only eigenvalue of T^* is 0 , i.e, $P\sigma\left(T^*\right) = \{0\}$.

(d) T *is not normal.*

Solution

(a) Now $\langle Te_n, e_n\rangle = \left\langle \frac{1}{n+1}e_{n+1}, e_n\right\rangle = 0 \forall n \in \mathbb{N}$. Hencc $\lim_{n\to\infty} < Te_n, e_n >= 0$. (This does not imply T is compact.) Define T_n on $\ell^2(\mathbb{N})$ by

$$T_n\left(\alpha_k\right)_{k=1}^{\infty} = \left(0, \frac{1}{2}\alpha_1, \frac{1}{3}\alpha_2, \cdots, \frac{1}{n+1}, 0, 0, \cdots\right) \forall n \in \mathbb{N}$$

Then if $x = \left(\alpha_k\right)_{k=1}^{\infty}$, we have

$$\begin{aligned}
\|T_n x - Tx\|^2 &= \left\|\left(0, \frac{1}{2}\alpha_1, \frac{1}{3}\alpha_2, \cdots, \frac{1}{n+1}, 0, 0, \cdots\right) - \left(0, \frac{1}{2}\alpha_1, \frac{1}{3}\alpha_2, \cdots\right)\right\|^2 \\
&= \left\|\left(0, \cdots, \frac{1}{n+2}\alpha_{n+1}, \frac{1}{n+3}\alpha_{n+2}, \cdots\right)\right\|^2 \\
&= \frac{1}{(n+2)}^2 \left|\alpha_{n+1}\right|^2 + \frac{1}{(n+3)}\left|\alpha_{n+2}\right|^2 + \cdots \\
&= \frac{1}{(n+2)}^2 \left[\left|\alpha_{n+1}\right|^2 + \frac{n+2^2}{n+3} + \left|\alpha_{n+2}\right|^2 + \frac{n+2^2}{n+4} + \left|\alpha_{n+3}\right|^2 + \cdots\right] \\
&\leq \frac{1}{(n+2)}^2 \left[\left|\alpha_1\right|^2 + \left|\alpha_2\right|^2 + \cdots\right] = \frac{1}{(n+2)}\|x\|^2, \forall x \in \ell^2(\mathbb{N})
\end{aligned}$$

Hence $\|T_n - T\| \leq \frac{1}{n+2} \forall n \in \mathbb{N}$. Now each T_n being of finite rank $(n+1)$, is compact and $\lim_{n\to\infty}\|T_n - T\| = 0$. Hence T is compact (by proposition 2.11).

(b) Let $x = \left(\alpha_n\right)_{n=1}^{\infty}$. So $Tx = \left(0, \frac{\alpha_1}{2}, \frac{\alpha_2}{3}, \cdots\right)$. Now if $\lambda \in \mathbb{K}$, then $Tx = \lambda x$ implies $\left(0, \frac{\alpha_1}{2}, \frac{\alpha_2}{3}, \cdots\right) = \left(\lambda\alpha_1, \lambda\alpha_2, \lambda\alpha_3, \cdots\right)$. So $0 = \lambda\alpha_1, \frac{\alpha_1}{2} = \lambda\alpha_2, \frac{\alpha_2}{3} = \lambda\alpha_3, \frac{\alpha_{n-1}}{n} = \lambda\alpha_n, \cdots$. If $\lambda = 0$, then $\alpha_1 = a_2 = \cdots = 0$, i.e $x = \overline{0}$, and hence $\lambda = 0$ is not an eigenvalue of 'T. If $\lambda \neq 0$ then again $\alpha_1 = \alpha_2 = \cdots = 0$ and again since $x = \overline{0}$ necessarily, it follows that λ is not in $P\sigma(T)$. Thus $P\sigma(T) = 0$.

(c) Let $x = \left(\alpha_1, \alpha_2, \cdots\right), y = \left(\beta_1, \beta_2, \beta_3, \cdots\right) \in \ell^2(\mathbb{N})$. Then $\langle Tx, y\rangle = \left\langle\left(0, \frac{\alpha_1}{2}, \frac{\alpha_2}{3}, \cdots\right), \left(\beta_1, \beta_2, \cdots\right)\right\rangle$

$$\begin{aligned}
&= \frac{\omega_1}{2}\overline{\beta_2} + \frac{\omega_2}{3}\overline{\beta_3} + \cdots = \left\langle\left(\alpha_1, \alpha_2, \cdots\right), \left(\frac{\beta_2}{2}, \frac{\beta_1}{3}, \cdots\right)\right\rangle \\
&= \langle x, T^* y
\end{aligned}$$

and this shows that

$$T^*\left(\beta_1, \beta_2, \beta_3, \cdots\right) = \left(\frac{\beta_2}{2}, \frac{\beta_1}{3}, \frac{\beta_4}{4}, \cdots\right)$$

Now $\lambda \neq 0$ cannot be in $P\sigma\left(T^*\right)$ for then $\bar{\lambda}$ would be in $P\sigma(T)$ (see Proposition 2.9). But $P\sigma(T)=\sigma$. So $\lambda \neq 0$ cannot be in $P\sigma\left(T^*\right)$. Consider $\lambda=0$. Then $T^* y=\lambda y$ implies $T^* y=\overline{0}$, i.e all $n \geq 2$. Thus for any $\beta_1 \neq 0$, the vector $\left(\beta_1, 0,0, \cdots\right)=\beta_1 e_1$ is an eigenvector corresponding to the eigenvalue 0 . It follows that $P\sigma\left(T^*\right)=\{0\}$.

(d) If T was normal, then $\overline{P\sigma(T)}=P\sigma\left(T^*\right)$ and so $\lambda=0$ must also be an eigenvalue of T. But $P\sigma(T)=\emptyset$. Hence T is not normal.

Definition 1. An operator $T \in B(H)$, where H is a complex Hilbert space is said to be quasi-nilpotent if its spectral radius is 0 .
It is clear that $T \in B(H)$) is quasi-nilpotent if and only if $\sigma(T)$ consists of the single point 0 (Note: $\sigma(T)$ is nonvoid).

Proposition 2.14. *Let H be a Hilbert space and $T \in B(H))$ be quasi-nilpotent. If ImT: $T=A+iB$ where $A=\frac{1}{2}\left(T+T^*\right), B=\frac{1}{2i}\left(T-T^*\right), i=\sqrt{-1}$ and A, B are bounded self-adjoint; so $\operatorname{Im} T=\left.\frac{1}{2i}\left(T-T^*\right)\right)$ is compact, then T is compact.*

Proof. Let $A=\operatorname{Re} T\left(=\frac{1}{2}\left(T+T^*\right)\right), B=ImT$, so that A and B are self-adjoint. Now B is compact. We have to show that A is compact. Suppose the contrary. Then, there is an orthonormal system $\left\{e_{ar}: \alpha \in \Lambda\right\}$ such that $\left\langle A e_r, e_r\right\rangle \nrightarrow 0$ as $n \rightarrow \infty$
For some $\delta>0$, the set

$$\Lambda_0=\left\{\alpha \in \Lambda:\left|\left\langle A e_\alpha, e_\alpha\right\rangle\right| \geq \delta\right\}$$

is infinite. If $\left\{f_n: n \in \mathbb{N}\right\}$ is a countable infinite subset of $\left\{e_n: \alpha \in \Lambda\right\}$, then $\left(f_n\right)$ is an orthonormal sequence and

$$\left|\left\langle A f_n, f_n\right\rangle\right| \geq \delta(n \in \mathbb{N}) \tag{2.20}$$

Since $\left|\left\langle A^m f_n, f_n\right\rangle\right|^2 \leq\left\|A^m f_n\right\|^2=\left\langle A^m f_n, A^m f_n\right\rangle=\left\langle A^2 f_n, f_n\right\rangle$ for all positive integers m and n, it follows from (2.20) that

$$\mid\left\langle A^m f_n, f_n\right\rangle 1 \geq \delta^m(n \in \mathbb{N})$$

whenever $m=2^q$ for some $q=0,1,2, \cdots$. (Indeed, $\left\langle A^2 f_n, f_n\right\rangle \geq\left|\left\langle A f_n, f_n\right\rangle\right|^2$, $\left\langle A^4 f_n, f_n\right\rangle \geq \left|\left\langle A^2 f_n, f_n\right\rangle\right|^2 \geq\left|\left\langle A f_n, f_n\right\rangle\right|^4 \geq \delta^4$, e.t.c). For sufficiently large m of the form 2^q

$$\left\|T^m\right\|^{\frac{1}{m}}<\delta$$

(For T is quasi-nilpotent. Note $r(T)=\lim_{n \rightarrow \infty}\left\|T^n\right\|^{\frac{1}{n}}=0$) and hence..... Now $(A+iB)^m= A^m+m\left(A^{m-1}\right)(iB)+\cdots+(iB)^m$ (Binomial expansion)

$$=A^m+\left(imA^{m-1}+\cdots 1 i^m B^{m-1}\right) B$$

Since $\left(imA^{m-1}+\cdots+i^m B^{m-1}\right) \in B(H)$ and B is compact, it follows that $\left(imA^{m-1}+\cdots+i^m B^{m-1}\right) B= C$ (say) is compact. Thus $\delta^m>\left\|T^m\right\|=\left\|A^m-C\right\|$, for a compact linear operator C. We have $\left\langle C \int_n, \int_n\right\rangle \rightarrow 0$ ns $n \rightarrow \infty$. Hence:

$$\begin{aligned}
\left|\left\langle A^m f_n, f_n\right\rangle\right| & \leq 1\left\langle\left(A_C^m\right) f_n, \int_n\right\rangle\left|+\left|\left\langle C \int_n, \int_n\right\rangle 1\right.\right. \\
& \leq\left\|A^m-C\right\|+\left|\left\langle C \int_n, f_n\right\rangle\right|<\delta^m
\end{aligned}$$

for sufficiently large n. This contradicts (2.20) and completes the proof of the proposition. □

Next we discuss a generalization of Example 2 by describing a class of compact linear operators.

Example 3. *Suppose that (X, S, μ) is a σ-finite measure space, $(X\times X, S\times S, \mu\times\mu)$ is the product of this measure space with itself and $K \in L^2(X\times X, S\times S, \mu\times\mu)$. By Fubini's theorem*

$$\|k\|^2 = \int_x \left[\int_x |k(s,t)|^2 d\mu(t)\right] d\mu(s) \tag{2.21}$$

(here, and subsequently, the norm of any function refers to the usual norm in the appropriate L^2 space). For almost all $s \in X, K(s,t)$ is of class $L^2(X, S, \mu)$ as a function of t. Let Z denote the exceptional set of measure 0 . If $f \in L^2(X, S, \mu)$, it follows from the Cauchy–Bunyakovsky–Schwarz inequality that

$$(Tf)(s) = \int_x K(s,t)f(t)d\mu(t)$$

exists whenever $s \in X - Z$ and

$$|(Tf)(s)| \leq \|f\|^2 \int_x |K(s,t)f(t)|^2 d\mu(t) \tag{2.22}$$

It is easily verified that the function Tf (defined arbitrarily on Z) is measurable. From (2.21) and (2.22)

$$\left|\int_x (Tf)(s)\right|^2 d\mu(s) \leq \|f\|^2 \int_x \left[\int_x |K(s,t)f(t)|^2 d\mu(t)\right] d\mu(s)$$
$$= \|f\|^2 \|K\|^2$$

Hence $Tf \in L^2(X, S, \mu)$ and $\|Tf\| \leq \|K\|\|f\|$, i.e, T is a boumded lincar operator on the Hilbert space $L^2(X, S, \mu)$ with $\|T\| \leq \|K'\|$. We shall refer to K *as an L^2 kernel and to 'T as its associated.... if $f, g \in L^2(X, S, \mu)$, then*

$$\langle Tf, g\rangle = \iint_{X\times X} K(s,l)\overline{g(s)}f(t)d\mu(s)d\mu(t) \tag{2.23}$$

From this, it is easily verified that the adjoint T^* is the integral operator associated with the L^2 kernel $K^*(s,t)$, where $K^*(s,t) = \overline{K(t,s)}$.

We assert that $\{e_\alpha : \alpha \in \Lambda\}$ is an orthonormal base in $L^2(X, S, \mu)$. The functions Ψ_α on $X \times X$ defined by

$$\Psi_\alpha(s,t) = e_\alpha(s)\overline{e_{ux}(t)}$$

form an orthonormal system in $L^2(X\times X, S\times S, \mu\times\mu)$. With $f = g = e_{\mathrm{a}}$, it follows that

$$\langle Te_\alpha, c_\alpha\rangle = \langle K, \Psi_\alpha\rangle$$

and Bessel's inequality asserts that

$$\sum_{\alpha\in\Lambda} |\langle Te_\alpha, e_\alpha\rangle|^2 = \sum_{\alpha\in\Lambda} |\langle K, \Psi_\alpha\rangle|^2 \leq \|K\|^2.$$

Thus $\langle T\Psi_\alpha, \Psi_\alpha\rangle \to 0$ as $\alpha \to \infty$ (Note the interpretation of this in the Remark following Proposition 2.1). Thus it follows that T is a compact linear operator. Finally, we assert that $T = 0$ if and only if $K(s,t) = 0$ a.e. on $X \times X$. The 'if' part of the statement is an immediate consequence of the inequality $\|T\| \leq \|K\|$. Now suppose that $\|T\| = 0$. To show that $K(s,t) = 0$ i.e. on $X \times X$, it is sufficient to show that $K(s,t) = 0$ i.e. on $X_0 \times X_0$ where X_0 is a measurable subset of X with $\mu(X_0) < \infty$. Let S *denote the class of all measurable subsets s of $X_0 \times X_0$ which satisfy*

$$\iint_s K(s,t)d\mu(s)d\mu(t) = 0 \tag{2.24}$$

As a function on $X_0 \times X_0$, K *is of class L^2. and therefore (sines $X_0 \times X_0$ has finite: measure) of class L^1. If $s_1, s_2, s_3, \cdots \in S$ and the sequence (S_n) is either increasing or decreasing, it follows* *easily from the dominated convergence theorem that lim* $s_n \in S(S$ *is monotone class). If* A *and* B *are measurable subsets of* X_0*, we can take* f *and* g *in (2.24) to be the characteristic functions of* S *contains the algebra consisting of all finite disjoint unions of such sets* $A \times B$*. Since* S *is monotone, it contains the* σ*-algebra generated by this last algebra; that, is,* S *contains of all measurable subsets* s *of* $X_0 \times X_0$ *and (2.24) is satisfied for all such* s*. By taking for* s*, in turn, the four sets on which the real and imaginary parts of* $K(s,t)$ *both have constant sign, it follows from (2.24) that* $K(s,t) = 0$ *i.e. on* $X_0 \times X_0$*.*

3 Conclusion

Spectral properties provide a powerful way to understand linear operators by decomposing the space on which they act into invariant subspaces. The spectral properties of a compact operator on a normed linear space resemble those of square matrices. For a compact operator, the spectral properties can be treated fairly completely in the sense that Fredholm's famous theory of integral equations may be extended to linear functional equations with a complex parameter λ. In this paper, the spectral properties of compact operators in Hilbert spaces have been studied and investigated. Also, it has been observed that on finite dimensional vector space, the spectrum of an operator consists of all its eigenvalues while on infinite dimensional vector space. The spectrum consists of the continuous, residual and the point spectrum. Also, it has been shown that the spectral properties of compact linear operators are relatively simple generalization of the eigenvalues of finite matrices. The paper has given a number of corresponding propositions and interesting facts which are used to prove basic properties of compact operators. The main results of this paper have been captured in propositions $2.1, 2.2, 2.3, 2.4, 2.5, 2.6, 2.7$. The paper also introduced the Fredholm theory to investigate the solvability of linear integral equations involving compact operators. The main Fredholm Theorem is captured in proposition 2.13.

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