

On The Diophantine Equation $ab(cd + 1) + L = u^2 + v^2$

Abstract

The search for integer sums of two squares is still an open problem. A number of research conducted so far has put more attention on Fermat sums of two squares with little attention given to other methods of generating sums of two squares. Let a, b, c, d, u, v and L be positive integers. This study introduces the relation $ab(cd + 1) + L = u^2 + v^2$ for generating integer sums of two squares. The main objective of this study is to develop general formulae by determining the integer values a, b, c, d, u, v and L such that $ab(cd + 1) + L = u^2 + v^2$. Moreover, this research provides conjecture for the title equation.

Keywords: Diophantine Equation; Sums of Two Squares

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1 Introduction

The classification of integer sums of two squares is still an open problem. A number of research done so far has only provided partial solutions with no universal method of generating all sums of two squares. Let a, b, c, d and K be positive integers. In this study we develop and introduce the relation $ab(cd + 1) + K = u^2 + v^2$ for generating integer sums of two squares. The theory of sums of two squares has a rich history and goes back to early 16th century pioneered by Fermat 1640.

He stated that an odd prime p is a sum of two squares if and only if $p \equiv 1 \pmod{4}$. This theorem was stated without any formal proof. The major contribution to this theorem was due to Heath-Brown who gave a complete elementary proof in [2] and Zagier[3]. This proof, due to Zagier, is a simplification of an earlier proof by Heath-Brown, which in turn was inspired by a proof of Liouville. The technique of the proof is a combinatorial analogue of the topological principle that the Euler characteristics of a topological space with an involution and of its fixed-point set have the same parity and is reminiscent of the use of sign-reversing involutions in the proofs of combinatorial bijections. In [1], A. David Christopher gave a partition-theoretic proof by considering partitions of the odd prime n having exactly two sizes each occurring exactly and by showing that at least one such partition exists if n is congruent to 1 modulo 4. For recent survey on integer sums of two squares reference can be made to [1,2,3,4,7,8]. This research seeks to explore the diophantine equation $ab(cd+1)+L = u^2+v^2$ thereby adding some more general formulae to existing results on integer sums of two squares.

2 Main Results

Next, we present some general results for integer sums of two squares.

Proposition 2.1. $ab(cd+1) + b + 1 = b^2 + [(a+1)^2 + a]^2$ has solution in integers if $a = m, b = m+1, c = m+2, d = m+3$ where $m \geq 0 \in \mathbb{Z}$ and $L = b+1$.

Proof. Suppose $a = m, b = m+1, c = m+2, d = m+3$ where m is any non-negative integer. Then, $ab(cd+1) + b + 1 = m(m+1) = (m^2 + m)[m(m+3) + 2(m+3) + 1] + m + 2 = (m^2 + m)[m^2 + 5m + 7] + m + 2 = m^2(m^2 + 5m + 7) + m(m^2 + 5m + 7) + m + 2 = m^4 + 6m^3 + 12m^2 + 8m + 2 = m^4 + 6m^3 + 11m^2 + 6m + 1 + m^2 + 2m + 1 = [(m+1)^2 + m]^2 + [m+1]^2$. Since $a = m$ and $b = m+1$ the result easily follows. $\square$

Proposition 2.2. $ab(cd+1) + 3b + 2 = [a+2]^2 + [(a+1)^2 + a]^2 = [b+1]^2 + [(b-1) + b^2]^2 = c^2 + [(c-2) + (c-1)^2]^2$ has solution in integers if $a = m, b = m+1, c = m+2, d = m+3$ where $m \geq 0 \in \mathbb{Z}$ and $L = 3b+2$.

Proof. Assume $a = m, b = m+1, c = m+2, d = m+3$ where m is any positive integer. Then, $ab(cd+1) + 3b + 2 = m(m+1)((m+2)(m+3) + 1) + 3(m+1) + 2 = (m^2 + m)(m(m+3) + 2(m+3) + 1) + 3m + 3 + 2 = (m^2 + m)(m^2 + 3m + 2m + 6 + 1) + 3m + 5 = (m^2 + m)(m^2 + 5m + 7) + 3m + 5 = m^2(m^2 + 5m + 7) + m(m^2 + 5m + 7) + 3m + 5 = m^4 + 5m^3 + 7m^2 + m^3 + 5m^2 + 7m + 3m + 5 = m^4 + 6m^3 + 12m^2 + 10m + 5 = m^2 + 4m + 4 + m^4 + 6m^3 + 11m^2 + 6m + 1 = [a+2]^2 + [(a+1)^2 + a]^2$. Since $b = m+1$ and $c = m+2$. It follows that $ab(cd+1) + 3b + 2 = [a+2]^2 + [(a+1)^2 + a]^2 = [b+1]^2 + [(b-1) + b^2]^2 = c^2 + [(c-2) + (c-1)^2]^2$. $\square$

Proposition 2.3. $ab(cd+1) + 17 = [a+1]^2 + [(b-1)^2 + 4(b-1) - 1]^2 = [\frac{a+b}{2}]^2 + [(b-1)^2 + 4(b-1) - 1]^2$ has solution in integers if $a = 2m+1, b = 2m+3, c = 2m+5, d = 2m+7$ where $m \geq 0$ and $L = 17$.

Proof. Let $a = 2m+1, b = 2m+3, c = 2m+5, d = 2m+7$ where m is any non-negative integer. Then, $ab(cd+1) + 17 = (2m+1)(2m+3)((2m+5)(2m+7) + 1) + 17 = 2m(2m+3) + 1(2m+3)(2m(2m+7) + 5(2m+7) + 1) + 17 = (4m^2 + 6m + 2m + 3)(4m^2 + 14m + 10m + 35 + 1) + 17 = (4m^2 + 8m + 3)(4m^2 + 24m + 36) + 17 = 4m^2(4m^2 + 24m + 36) + 8m(4m^2 + 24m + 36) + 3(4m^2 + 24m + 36) + 17 = 16m^4 + 96m^3 + 144m^2 + 32m^3 + 192m^2 + 288m + 12m^2 + 72m + 108 + 17 = 16m^4 + 128m^3 + 348m^2 + 360m + 125 = m^2 + 4m + 4 + m^4 + 6m^3 + 11m^2 + 6m + 1 = [2m+2]^2 + [(2m+2)^2 + 4(2m+2) - 1]^2$. Since $2m+2 = [a+1] = [\frac{a+b}{2}] = b-1$ it follows that $ab(cd+1) + 17 = [a+1]^2 + [(b-1)^2 + 4(b-1) - 1]^2 = [\frac{a+b}{2}]^2 + [(b-1)^2 + 4(b-1) - 1]^2 = [b-1]^2 + [(c-1)^2 - 5]^2$. $\square$

Proposition 2.4. $ab(cd+1)+17=[a+1]^2+[a^2+6a+4]^2=[\frac{a+b}{2}]^2+[a^2+6a+4]^2$ has solution in integers if $a=2m+2, b=2m+4, c=2m+6, d=2m+8$ where $m \geq 0$ and $L=17$.

Proof. Suppose $a=2m+2, b=2m+4, c=2m+6, d=2m+8$. Then, $ab(cd+1)+17=(2m+2)(2m+4)((2m+6)(2m+8)+1)+17=2m(2m+4)+2(2m+4)(2m(2m+8)+6(2m+8)+1)+17=(4m^2+8m+4m+8)(4m^2+16m+12m+48+1)+17=(4m^2+12m+8)(4m^2+28m+49)+17=4m^2(4m^2+28m+49)+12m(4m^2+28m+49)+8(4m^2+28m+49)+17=16m^4+112m^3+196m^2+48m^3+336m^2+588m+32m^2+224m+392+17=16m^4+160m^3+564m^2+812m+409=4m^2+12m+9+16m^4+160m^3+562m^2+800m+400=[2m+3]^2+[(2m+2)^2+6(2m+2)+4]^2$. Since $a=2m+2, b=2m+4$ it is clear that $ab(cd+1)+17=[a+1]^2+[a^2+6a+4]^2=[\frac{a+b}{2}]^2+[a^2+6a+4]^2$. $\square$

Proposition 2.5. $ab(cd+1)+2(b+8)=b^2+[(b-1)^2+4(b-1)-1]^2=[\frac{a+c}{2}]^2+[(b-1)^2+4(b-1)-1]^2$ has solution in integers if $a=2m+1, b=2m+3, c=2m+5, d=2m+7$ where $m \geq 0$ and $L=2(b+8)$.

Proof. Assume $a=2m+1, b=2m+3, c=2m+5, d=2m+7$. Then, $ab(cd+1)+2(b+8)=(2m+1)(2m+3)((2m+5)(2m+7)+2)+2(2m+3+8)=2m(2m+3)+1(2m+3)(2m(2m+7)+5(2m+7)+1)+2(2m+11)=(4m^2+6m+2m+3)(4m^2+14m+10m+35+1)+4m+22=(4m^2+8m+3)(4m^2+24m+36)+4m+22=4m^2(4m^2+24m+36)+8m(4m^2+24m+36)+3(4m^2+24m+36)+4m+22=16m^4+96m^3+144m^2+32m^3+192m^2+288m+12m^2+72m+108+4m+22=16m^4+128m^3+348m^2+364m+130=4m^2+12m+9+16m^4+128m^3+344m^2+352m+121=[2m+3]^2+[(2m+3)^2+4(2m+2)-1]^2$. Since $b=2m+3=\frac{a+c}{2}$ it follows that $ab(cd+1)+2(b+8)=b^2+[(b-1)^2+4(b-1)-1]^2$. $\square$

Proposition 2.6. $ab(cd+1)+2(b+8)=b^2+[(a^2+6a+4)]^2$ has solution in integers if $a=2m+2, b=2m+4, c=2m+6, d=2m+8$ where $m \geq 0$ and $L=2(b+8)$.

Proof. Let $a=2m+2, b=2m+4, c=2m+6, d=2m+8$. Then, $ab(cd+1)+2(b+8)=(2m+2)(2m+4)((2m+6)(2m+8)+1)+2(2m+4+8)=2m(2m+4)+2(2m+4)(2m(2m+8)+6(2m+8)+1)+2(2m+12)=(4m^2+8m+4m+8)(4m^2+16m+12m+48+1)+4m+24=4m^2(4m^2+28m+49)+12m(4m^2+28m+49)+8(4m^2+28m+49)+4m+24=16m^4+112m^3+196m^2+48m^3+336m^2+588m+32m^2+224m+392+4m+24=16m^4+160m^3+564m^2+816m+416=4m^2+16m+16+16m^4+160m^3+560m^2+800m+400=[2m+4]^2+[(2m+2)^2+6(2m+2)+4]^2=b^2+[(a^2+6a+4)]^2$ establishing the proof. $\square$

Proposition 2.7. $ab(cd+1)+b+82=[b-1]^2+[(b-1)^2+5(b-1)-5]^2=[\frac{a+b+1}{2}]^2+[(b-1)^2+5(b-1)-5]^2$ has solution in integers if $a=3m+1, b=3m+4, c=3m+7, d=3m+10$ where $m \geq 0$ and $L=b+82$.

Proof. Suppose $a=3m+1, b=3m+4, c=3m+7, d=3m+10$. Then, $ab(cd+1)+b+82=(3m+1)(3m+4)((3m+7)(3m+10)+1)+b+82=3m(3m+4)+1(3m+4)(3m(3m+10)+7(3m+10)+1)+3m+86=(9m^2+12m+3m+4)(9m^2+30m+21m+70+1)+3m+86=(9m^2+15m+4)(9m^2+51m+71)+3m+86=9m^2(9m^2+51m+71)+15m(9m^2+51m+71)+4(9m^2+51m+71)+3m+86=81m^4+459m^3+639m^2+135m^3+765m^2+1065m+36m^2+204m+284+3m+86=9m^2+18m+9+81m^4+297m^3+171m^2+297m^3+1089m^2+627m+171m^2+627m+361=[3m+3]^2+[(3m+3)^2+5(3m+3)-5]^2$. Since $b-1=\frac{a+b+1}{2}=3m+3$ we have $ab(cd+1)+b+82=[b-1]^2+[(b-1)^2+5(b-1)-5]^2=[\frac{a+b+1}{2}]^2+[(b-1)^2+5(b-1)-5]^2$. $\square$

Proposition 2.8. $ab(cd+1)+b+82=[b-1]^2+[(b-2)^2+7(b-2)+1]^2=[\frac{a+b+1}{2}]^2+[(b-2)^2+7(b-2)+1]^2$ has solution in integers if $a=3m+2, b=3m+5, c=3m+8, d=3m+11$ where $m \geq 0$ and $L=b+82$.

Proof. Assume $a = 3m + 2, b = 3m + 5, c = 3m + 8, d = 3m + 11$. Then, $ab(cd + 1) + b + 82 = (3m + 2)(3m + 5)((3m + 8)(3m + 11) + 1) + b + 82 = 3m(3m + 5) + 2(3m + 5)(3m(3m + 11) + 8(3m + 11) + 1) + 3m + 87 = (9m^2 + 21m + 10)(9m^2 + 33m + 24m + 88 + 1) + 3m + 87 = 9m^2(9m^2 + 57m + 89) + 21m(9m^2 + 57m + 89) + 10(9m^2 + 57m + 89) + 3m + 87 = 81m^4 + 513m^3 + 801m^2 + 189m^3 + 1197m^2 + 1869m + 90m^2 + 570m + 890 + 3m + 87 = 9m^2 + 24m + 16 + 81m^4 + 351m^3 + 279m^2 + 351m^3 + 1521m^2 + 1209m + 279m^2 + 1209m + 961 = [3m + 4]^2 + [(3m + 3)^2 + 7(3m + 3) + 1]^2$. Since $b - 1 = 3m + 4$ and $b - 2 = 3m + 3$ the results easily follows. $\square$

Proposition 2.9. $ab(cd + 1) + b + 82 = [b - 1]^2 + [(b - 3)^2 + 9(b - 3) + 9]^2 = [\frac{a+b+1}{2}]^2 + [(b - 3)^2 + 9(b - 3) + 9]^2$ has solution in integers if $a = 3m + 3, b = 3m + 6, c = 3m + 9, d = 3m + 12$ where $m \geq 0$ and $L = b + 82$.

Proof. Let $a = 3m + 3, b = 3m + 6, c = 3m + 9, d = 3m + 12$. Then, $ab(cd + 1) + b + 82 = (3m + 3)(3m + 6)((3m + 9)(3m + 12) + 1) + b + 82 = 3m(3m + 6) + 3(3m + 6)(3m(3m + 12) + 9(3m + 12) + 1) + 3m + 88 = (9m^2 + 18m + 9m + 18)(9m^2 + 36m + 27m + 109) + 3m + 88 = 9m^2(9m^2 + 63m + 109) + 27m(9m^2 + 63m + 109) + 18(9m^2 + 63m + 109) + 3m + 88 = 81m^4 + 567m^3 + 981m^2 + 243m^3 + 1701m^2 + 2943m + 162m^2 + 1134m + 1962 + 3m + 88 = 9m^2 + 30m + 25 + 81m^4 + 405m^3 + 405m^2 + 405m^3 + 2025m^2 + 2025m + 405m^2 + 2025m + 2025 = [3m + 5]^2 + [(3m + 3)^2 + 9(3m + 3) + 9]^2 = [b - 1]^2 + [(b - 3)^2 + 9(b - 3) + 9]^2 = [\frac{a+b+1}{2}]^2 + [(b - 3)^2 + 9(b - 3) + 9]^2$ since $b - 1 = \frac{a+b+1}{2} = 3m + 5$ and $b - 3 = 3m + 3$ the results easily follows. $\square$

Proposition 2.10. $ab(cd + 1) + 3a(a - 1) + 81 = [2a]^2 + [(b - 1)^2 + 5(b - 1) - 5]^2$ has solution in integers if $a = 3m + 1, b = 3m + 4, c = 3m + 7, d = 3m + 10$ where $m \geq 0$ and $L = 3a(a - 1) + 81$.

Proof. Assume $a = 3m + 1, b = 3m + 4, c = 3m + 7, d = 3m + 10$. Then, $ab(cd + 1) + 3a(a - 1) + 81 = (3m + 1)(3m + 4)((3m + 7)(3m + 10) + 1) + 3(3m + 1)(3m + 1 - 1) + 81 = 3m(3m + 4) + 1(3m + 4)(3m(3m + 10) + 7(3m + 10) + 1) + (9m + 3)3m + 81 = (9m^2 + 12m + 3m + 4)(9m^2 + 30m + 21m + 70 + 1) + 27m^2 + 9m + 81 = (9m^2 + 15m + 4)(9m^2 + 51m + 71) + 27m^2 + 9m + 81 = 9m^2(9m^2 + 51m + 71) + 15m(9m^2 + 51m + 71) + 4(9m^2 + 51m + 71) + 27m^2 + 9m + 81 = 81m^4 + 459m^3 + 639m^2 + 135m^3 + 765m^2 + 1065m + 36m^2 + 204m + 284 + 27m^2 + 9m + 81 = 36m^2 + 24m + 4 + 81m^4 + 297m^3 + 171m^2 + 297m^3 + 1089m^2 + 627m + 171m^2 + 627m + 361 = [2(3m + 1)]^2 + [(3m + 3)^2 + 5(3m + 3) - 5]^2$. Since $a = 3m + 1, b = 3m + 4$ we have $ab(cd + 1) + 3a(a - 1) + 81 = [2a]^2 + [(b - 1)^2 + 5(b - 1) - 5]^2$. $\square$

Proposition 2.11. $ab(cd + 1) + 3a(a - 1) + 81 = [a + b - 3]^2 + [(b - 2)^2 + 7(b - 2) + 1]^2$ has solution in integers if $a = 3m + 2, b = 3m + 5, c = 3m + 8, d = 3m + 11$ where $m \geq 0$ and $L = 3a(a - 1) + 81$.

Proof. Let $a = 3m + 2, b = 3m + 5, c = 3m + 8, d = 3m + 11$. Then, $ab(cd + 1) + 3a(a - 1) + 81 = (3m + 2)(3m + 5)((3m + 8)(3m + 11) + 1) + 3(3m + 2)(3m + 2 - 1) + 81 = 3m(3m + 5) + 2(3m + 5)(3m(3m + 11) + 8(3m + 11) + 1) + (9m + 6)(3m + 1) + 81 = (9m^2 + 15m + 6m + 10)(9m^2 + 33m + 24m + 88 + 1) + 9m(3m + 1) + 6(3m + 1) + 81 = (9m^2 + 21m + 10)(9m^2 + 57m + 89) + 27m^2 + 9m + 18m + 6 + 81 = 9m^2(9m^2 + 57m + 89) + 21m(9m^2 + 57m + 89) + 10(9m^2 + 57m + 89) + 27m^2 + 27m + 87 = 81m^4 + 513m^3 + 801m^2 + 189m^3 + 1197m^2 + 1869m + 90m^2 + 570m + 890 + 27m^2 + 27m + 87 = 36m^2 + 48m + 16 + 81m^4 + 351m^3 + 279m^2 + 351m^3 + 1521m^2 + 1209m + 279m^2 + 1209m + 961 = [6m + 4]^2 + [(3m + 3)^2 + 7(3m + 3) + 1]^2$. Since $a + b - 3 = 6m + 4, b - 2 = 3m + 3$ we have $ab(cd + 1) + 3a(a - 1) + 81 = [2a]^2 + [a + b - 3]^2 + [(b - 2)^2 + 7(b - 2) + 1]^2$. $\square$

Proposition 2.12. $ab(cd + 1) + 3a(a - 1) + 81 = [a + b - 3]^2 + [(b - 3)^2 + 9(b - 3) + 9]^2$ has solution in integers if $a = 3m + 3, b = 3m + 6, c = 3m + 9, d = 3m + 12$ where $m \geq 0$ and $L = 3a(a - 1) + 81$.

Proof. Suppose $a = 3m + 3, b = 3m + 6, c = 3m + 9, d = 3m + 12$. Then, $ab(cd + 1) + 3a(a - 1) + 81 = (3m + 3)(3m + 6)((3m + 9)(3m + 12) + 1) + 3(3m + 3)(3m + 3 - 1) + 81 = 3m(3m + 6) + 3(3m + 6)(3m(3m + 12) + 9(3m + 12) + 1) + (9m + 9)(3m + 2) + 81 = (9m^2 + 18m + 9m + 18)(9m^2 + 36m + 27m + 108 + 1) + 9m(3m + 2) + 9(3m + 2) + 81 = (9m^2 + 27m + 18)(9m^2 + 63m + 109) + 27m^2 + 18m + 27m + 18 + 81 =$

$$\begin{aligned}
& 9m^2(9m^2 + 63m + 109) + 27m(9m^2 + 63m + 109) + 18(9m^2 + 63m + 109) + 27m^2 + 45m + 99 = \\
& 81m^4 + 567m^3 + 981m^2 + 243m^3 + 1701m^2 + 2943m + 162m^2 + 1134m + 1962 + 27m^2 + 45m + 99 = \\
& 36m^2 + 72m + 36 + 81m^4 + 405m^3 + 405m^2 + 405m^3 + 2025m^2 + 2025m + 405m^2 + 2025m + 2025 = \\
& [6m + 6]^2 + [(3m + 3)^2 + 9(3m + 3) + 9]^2. \text{ Since } a + b - 3 = 6m + 6, b - 2 = 3m + 3 \text{ consequently} \\
& ab(cd + 1) + 3a(a - 1) + 81 = [2a]^2 + [a + b - 3]^2 + [(b - 2)^2 + 7(b - 2) + 1]^2.
\end{aligned}$$

□

Proposition 2.13. $ab(cd + 1) + 3a^2 + a + 82 = [a + b - 2]^2 + [(b - 1)^2 + 5(b - 1) - 5]^2$ has solution in integers if $a = 3m + 1, b = 3m + 4, c = 3m + 7, d = 3m + 10$ where $m \geq 0$ and $L = 3a^2 + a + 82$.

Proof. Let $a = 3m + 1, b = 3m + 4, c = 3m + 7, d = 3m + 10$. Then, $ab(cd + 1) + 3a^2 + a + 82 = (3m + 1)(3m + 4)((3m + 7)(3m + 10) + 1) + 3(3m + 1)^2 + (3m + 1) + 82 = 3m(3m + 4) + 1(3m + 4)(3m(3m + 10) + 7(3m + 10) + 1) + 3(3m + 1)^2 + 3m + 1 + 82 = (9m^2 + 12m + 3m + 4)(9m^2 + 30m + 21m + 70 + 1) + 27m^2 + 21m + 86 = (9m^2 + 15m + 4)(9m^2 + 51m + 71) + 27m^2 + 21m + 86 = 9m^2(9m^2 + 51m + 71) + 15m(9m^2 + 51m + 71) + 4(9m^2 + 51m + 71) + 27m^2 + 21m + 86 = 81m^4 + 459m^3 + 639m^2 + 135m^3 + 765m^2 + 1065m + 36m^2 + 204m + 284 + 27m^2 + 21m + 86 = 36m^2 + 36m + 9 + 81m^4 + 297m^3 + 171m^2 + 297m^3 + 1089m^2 + 627m + 171m^2 + 627m + 361 = [6m + 3]^2 + [(3m + 3)^2 + 5(3m + 3) - 5]^2. Since $a + b - 2 = 6m + 3, b - 1 = 3m + 3$ it follows that $ab(cd + 1) + 3a(a - 1) + 81 = [a + b - 2]^2 + [(b - 1)^2 + 5(b - 1) - 5]^2$. □$

Proposition 2.14. $ab(cd + 1) + 3a^2 + a + 82 = [a + b - 2]^2 + [(b - 2)^2 + 7(b - 2) + 1]^2$ has solution in integers if $a = 3m + 2, b = 3m + 5, c = 3m + 8, d = 3m + 11$ where $m \geq 0$ and $L = 3a^2 + a + 82$.

Proof. Assume $a = 3m + 2, b = 3m + 5, c = 3m + 8, d = 3m + 11$. Then, $ab(cd + 1) + 3a^2 + a + 82 = (3m + 2)(3m + 5)((3m + 8)(3m + 11) + 1) + 3(3m + 2)^2 + (3m + 2) + 82 = 3m(3m + 5) + 2(3m + 5)(3m(3m + 11) + 8(3m + 11) + 1) + 3(3m + 2)^2 + (3m + 2) + 82 = (9m^2 + 15m + 6m + 10)(9m^2 + 33m + 24m + 88 + 1) + 3(9m^2 + 12m + 4) + 3m + 84 = (9m^2 + 21m + 10)(9m^2 + 57m + 89) + 27m^2 + 39m + 96 = 9m^2(9m^2 + 57m + 89) + 21m(9m^2 + 57m + 89) + 10(9m^2 + 57m + 89) + 27m^2 + 39m + 96 = 81m^4 + 513m^3 + 801m^2 + 189m^3 + 1197m^2 + 1869m + 90m^2 + 570m + 890 + 27m^2 + 39m + 96 = 36m^2 + 60m + 25 + 81m^4 + 351m^3 + 279m^2 + 351m^3 + 1521m^2 + 1209m + 279m^2 + 1209m + 961 = [6m + 5]^2 + [(3m + 3)^2 + 7(3m + 3) + 1]^2. Since $a + b - 2 = 6m + 5, b - 2 = 3m + 3$ it can be seen that $ab(cd + 1) + 3a^2 + a + 82 = [a + b - 2]^2 + [(b - 2)^2 + 7(b - 2) + 1]^2$. □$

Proposition 2.15. $ab(cd + 1) + 3a^2 + a + 82 = [a + b - 2]^2 + [(b - 3)^2 + 7(b - 3) + 9]^2$ has solution in integers if $a = 3m + 3, b = 3m + 6, c = 3m + 9, d = 3m + 12$ where $m \geq 0$ and $L = 3a^2 + a + 82$.

Proof. Suppose $a = 3m + 3, b = 3m + 6, c = 3m + 9, d = 3m + 12$. Then, $ab(cd + 1) + 3a^2 + a + 82 = (3m + 3)(3m + 6)((3m + 9)(3m + 12) + 1) + 3(3m + 3)^2 + (3m + 3) + 82 = 3m(3m + 6) + 3(3m + 6)(3m(3m + 12) + 9(3m + 12) + 1) + 27m^2 + 54m + 27 + 3m + 85 = (9m^2 + 18m + 9m + 18)(9m^2 + 36m + 27m + 108 + 1) + 27m^2 + 57m + 112 = (9m^2 + 27m + 18)(9m^2 + 63m + 109) + 27m^2 + 57m + 112 = 9m^2(9m^2 + 63m + 109) + 27m(9m^2 + 63m + 109) + 18(9m^2 + 63m + 109) + 27m^2 + 57m + 112 = 81m^4 + 567m^3 + 981m^2 + 243m^3 + 1701m^2 + 2943m + 162m^2 + 1134m + 1962 + 27m^2 + 57m + 112 = 36m^2 + 84m + 49 + 81m^4 + 405m^3 + 405m^2 + 405m^3 + 2025m^2 + 2025m + 405m^2 + 2025m + 2025 = [6m + 7]^2 + [(3m + 3)^2 + 9(3m + 3) + 9]^2. Since $a + b - 2 = 6m + 7, b - 3 = 3m + 3$ we get $ab(cd + 1) + 3a^2 + a + 82 = [a + b - 2]^2 + [(b - 3)^2 + 7(b - 3) + 9]^2$. □$

The next result provides conjecture for the title equation.

Conjecture 2.1. $ab(cd + 1) + L = u^2 + v^2$ has no general solution in integers if $b - a = c - b = d - c \neq \pm u$ where u is an integer and L some integer.

Conclusion

This paper has researched on diophantine equation $ab(cd + 1) + L = u^2 + v^2$. The study revealed some novel formulas for generating sums of two squares. This was achieved by determining integers a, b, c, d, u, v and L together with factorization argument and application of modular arithmetic. To this far, research in this area is still minimal and we recommend other researchers to carry out more studies regarding the title equation.

Competing Interests

Author has declared no competing interest.

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