Estimation of Finite Population Mean Using An Improved Class of Mixed Estimators with Two Auxiliary Variables

Abstract

This paper deals with a class of estimators of finite population mean using a combination of two mixed classes of estimators by exploring the information on two auxiliary variables. We have assumed that the study variable y is highly correlated with both the auxiliary variables x and z. The optimum properties of the proposed class of estimators is studied both theoretically and empirically. The minimum variance bound(MVB) estimator of this class is also derived and compared with several other competing estimators in terms of its bias and percent relative efficiency.

keywords: Population mean, Class of estimators, Ratio estimator, Dual to product estimator, Relative bias, Minimum variance bound(MVB), Percent relative efficiency(PRE)

1 Introduction

Supplementary information is engaged in various ways in order to design more precise estimators for a finite population of size N. The supplementary information can be obtained readily or can be made available by utilizing a very minimum cost from the total cost of survey in order to achieve considerable gain in precision of estimators. Sometimes, the information on more than one auxiliary variable is available which helps the researcher a lot in designing precise estimators in the form of mixing of different estimators. These mixed estimators can be applied in more general conditions than the individual ones in terms of correlation structure between the study variable (y) and auxiliary variables (x, z). For example; to estimate the average cotton output, the proportion of good seeds and the planting area are two principal auxiliary variables in agricultural engineering and both of these variables can be utilized to estimate the average cotton output more precisely. Hence, the auxiliary information is frequently used in the field of medical sciences, education, biostatistics, biology, economics, management, sociology and many more. Estimation of finite population parameters, specifically population mean, gained much interest among the survey samplers. Some of the recent studies can be seen with Mishra et al. [2018], Akingbade and Okafor [2019], Adichwal et al. [2022], Yadav [2022], Bhushan and Pandey [2022], Dash and Sunani [2022], etc. In the following, we have noted down some important mixed-type estimators using two auxiliary variables for estimating finite population mean $\overline{Y} = N^{-1}\sum_{i=1}^{N} Y_i$.

When in information on only as single auxiliary variable is available, Cochran [1940] proposed ratio estimator on the basis of a sample of size n drawn by SRSWOR scheme for estimating $\overline{Y}$ as

$$t_{1_x} = \overline{y}\frac{\overline{X}}{\overline{x}}, \quad \text{where } \overline{X} = \frac{1}{N}\sum_{i=1}^{N} X_i, \tag{1}$$

and $\overline{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ is the sample mean of study variable y; whose bias and MSE are respectively given by

$$\begin{aligned} B(t_{1_x}) &= \overline{Y}\theta\,(C_{xx} - C_{yx}) \\ MSE(t_{1_x}) &= \overline{Y}^2\theta\,[C_{yy} + C_{xx} - 2C_{yx}]\,. \end{aligned} \tag{2}$$

But, in presence of information on two auxiliary variables, Olkin [1958] proposed a class of estimators of $\overline{Y}$ by mixing two ratio estimators as

$$t_1 = [wt_{1_x} + (1-w)t_{1_z}]\,, \tag{3}$$

where, w is a positive constant and $t_{1_x} = \overline{y}\frac{\overline{X}}{\overline{x}}$ and $t_{1_z} = \overline{y}\frac{\overline{Z}}{\overline{z}}$ are two usual ratio estimators of population mean $\overline{Y}$ using two auxiliary variables x and z respectively at the estimation stage where $\overline{Z} = \frac{1}{N}\sum_{i=1}^{N} Z_i$. The bias and mean square error of this class of estimators are

$$B(t_1) = \overline{Y}\theta\,[(C_{zz} - C_{yz}) + w(C_{xx} - C_{zz} + C_{yz} - C_{yx})] \tag{4}$$

$$\begin{aligned} MSE(t_1) = \overline{Y}^2\theta\,\big[&C_{yy} + C_{zz} - 2C_{yz} + w^2(C_{xx} + C_{zz} - C_{xz}) \\ &+2w(C_{yz} - C_{zz} - C_{yx} + C_{xz})\big]\,. \end{aligned} \tag{5}$$

The value of w which results a MVB estimator of this class is

$$w^{(o)} = -\frac{(C_{yz} - C_{zz} - C_{yx} + C_{xz})}{C_{xx} + C_{zz} - 2C_{xz}}$$

and the MVB of this class is

$$MSE\left(t_1^{(o)}\right) = \overline{Y}^2\theta\left[C_{yy} + C_{zz} - 2C_{xz} - \frac{(C_{yz} - C_{zz} - C_{yx} + C_{xz})^2}{C_{xx} + C_{zz} - 2C_{xz}}\right]. \tag{6}$$

Singh [1967] proposed three such estimators as

$$\begin{aligned} t_2 &= \overline{y}\left(\frac{\overline{z}}{\overline{x}}\right)\left(\frac{\overline{X}}{\overline{Z}}\right) \\ t_3 &= \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)\left(\frac{\overline{Z}}{\overline{z}}\right) \\ t_4 &= \overline{y}\left(\frac{\overline{x}}{\overline{X}}\right)\left(\frac{\overline{z}}{\overline{Z}}\right) \end{aligned} \tag{7}$$

The bias and MSE of these estimators are

$$B(t_2) = \overline{Y}\theta\,[C_{xx} + C_{yz} - C_{xz} - C_{yx}] \tag{8}$$

$$B(t_3) = \overline{Y}\theta\,[C_{xx} + C_{zz} + C_{xz} - C_{yx} - C_{yz}] \tag{9}$$

$$B(t_4) = \overline{Y}\theta\,[C_{yx} + C_{yz} + C_{xz}] \tag{10}$$

$$MSE(t_2) = \overline{Y}^2\theta\,[C_{yy} + C_{xx} + C_{zz} - 2C_{yx} + 2C_{yz} - 2C_{xz}] \tag{11}$$

$$MSE(t_3) = \overline{Y}^2\theta\,[C_{yy} + C_{xx} + C_{zz} - 2C_{yx} - 2C_{yz} + 2C_{xz}] \tag{12}$$

$$MSE(t_4) = \overline{Y}^2\theta\,[C_{yy} + C_{xx} + C_{zz} + 2C_{yx} + 2C_{yz} + 2C_{xz}] \tag{13}$$

Srivenkataramana [1980] proposed dual to ratio estimator basing on the auxiliary variable x, which is negatively correlated with y as

$$t_5 = \overline{y}\frac{\overline{x}^*}{\overline{X}}, \quad \text{where} \quad \overline{x}^* = (1+g)\overline{X} - g\overline{x}, \quad g = \frac{n}{N-n} \tag{14}$$

The bias and mean square error of this estimator are given by

$$B(t_5) = -\overline{Y}\theta g C_{yx}, \quad \theta = \frac{1}{n} - \frac{1}{N}. \tag{15}$$

$$MSE(t_5) = \overline{Y}^2\theta\left[C_{yy} + g^2C_{xx} - 2gC_{yx}\right] \tag{16}$$

In the similar manner, Bandopadhyay [1980] suggested dual to product estimator for estimating $\overline{Y}$, when y and x are positively correlated, as

$$t_6 = \overline{y}\frac{\overline{X}}{\overline{x}^*}. \tag{17}$$

The bias and mean square error of this estimator are given by

$$B(t_6) = \overline{Y}\theta g\left[C_{yx} + gC_{xx}\right] \tag{18}$$

$$MSE(t_6) = \overline{Y}^2\theta\left[C_{yy} + g^2C_{xx} + 2gC_{yx}\right]. \tag{19}$$

Abu-Dayyeh et al. [2003] proposed a class of estimators of population mean when the population means $\overline{X}$ and $\overline{Z}$ of the auxiliary variables x and z are known in advance, which is given by

$$t_7 = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha_1}\left(\frac{\overline{Z}}{\overline{z}}\right)^{\alpha_2}. \tag{20}$$

where α_1, α_2 are two unknown constant. The bias and mean square error of t_7 becomes

$$\begin{aligned} B(t_7) &= \overline{Y}\theta\left[\alpha_1C_{yx} + \alpha_2C_{yz} + \alpha_1\alpha_2C_{xz}\right. \\ &\qquad \left. + \frac{\alpha_1(\alpha_1-1)}{2}C_{xx} + \frac{\alpha_2(\alpha_2-1)}{2}C_{zz}\right] \end{aligned} \tag{21}$$

$$\begin{aligned} MSE(t_7) &= \overline{Y}^2\theta\left[C_{yy} + \alpha_1^2C_{xx} + \alpha_2^2C_{zz}\right. \\ &\qquad \left. + 2\alpha_1C_{yx} + 2\alpha_2C_{yz} + 2\alpha_1\alpha_2C_{xz}\right] \end{aligned} \tag{22}$$

respectively and the values of α_1 and α_2 which minimizes the $MSE(t_7)$ are

$$\alpha_1^{(o)} = -\frac{(\rho_{yx} - \rho_{yz}\rho_{xz})}{(1-\rho_{xz}^2)}\sqrt{\frac{C_{yy}}{C_{xx}}}, \text{ and } \alpha_2^{(o)} = -\frac{(\rho_{yz} - \rho_{yx}\rho_{xz})}{(1-\rho_{xz}^2)}\sqrt{\frac{C_{yy}}{C_{zz}}} \tag{23}$$

respectively, where $\rho_{yx}, \rho_{yz}, \rho_{xz}$ are the simple coefficient of Correlation between the variables y, x and z, $\rho_{y.xz}$ is the multiple correlation coefficient between y on x and z. The MVB of this class is given by

$$MSE\,(t_7)^{(o)} = \overline{Y}^2\theta C_{yy}\left(1 - \rho_{y.xz}^2\right). \tag{24}$$

It can be easily verified that the estimators t_2, t_3 and t_4 proposed by Singh [1967] are the particular members of this class t_7.

Perri [2004] proposed bivariate dual to product estimator for estimating $\overline{Y}$ using two auxiliary variables as

$$t_8 = \overline{y}\frac{\overline{X}}{\overline{x}^*}\frac{\overline{Z}}{\overline{z}^*}, \qquad \text{where } \ \overline{z}^* = (1+g)\overline{Z} - g\overline{z}. \tag{25}$$

whose bias and mean square error are given by

$$B(t_8) = \overline{Y}\theta\left[g^2\left(C_{xx} + C_{zz} + C_{xz}\right) + g(C_{yx} + C_{yz})\right] \tag{26}$$

$$MSE(t_8) = \overline{Y}^2\theta\left[C_{yy} + g^2\left(C_{xx} + C_{zz} + 2C_{xz}\right) + 2g\left(C_{yz} + C_{yx}\right)\right] \tag{27}$$

respectively. Singh et al. [2005] proposed dual to ratio cum product estimator for estimating $\overline{Y}$ as

$$t_9 = \overline{y}\frac{\overline{x}^*}{\overline{X}}\frac{\overline{Z}}{\overline{z}^*}, \tag{28}$$

whose bias and mean square error are given by

$$B(t_9) = \overline{Y}\theta\left[gC_{yz} + g^2C_{zz} - gC_{yx} - g^2C_{xz}\right] \tag{29}$$

$$MSE(t_9) = \overline{Y}^2\theta\left[C_{yy} + g^2(C_{xx} + C_{zz} - 2C_{xz}) + 2gC_{yz} - 2gC_{yx}\right] \tag{30}$$

respectively. Singh et al. [2011] proposed a general family of dual to ratio cum product estimators for estimating population mean $\overline{Y}$ following Srivastava [1967] as

$$t_{10} = \overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{\psi_1}\left(\frac{\overline{Z}}{\overline{z}^*}\right)^{\psi_2} \tag{31}$$

where ψ_1, ψ_2 are two suitably chosen constants. The bias and MSE of this class t_{10} are

$$B(t_{10}) = \overline{Y}\theta\left[\psi_2 gC_{yz} - \psi_1 gC_{yx} - \psi_1\psi_2 g^2C_{xz} + \frac{\psi_1(\psi_1-1)}{2}g^2C_{xx} + \frac{\psi_2(\psi_2-1)}{2}g^2C_{zz}\right], \tag{32}$$

$$MSE(t_{10}) = \overline{Y}^2\theta\left[C_{yy} + g^2\{\psi_1^2C_{xx} + \psi_2^2C_{zz} - 2\psi_1\psi_2C_{xz}\} - 2g(\psi_1C_{yx} - \psi_2C_{yz})\right]. \tag{33}$$

The optimum values of ψ_1 and ψ_2 which minimizes $MSE(t_{10})$ are

$$\psi_1^{(o)} = \frac{(\rho_{yx} - \rho_{xz}\rho_{yz})}{g(1-\rho_{xz}^2)}\sqrt{\frac{C_{yy}}{C_{xx}}}, \quad \psi_2^{(o)} = \frac{-(\rho_{yz} - \rho_{xz}\rho_{yx})}{g(1-\rho_{xz}^2)}\sqrt{\frac{C_{yy}}{C_{zz}}}. \tag{34}$$

The MVB of this class is

$$MSE\left(t_{10}^{(o)}\right) = \overline{Y}^2C_{yy}\theta\left[1 - \rho_{y.xz}^2\right]. \tag{35}$$

It can be easily verified that the estimators proposed by Perri [2004] and Singh et al. [2005] are particular members of this class t_{10}. Choudhury and Singh [2012] proposed a class of ratio cum dual to product estimator for estimating $\overline{Y}$ as

$$t_{11} = \overline{y}\left[\alpha\frac{\overline{X}}{\overline{x}} + (1-\alpha)\frac{\overline{X}}{\overline{x}^*}\right] \tag{36}$$

where, α is a real constant. The bias and mean square error of t_{11} becomes

$$B(t_{11}) = \overline{Y}\theta\left[g(C_{yx}+gC_{xx})+\alpha(1+g)\{(1-g)C_{xx}-C_{yx}\}\right] \quad (37)$$

$$MSE(t_{11}) = \overline{Y}^2\theta\left[C_{yy}+\{g-\alpha(1+g)\}^2C_{xx}+2\{g-\alpha(1+g)\}C_{yx}\right] \quad (38)$$

and the optimum value of 'α' which results an MVB for this class is given by

$$\alpha^{(o)} = \frac{C_{yx}+gC_{xx}}{(1+g)C_{xx}}$$

and the MVB of this class is given by

$$MSE(t_{11}^{(o)}) = \overline{Y}^2\theta C_{yy}\left[1-\rho_{yx}^2\right]. \quad (39)$$

Again, Choudhury and Singh [2012] suggested dual to product cum dual to ratio estimator as

$$t_{12} = \overline{y}\left[\alpha\frac{\overline{X}}{\overline{x}^*}+(1-\alpha)\frac{\overline{x}^*}{\overline{X}}\right], \quad (40)$$

where α is any scalar. The bias and mean square error of the estimator t_{12} is given by

$$B(t_{12}) = \overline{Y}\theta g\left[(2\alpha-1)C_{yx}+\alpha gC_{xx}\right] \quad (41)$$

$$MSE(t_{12}) = \overline{Y}^2\theta\left[C_{yy}+g(2\alpha-1)\{g(2\alpha-1)C_{xx}+2C_{yx}\}\right] \quad (42)$$

The optimum value of 'α' which results an MVB for this class is given by

$$\alpha^{(o)} = -\frac{C_{yx}-gC_{xx}}{2gC_{xx}}$$

and the MVB of this class is the MVB of the class t_{11}.

2 The Proposed Class of Estimators

Motivated by the above estimators, we propose a class of estimators t for the finite population mean $\overline{Y}$ by combining the ratio estimator along with dual to product estimator as

$$t = \overline{y}\left[\alpha\left(\frac{\overline{X}}{\overline{x}}\right)+(1-\alpha)\left(\frac{\overline{X}}{\overline{x}^*}\right)\right]\left[\beta\left(\frac{\overline{Z}}{\overline{z}}\right)+(1-\beta)\left(\frac{\overline{Z}}{\overline{z}^*}\right)\right], \quad (43)$$

where α and β are two real constants or parameters. In order to study the large sample behaviour of this estimator, we consider

$$\overline{y} = \overline{Y}(1+e_0), \quad \overline{x} = \overline{X}(1+e_1), \quad \overline{z} = \overline{Z}(1+e_2), \quad E(e_i) = 0,\ i = 0,1,2. \quad (44)$$

So, e_i's are the sampling errors associated with respective statistics. Thus, we have

$$E(e_0^2) = \theta C_{yy}, \quad E(e_1^2) = \theta C_{xx}, \quad E(e_2^2) = \theta C_{zz}$$
$$E(e_0e_1) = \theta C_{yx}, \quad E(e_0e_2) = \theta C_{yz}, \quad E(e_1e_2) = \theta C_{xz}. \quad (45)$$

Using (44) and (45) in (43) we get

$$\begin{aligned} t = \overline{Y}\big[&1+e_0+ge_1+ge_2+ge_0e_1+ge_0e_2+g^2e_1^2+g^2e_2^2+g^2e_1e_2 \\ &+\alpha\left(e_1^2-g^2e_1^2-e_1-e_0e_1-ge_1-ge_0e_1-ge_1e_2-g^2e_1e_2\right) \\ &+\beta\left(e_2^2-g^2e_2^2-e_2-e_0e_2-ge_2-ge_0e_2-ge_1e_2-g^2e_1e_2\right) \\ &+\alpha\beta\left(ge_1e_2+g^2e_1e_2+e_1e_2+ge_1e_2\right)\big]+o\left(e_i^2\right). \end{aligned} \quad (46)$$

3 Properties of the Proposed Class

The equation (46) immediately implies the bias of the proposed class t as

$$\begin{aligned} B(t) &= \overline{Y}\theta \left[g^2(C_{xx} + C_{zz} + C_{xz}) + g(C_{yz} + C_{yx})\right. \\ &\quad + \beta(1+g)\{(1-g)C_{zz} - C_{yz} - gC_{xz}\} \\ &\quad + \alpha(1+g)\{(1-g)C_{xx} - C_{yx} - gC_{xz}\} \\ &\quad \left. + \alpha\beta(1+g)\{(1+g)C_{xz}\}\right] \end{aligned} \tag{47}$$

and MSE up to $o(n^{-1})$ as

$$\begin{aligned} MSE(t) &= \overline{Y}^2\theta \left[C_{yy} + g^2(C_{xx} + C_{zz} + 2C_{xz})\right. \\ &\quad + (1+g)^2(\beta^2 C_{zz} + \alpha^2 C_{xx} + 2\alpha\beta C_{xz}) \\ &\quad + 2g(C_{yx} + C_{yz}) \\ &\quad - 2g(1+g)(\beta C_{xz} + \alpha C_{xx} + \beta C_{zz} + \alpha C_{xz}) \\ &\quad \left. - 2(1+g)\{\beta C_{yz} + \alpha C_{yx}\}\right] \end{aligned} \tag{48}$$

The optimum values of α and β which minimizes the $MSE(t)$ are given by

$$\alpha^{(o)} = \frac{g}{1+g} + \frac{(\rho_{yx} - \rho_{yz}\rho_{xz})}{(1+g)(1-\rho_{xz}^2)}\sqrt{\frac{C_{yy}}{C_{xx}}} \tag{49}$$

$$\beta^{(o)} = \frac{g}{1+g} + \frac{(\rho_{yz} - \rho_{yx}\rho_{xz})}{(1+g)(1-\rho_{xz}^2)}\sqrt{\frac{C_{yy}}{C_{zz}}}. \tag{50}$$

Using the optimum values of $\alpha^{(o)}$ and $\beta^{(o)}$ in place of α and β in equation (48), we get the MVB of the class t as

$$MSE(t^{(o)}) = \overline{Y}^2 C_{yy}\theta\left[1 - \rho_{y.xz}^2\right], \tag{51}$$

where $\rho_{y.xz}$ is the multiple correlation coefficient between y on x and z and the corresponding MVB estimator is

$$\begin{aligned} t^{(o)} &= \frac{\overline{y}}{(1+g)}\left[\left\{g + \frac{(\rho_{yx} - \rho_{yz}\rho_{xz})C_y}{(1-\rho_{xz}^2)C_z}\right\}\frac{\overline{X}}{\overline{x}} + \left\{1 - \frac{(\rho_{yx} - \rho_{yz}\rho_{xz})C_y}{(1-\rho_{xz}^2)C_x}\right\}\frac{\overline{X}}{\overline{x}^*}\right] \\ &\quad \left[\left\{g + \frac{(\rho_{yz} - \rho_{yx}\rho_{xz})C_y}{(1-\rho_{xz}^2)C_z}\right\}\frac{\overline{Z}}{\overline{z}} + \left\{1 - \frac{(\rho_{yz} - \rho_{yx}\rho_{xz})C_y}{(1-\rho_{xz}^2)C_z}\right\}\frac{\overline{X}}{\overline{x}^*}\right]. \end{aligned} \tag{52}$$

4 Some Particular Cases

Some of the popular estimators as the members of proposed class of estimators t are discussed below along with their bias and MSE.

a. Bivariate Ratio Estimator

When $\alpha = \beta = 1$, the proposed class of estimators 't' reduces to ratio estimator using two auxiliary variables proposed by Singh [1967]

$$t_1^* = \overline{y}\frac{\overline{X}}{\overline{x}}\frac{\overline{Z}}{\overline{z}}. \tag{53}$$

The bias of t_1^* is

$$B(t_1^*) = \overline{Y}\theta \left[C_{xx} + C_{zz} + C_{xz} - C_{yx} - C_{yz}\right] \tag{54}$$

and the mean squared error is

$$MSE(t_1^*) = \overline{Y}^2\theta \left[C_{yy} + C_{xx} + C_{zz} + 2C_{xz} - 2C_{yz} - 2C_{yx}\right]. \tag{55}$$

b. Bivariate Dual to Product Estimator

When $\alpha = \beta = 0$, 't' reduces to usual dual to product estimator with two auxiliary variables proposed by Perri [2004]

$$t_2^* = \overline{y}\frac{\overline{X}}{\overline{x}^*}\frac{\overline{Z}}{\overline{z}^*} \tag{56}$$

Its bias and MSE are given as

$$B(t_2^*) = \overline{Y}\theta g \left[g\left(C_{xx} + C_{zz} + C_{xz}\right) + C_{yx} + C_{yz}\right] \tag{57}$$

and

$$MSE(t_2^*) = \overline{Y}^2\theta \left[C_{yy} + g^2\left(C_{xx} + C_{zz} + 2C_{xz}\right) + 2\left(C_{yz} + C_{yx}\right)\right]. \tag{58}$$

c. Ratio cum Dual to Product Estimator

When $\alpha = 1, \beta = 0$, the class of estimators 't' reduces to the ratio cum dual to product estimator with two auxiliary variables as

$$t_3^* = \overline{y}\frac{\overline{X}}{\overline{x}}\frac{\overline{Z}}{\overline{z}^*}. \tag{59}$$

The bias and mean square error of this estimators are

$$\begin{aligned} B(t_3^*) &= \overline{Y}\theta \left[C_{xx} + g^2\left(C_{zz} + C_{xz}\right) + g\left(C_{yz} - C_{xz}\right)\right. \\ &\quad \left. - \left(C_{yx} + C_{xz}\right)\right], \qquad (60) \\ MSE(t_3^*) &= \overline{Y}^2\theta \left[C_{yy} + C_{xx} + g^2 C_{zz} + 2gC_{yz} - 2gC_{xz} - 2gC_{yx}\right] \qquad (61) \end{aligned}$$

respectively. The case when $\alpha = 0, \beta = 1$ is similar, so it is omitted.

5 Comparison with Different Estimators

We compare the mean square error of the proposed class of estimators 't' from (48) with the different competing estimators proposed by different authors. Again, minimum variance bound (MVB) estimators is a focus among the estimators of any class, so we also compare the MVB of the proposed class of estimators 't' with other estimators as well as the MVB of other competing classes.

a. With Mean per Unit Estimator

The variance of mean per unit estimator $t_0 = \overline{y}$ is

$$V(t_0) = \overline{Y}^2 \theta C_{yy} \tag{62}$$

From (51) and (62), the proposed class of estimators t is preferred to mean per unit estimator t_0 if

$$\rho^2_{y.xz} > 0 \tag{63}$$

which is always true.

b. With Olkin (1958) Estimator

From (51) and (5), the proposed class of estimators t is preferred to Olkin (1958) estimator t_1 if

$$\frac{(C_{yz} - C_{zz} - C_{yx} + C_{xz})^2}{(C_{xx} + C_{zz} - 2C_{xz})} - C_{zz} + 2C_{yz} - \rho^2_{y.xz} > 0 \tag{64}$$

c. With Abu-Dayyeh et al. (2003) Estimator

From (51) and (22), the MVB of the proposed class of estimators t and Abu-Dayyeh (2003) Estimator t_7 are equal. so, $MSE(t) = MSE(t_7)$

d. With Singh et al. (2011) Estimator

From (51) and (33), the MVB of the proposed class of estimators t is equal to MVB of Singh (2011) Estimator t_{10}. So, $MSE(t) = MSE(t_{10})$.

e. With Choudhury and Singh (2012) Estimator

From (51) and (42), the proposed class of estimators t is preferred to Choudhury and Singh (2012) Estimator t_{12}, if

$$\rho^2_{y.xz} - \rho^2_{yx} > 0. \tag{65}$$

6 Empirical Study

In order to study the performance of the proposed class of estimators along with several competing estimators/class of estimators, we list some classes of estimators and their members in *Table.* (1). It is clear that the proposed class of estimators includes a number of popular estimator. Since, we are comparing the precision of various classes of estimators, it is desirable to consider the MVB estimators and their variance/MSE to arrive at a conclusion. So, we consider the comparison of the proposed class t with Olkin (1958) class of estimators t_1, Abu-Dayyeh et al. (2003) class t_7, Singh et al. (2011) class t_{10} and Choudhury and Singh (2012) class of estimators t_{12}, since the other estimator are particular members of these classes.

To examine the behaviour of proposed estimators t we have considered fourteen natural populations which are available in different text books. We consider different characteristics for the comparison between these estimators:

Table 1: Some Competing Classes of Estimators and Some popular estimators of these Classes

Sl. No.	Classes of Estimators	Some Popular Members
1.	Olkin [1958] Class	$t_{1_x} = \overline{y}\frac{\overline{X}}{\overline{x}}$, Cochran [1940]
	$t_1 = [wt_{1_x} + (1-w)t_{1_z}]$	$t_{1_z} = \overline{y}\frac{\overline{Z}}{\overline{z}}$, Cochran [1940]
2.	Abu-Dayesh(2003) Class	$t_2 = \overline{y}\left(\frac{\overline{z}}{\overline{x}}\right)\left(\frac{\overline{X}}{\overline{Z}}\right)$
	$t_7 = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)^{\alpha_1}\left(\frac{\overline{Z}}{\overline{z}}\right)^{\alpha_2}$	$t_3 = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right)\left(\frac{\overline{Z}}{\overline{z}}\right)$, Singh [1967]
		$t_4 = \overline{y}\left(\frac{\overline{x}}{\overline{X}}\right)\left(\frac{\overline{z}}{\overline{Z}}\right)$
3.	Singh (2011) Class	$t_5 = \overline{y}\frac{\overline{x}^*}{\overline{X}}$, Srivenkataramana [1980]
		$t_6 = \overline{y}\frac{\overline{X}}{\overline{x}^*}$, Bandopadhyay [1980]
	$t_{10} = \overline{y}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{\psi_1}\left(\frac{\overline{Z}}{\overline{z}^*}\right)^{\psi_2}$	$t_8 = \overline{y}\frac{\overline{X}}{\overline{x}^*}\frac{\overline{Z}}{\overline{z}^*}$, Perri [2004]
		$t_9 = \overline{y}\frac{\overline{x}^*}{\overline{X}}\frac{\overline{Z}}{\overline{z}^*}$, Singh et al. [2005]
4.	Choudhury (2012) Classes	
	$t_{11} = \overline{y}\left[\alpha\frac{\overline{X}}{\overline{x}} + (1-\alpha)\frac{\overline{X}}{\overline{x}^*}\right]$	t_{1x} and t_6
	$t_{12} = \overline{y}\left[\alpha\frac{\overline{X}}{\overline{x}^*} + (1-\alpha)\frac{\overline{x}^*}{\overline{X}}\right]$	t_5 and t_6
5.	Proposed Class	
	$t = \overline{y}\left[\alpha\left(\frac{\overline{X}}{\overline{x}}\right) + (1-\alpha)\left(\frac{\overline{X}}{\overline{x}^*}\right)\right]$	t_{1x}, t_{1z}, t_2, t_3,
	$\times\left[\beta\left(\frac{\overline{Z}}{\overline{z}}\right) + (1-\beta)\left(\frac{\overline{Z}}{\overline{z}^*}\right)\right]$	t_4, t_6, t_8, t_{11}

I. Percent Relative Bias of an estimator T:

$$PRB(T) = \frac{\left|E(T) - \overline{Y}\right|}{\overline{Y}} \times 100. \tag{66}$$

II. PRE's of an estimator T with respect to the simple mean estimator t_0 is given by

$$PRE(T) = \frac{V(t_0)}{MSE(T)} \times 100. \tag{67}$$

Table. (2) gives the sources and description of the variables (y, x, z) and *Table.* (3) describes some selected population parameters. *Table.* (4) gives the values of the constants (α and β) used in the proposed class of estimators t for which it leads to an MVB estimator and *Table.* (5) gives the bias and *Table.* (6) gives percent relative efficiency (PRE's) of different classes of estimators (MVB case) with respect to simple mean estimator t_0.

From *Table.* (4) we have seen that the bias of proposed estimator is minimum as compared to the other competitive estimators. From *Table.* (5) we have seen that the mean squared error of the proposed estimator is minimum for all the population as compared to the other existing estimators proposed by different authors. From *Table.* (6), the proposed estimator gains maximum percent relative efficiency (PRE) as compared to the other competitive estimators. So, it is found that the proposed estimator is more efficient than the existing estimators.

Table 2: Sources and descriptions

Pop. No.	Source	y	x	z
1	Gujarati [1945], p.203	Actual Inflation Rate	Unemployment rate	Unexpected inflation rate
2	Gujarati [1945], p.216	Real Gross Product (Millions of NT$)	Labour days (Millions of days)	Real capital input(Millions of NT$)
3	Gujarati [1945], p.224	Real Gross Product (millions of NT$)	Labour input(per thousand persons)	Real Capital input (Millions of NT$)
4	Gujarati [1945], p.227	Defense budget outlay for years t $/billions	GNP in different years, $/billions	U.S. military sales/assistance
5	Gujarati [1945], p.227	Defense budget outlay for years t $/billions	GNP in different years, $/billions	Average industry sales
6	Gujarati [1945], p.228	Per capita consumption of Chicken, Ibs	Real disposable income per capita ,$	Real retail price of chicken per Ib
7	Gujarati [1945], p.228	Per capita consumption of Chicken, Ibs	Real disposable income per capita ,$	Real retail price of pork per Ib,
8	Gujarati [1945], p.228	Per capita consumption of Chicken, Ibs	Real disposable income per capita ,$	Real retail price of beef per Ib
9	Gujarati [1945], p.228	Per capita consumption of Chicken, Ibs	Real disposable income per capita ,$	Composite real price of chicken substitutes per Ib,
10	Swain [2003], p.286	Mean yield of rice per plant	Number of tillers	Percentage of sterility
11	Murthy [1967], p.399	Area under wheat in 1964 (in acres)	Area under wheat in 1963(in acres)	Cultivated area in 1961(in acres)
12	Murthy [1967], p.228	Output of the Factory	The number of workers	Fixed capital
13	Singh [2003], p.1115	Season average price per pound during 1996	Season average price per pound during 1995	Season average price per pound during 1994
14	Cochran [1977], p.182	Number of ‘placebo’ children	Number of paralytic polio cases in the ‘placebo’ group	Number of paralytic polio cases in the ‘not inoculated’ group

Table 3: Some Selected Characteristics of the Populations

P. No.	N	n	g	$\overline{Y}$	$\overline{X}$	$\overline{Z}$	C_{yy}	C_{xx}	C_{zz}
1	13	4	0.300	7.757	6.654	6.686	0.167	0.051	0.152
2	15	3	0.250	24735.333	287.347	25506.633	0.042	0.003	0.089
3	15	5	0.500	24292.527	578.613	159919.333	0.349	0.191	0.062
4	20	7	0.538	83.860	1358.155	6.287	0.126	0.299	1.051
5	20	6	0.429	83.860	1358.155	29.145	0.126	0.299	0.243
6	23	5	0.278	39.670	1035.065	47.996	0.036	0.373	0.056
7	23	6	0.353	39.670	1035.065	90.400	0.036	0.373	0.159
8	23	7	0.438	39.670	1035.065	124.448	0.036	0.373	0.179
9	23	5	0.278	39.670	1035.065	107.857	0.036	0.373	0.142
10	50	8	0.190	12.830	9.040	18.762	0.164	0.072	0.010
11	34	10	0.417	199.441	747.588	208.882	0.584	0.363	0.535
12	80	18	0.290	5182.637	285.125	1126.463	0.127	0.911	0.571
13	36	12	0.500	0.203	0.186	0.171	0.161	0.169	0.142
14	34	8	0.308	4.924	2.588	2.912	1.079	1.567	1.358

Table 4: Optimum Values of $\alpha^{(o)}$ and $\beta^{(o)}$

Pop No	1	2	3	4	5	6	7	8	9	10	11	12	13	14
α_{opt}	-0.688	1.575	0.372	0.853	0.408	0.521	0.464	0.498	0.402	1.070	0.362	0.031	0.687	0.585
β_{opt}	1.205	0.533	1.843	0.290	0.680	0.015	0.285	0.322	0.293	-0.754	0.967	0.811	0.603	0.417

Table 5: Relative Bias of the MVB Estimator of Different Classes of Estimators

P. No.	t_1	t_7	t_{10}	t_{12}	t
1	0.012	0.304	0.069	0.006	0.209
2	544.961	236.654	98.426	14.855	25.902
3	361.020	1433.869	455.129	438.853	983.780
4	8.327	0.831	1.138	0.788	1.788
5	0.710	0.912	0.415	0.799	0.072
6	1.631	0.098	0.239	0.190	0.621
7	0.461	0.025	0.163	0.187	0.362
8	0.414	0.021	0.149	0.182	0.405
9	0.528	0.014	0.165	0.190	0.273
10	0.016	0.146	0.075	0.020	0.034
11	0.139	8.036	2.564	2.764	3.852
12	66.896	15.375	8.661	20.555	49.925
13	0.000	0.001	0.001	0.001	0.000
14	0.488	0.193	0.174	0.163	0.095

Table 6: PRE of MVB Estimators of Different Classes

P. No.	t_1	t_7	t_{10}	t_{12}	t
1	230.661	810.304	810.304	101.120	810.304
2	80.193	1105.691	1105.691	111.603	1105.691
3	152.027	10012.618	10012.618	254.717	10012.618
4	14.979	1259.079	1259.079	944.348	1259.079
5	482.519	3015.545	3015.545	590.349	3015.545
6	202.332	1121.340	1121.340	945.844	1121.340
7	54.223	975.961	975.961	743.783	975.961
8	47.479	972.941	972.941	352.447	972.941
9	58.663	984.219	984.219	945.844	984.219
10	182.135	236.695	236.695	119.848	236.695
11	2499.545	2601.426	2601.426	211.289	2601.426
12	62.215	965.153	965.153	558.856	965.153
13	339.263	507.496	507.496	283.790	507.496
14	83.907	235.108	235.108	179.206	235.108

Remarks:–

I. The percent relative bias of the MVB estimator of the proposed class of estimators is very less indicates that this MVB estimator can be treated as almost unbiased for estimating population mean $\overline{Y}$.

II. The bias can be reduced by increasing the sample size.

III. The percent relative efficiency (PRE) of the MVB estimator of the proposed class is maximum for all populations.

IV. The MVB of three classes namely proposed class t, Abu-Dayyeh(2003) Class t_7 and Singh (2011) class t_{10} are equal but the MVB estimators of these classes are different.

IV. The PRE of the MVB estimators of the proposed class of estimators t is same as that of the class of estimators proposed by Singh et al. [2011].

V. But, in comparing the biases of the two MVB estimators of proposed class t and class of estimators proposed by Singh et al. [2011], we could not conclude in favor of any particular class as the minimum bias varies from one population to another.

VI. The proposed class of estimators can be easily extended to multi-auxiliary information case as

$$t = \overline{y} \prod_{i=1}^{p} \left[\alpha_i \left(\frac{\overline{X}_i}{\overline{x}_i} \right) + (1-\alpha_i) \left(\frac{\overline{X}_i}{\overline{x}_i^*} \right) \right] \left[\beta_i \left(\frac{\overline{Z}_i}{\overline{z}_i} \right) + (1-\beta_i) \left(\frac{\overline{Z}_i}{\overline{z}_i^*} \right) \right], \tag{68}$$

where $x_1, x_2, \cdots, x_p$ are p-auxiliary variables and the minimum variance bound for this class of estimators is equal to that of the multivariate linear regression estimator which is equal to

$$MSE\left(t^{o}\right)=\theta\overline{Y}C_{yy}\left(1-\rho_{1.23\cdots p}^{2}\right), \tag{69}$$

where $\rho_{1.23\cdots p}^{2}$ is the square of the multiple regression coefficient of y on $x_1, x_2, \cdots, x_p$.

7 Conclusion

The use of auxiliary information on a single auxiliary variable x, in the form of a mixed estimator by mixing the ratio estimator with the dual to product estimator for estimating $\overline{Y}$. In this paper, we extend and refined the idea to utilize the information on two auxiliary variables (x and z) in order to estimate $\overline{Y}$ and proposed a class of estimators t which includes many popular estimators along with classes of estimators as its members notably the classical ratio estimator t_{1x} of Cochran [1940], class of estimator t_1 of Olkin [1958], t_2, t_3 and t_4 proposed by Singh [1967], t_6 of Bandopadhyay [1980], t_8 of Perri [2004] and the class t_{11} of Choudhury and Singh [2012]. The MVB estimators of various classes were compared on the basis of their percent relative biases and percent gain in precision with respect to mean per unit estimator $\overline{y}$. Both the empirical and numerical studies advocated in favor of the proposed class of estimators.

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