The Influence of Total Catastrophe on
Simple Birth-Death-Immigration Processes

Abstract

Markov Models are very much popular in studying population dynamics subject to catastrophes. So, emphasis is given on a discrete state space models specifically when the population change occurs as a result of births, deaths and immigration in presence of catastrophes. A transient solution of simple birth-death-immigration process under the influence of total catastrophes has been found where the catastrophes occur at a constant rate and the population size reduces to zero if the catastrophes occur. Since, such processes are completely explained in terms of probability generating function, the mean and variance, we have analyzed the process focusing on these aspects. A simulation study was also carried out to gather evidences towards the results.

1   Introduction

Catastrophes effects on the population have been studied by Bartoszynski et. al. (1989), Brockwell et. al. (1982) and Kyriakidis (1994) has considered the influence of catastrophes on the population for the catastrophes rate $\gamma = 1$ as a special case.When $\lambda = 0, \mu = 0$,the transient solution of simple immigration-catastrophe process was obtained by Swift (2000). Swift (2001) has considered a simple birth-death-immigration process under the influence of total catastrophes using the forward Kolmogorov equations. Stirzaker (2006) highlighted on the processes affected by catastrophes. Stirzaker (2007) considered a class of stochastic models for systems subject to random regulation and derived expressions for the distribution of the intervals between regulating instants and for the transient and equilibrium properties of the process. Kyriakidis (2004) considered a simple immigration birth-death process with total catastrophes and we obtain the transient probabilities basing on a renewal argument. Pradhan et. al. (2021) made generalization of the simplest time-dependent discrete Markov process as a linear growth process with immigration-emigration. Dash et. al. (2018) highlighted the process in presence of a single birth or twin births. Swift (2001) found $P_n'(t)$ for $n \geq 1$ and also $P_n'(t)$ for $n = 0$. But the expression of $P_n'(t)$ for for $n = 0$ is not correct one. He also considered the initial condition $P_0(0) = 1$ for simplicity, which may not always hold because if the population size is zero at instant $t = 0$, then it is completely immigration process. Again, Swift (2001) could not obtained an explicit for of the probabilities $P_n(t)$ and also has not given any idea about the mean and variances of the process. In this paper, we considered simple birth-death-immigration process under

the influence of total catastrophes if the process stats with n_0 individuals at instant $t = 0$, which is more generalized assumption and found a transient solution of this process. Also the mean and variance of this process have been calculated. Here, we considered birth rate $= \lambda$, death rate$= \mu$, immigration rate$= \nu$ and catastrophe rate $= \gamma$.

2   The Transient Solution $(\lambda \neq \mu)$.

Let $P_n(t)$ be the probability that the process starts with 'n' individuals at instant 't' i.e.,

$$P_n(t) = Pr\left[N(t) = n\right],$$

where $\{N(t)\}$ represents the size of the population at instant time 't'.

Considered two contiguous intervals $(0, t)$ and $(t, t + h)$. If exactly $n \; (\geq 1)$ individuals present in the interval $(0, t + h)$, then

$$
\begin{aligned}
P_n(t + h) &= P_n(t) - n\lambda h P_n(t) - n\mu h P_n(t) - \nu h P_n(t) - \gamma h P_n(t) + (n-1)\lambda h P_{n-1}(t) \\
&+ \nu h P_{n-1}(t) + (n+1)\mu h P_{n+1}(t) + o(h), \quad n \geq 1
\end{aligned}
\tag{2.1}
$$

Hence,

$$
\begin{aligned}
P_n'(t) &= -n\lambda P_n(t) - n\mu P_n(t) - \nu P_n(t) - \gamma P_n(t) + (n-1)\lambda P_{n-1}(t) \\
&+ \nu P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \qquad \text{as } \lim_{h \to 0} \frac{o(h)}{h} \geq 0
\end{aligned}
\tag{2.2}
$$

Similarly, we can found for $n = 0$,

$$
\begin{aligned}
P_0'(t) &= -\nu P_0(t) - \gamma P_0(t) + \mu P_1(t) + \gamma \sum_{i=1}^{\infty} P_i(t) \\
&= -\nu P_0(t) - \gamma P_0(t) + \mu P_1(t) + \gamma[1 - P_0(t)] \\
&= -\mu P_1(t) - \nu P_0(t) + \gamma
\end{aligned}
\tag{2.3}
$$

Note: Catastrophes have no effect to the population if there is no population at instant time 't'. Hence, we get $\gamma P_0(t) = 0$.

Let $\pi(z, t)$ be the probability generating function of $P_n(t)$ i.e,

$$
\pi(z, t) = \sum_{n=0}^{\infty} P_n(t) z^n.
\tag{2.4}
$$

Hence,

$$
\begin{aligned}
\frac{\partial \pi(z, t)}{\partial t} &= \sum_{n=0}^{\infty} \frac{dP_n(t)}{dt} z^n \\
&= P_0'(t) + \sum_{n=1}^{\infty} \frac{dP_n(t)}{dt} z^n \\
&= P_0'(t) - \lambda z \frac{\partial \pi(z, t)}{\partial z} - \mu z \frac{\partial \pi(z, t)}{\partial z} - \nu[\pi(z, t) - P_0(t)] \\
&- \gamma[\pi(z, t) - P_0(t)] + \lambda z^2 \frac{\partial \pi(z, t)}{\partial z} + \nu z \pi(z, t) \\
&+ \mu\left[\frac{\partial \pi(z, t)}{\partial z} - P_1(t)\right].
\end{aligned}
\tag{2.5}
$$

Substituting the value of $P_0'(t)$ given by (2.3) in (2.5), we find, after simplification,

$$\frac{\partial \pi(z,t)}{\partial t} = (\lambda z - \mu)(z-1)\frac{\partial \pi(z,t)}{\partial z} + [\nu(z-1) - \gamma]\pi(z,t) + \gamma \text{ for } n \geq 0, \ (\lambda \neq \mu). \tag{2.6}$$

Hence,

$$\frac{-(\lambda z - \mu)(z-1)}{\{\nu(z-1) - \gamma\}\pi(z,t) + \gamma}\frac{\partial \pi(z,t)}{\partial z} + \frac{1}{\{\nu(z-1) - \gamma\}\pi(z,t) + \gamma}\frac{\partial \pi(z,t)}{\partial t} = 1 \tag{2.7}$$

To solve the partial differential equation (2.7), we first form the subsidiary equation

$$\frac{d\pi(z,t)}{1} = \frac{dz}{\dfrac{-(\lambda z - \mu)(z-1)}{\{\nu(z-1) - \gamma\}\pi(z,t) + \gamma}} = \frac{dt}{\dfrac{1}{\{\nu(z-1) - \gamma\}\pi(z,t) + \gamma}}. \tag{2.8}$$

From the first and second parts of (2.8), we find

$$[\{\nu(z-1) - \gamma\}\pi(z,t) + \gamma](\lambda z - \mu)^{\frac{\nu}{\lambda}}\left(\frac{\lambda z - \mu}{z-1}\right)^{\frac{\gamma}{\lambda-\mu}} = \text{a Constant.} \tag{2.9}$$

From second and third parts of (2.8), we find

$$\left(\frac{\lambda z - \mu}{z-1}\right)e^{-(\lambda-\mu)t} = \text{a Constant.} \tag{2.10}$$

Hence,

$$[\{\nu(z-1) - \gamma\}\pi(z,t) + \gamma](\lambda z - \mu)^{\frac{\nu}{\lambda}}\left(\frac{\lambda z - \mu}{z-1}\right)^{\frac{\gamma}{\lambda-\mu}} = \psi_1\left[\left(\frac{\lambda z - \mu}{z-1}\right)e^{-(\lambda-\mu)t}\right], \tag{2.11}$$

where $\psi_1\left[\left(\frac{\lambda z - \mu}{z-1}\right)e^{-(\lambda-\mu)t}\right]$ is an arbitrary function to be determined by initial conditions.

If the process starts with n_0 individuals at time $t=0$, we have $P_{n_0}(0) = 1$ and $P_k(0) = 0$ for $k \neq n_0$, and hence $\pi(z,0) = z^{n_0}$.

Letting $t=0$ in (2.11), we get

$$[\{\nu(z-1) - \gamma\}z^{n_0} + \gamma](\lambda z - \mu)^{\frac{\nu}{\lambda}}\left(\frac{\lambda z - \mu}{z-1}\right)^{\frac{\gamma}{\lambda-\mu}} = \psi_1\left(\frac{\lambda z - \mu}{z-1}\right). \tag{2.12}$$

$$\text{Let } u = \frac{\lambda z - \mu}{z-1} \text{ and hence } z = \left(\frac{u-\mu}{u-\lambda}\right). \tag{2.13}$$

$$\begin{aligned}
\therefore \psi(u) &= \left[\frac{\lambda(u-\mu)}{(u-\lambda)} - \mu\right]^{\frac{\nu}{\lambda}}\left[\frac{\frac{\lambda(u-\mu)}{u-\lambda} - \mu}{\frac{u-\mu}{u-\lambda} - 1}\right]^{\frac{\gamma}{\lambda-\mu}}\left[\left\{\nu\left(\frac{u-\mu}{u-\lambda} - 1\right) - \gamma\right\}\left(\frac{u-\mu}{u-\lambda}\right)^{n_0} + \gamma\right] \\
&= \left[\frac{u(\lambda-\mu)}{u-\lambda}\right]^{\frac{\nu}{\lambda}}u^{\frac{\gamma}{\lambda-\mu}}\left[\left\{\frac{\nu(u-\mu)}{u-\lambda} - \gamma\right\}\left(\frac{u-\mu}{u-\lambda}\right)^{n_0} + \gamma\right]
\end{aligned} \tag{2.14}$$

From (2.11),(2.12),(2013) and (2.14), after simplification, we find

$$\begin{aligned}
\pi(z,t) &= \frac{1}{\nu(z-1) - \gamma}\left[\left\{\frac{\left(\frac{\lambda z - \mu}{z-1} - \lambda\right)e^{-(\lambda-\mu)t}}{\frac{\lambda z - \mu}{z-1}e^{-(\lambda-\mu)t} - \lambda}\right\}^{\frac{\nu}{\lambda}}e^{-\gamma t} \right. \\
&\quad \times \left.\left[\left\{\frac{\nu(\lambda-\mu)}{\frac{\lambda z - \mu}{z-1}e^{-(\lambda-\mu)t} - \lambda} - \gamma\right\}\left\{\frac{\frac{\lambda z - \mu}{z-1}e^{-(\lambda-\mu)t} - \mu}{\frac{\lambda z - \mu}{z-1}e^{-(\lambda-\mu)t} - \lambda}\right\}^{n_0} + \gamma\right] - \gamma\right] \text{ for } \lambda \neq \mu.
\end{aligned} \tag{2.15}$$

Corollary 2.1 (Transient Solution for $(\lambda = \mu)$). *When $\lambda = \mu$, we have*

$$\begin{aligned}
\pi(z,t) &= \frac{1}{\nu(z-1) - \gamma}\left[\{(1+\lambda t) - \lambda t z\}^{\frac{\nu}{\lambda}}e^{-\gamma t}\right. \\
&\quad \times \left.\left\{\left(\frac{\nu(z-1)}{(1+\lambda t) - \lambda t z} - \gamma\right)\left(\frac{z(1-\lambda t) + \lambda t}{(1+\lambda t) - \lambda t z}\right)^{n_0} + \gamma\right\} - \gamma\right]
\end{aligned} \tag{2.16}$$

3   Determination of Mean and Variance.

Case-I: For $(\lambda \neq \mu)$

For simple birth-death-immigration process under the influence of total catastrophes, we have

$$
\begin{aligned}
P_n'(t) \quad = \quad & -(n\lambda + n\mu + \nu + \gamma)P_n(t) + (n-1)\lambda P_{n-1}(t) \\
& + \nu P_{n-1}(t) + (n+1)\mu P_{n+1}(t) \qquad , n \geq 1
\end{aligned} \tag{3.1}
$$

and

$$
P_0'(t) = \mu P_1(t) - \nu P_0(t) + \gamma \tag{3.2}
$$

Therefore,

$$
\begin{aligned}
M'(t) \quad = \quad & \sum_{n=0}^{\infty} n P_n'(t) \\
= \quad & 0 P_0'(t) + \sum_{n=1}^{\infty} n P_n'(t) \\
= \quad & -\lambda M_2(t) - \mu M_2(t) - \nu M(t) - \gamma M(t) \\
+ \quad & \lambda \left[M_2(t) + M(t) \right] + \nu \left[M(t) + 1 \right] + \mu \left[M_2(t) - M(t) \right] \\
= \quad & \left\{ (\lambda - \mu) - \gamma \right\} M(t) + \nu
\end{aligned} \tag{3.3}
$$

Hence,

$$
\left\{ (\lambda - \mu) - \gamma \right\} M(t) + \nu = C_1 e^{(\lambda - \mu - \gamma)t} \tag{3.4}
$$

Where C_1 is a constant to be determined.

when $t = 0$, M(t)=$\sum_{n=0}^{\infty} n P_n(t)$ implies M(0) $= n_0$, if the process starts with n_0 individuals at time t=0, Letting t=0 in (4.4),we find

$$
C_1 = (\lambda - \mu - \gamma)n_0 + \nu \tag{3.5}
$$

From (4.4) and (4.5), after simplification,we find

$$
M(t) = \frac{\nu}{(\lambda - \mu - \gamma)} \left[e^{(\lambda - \mu - \gamma)t} - 1 \right] + n_0 e^{(\lambda - \mu - \gamma)t} \tag{3.6}
$$

Similarly,We find

$$
\begin{aligned}
M'_{(t)} \quad = \quad & \sum_{n=0}^{\infty} n^2 P_n'(t) \\
= \quad & (2\lambda - 2\mu - \gamma)M_2(t) + \lambda + \mu + 2\nu M(t) + \nu
\end{aligned} \tag{3.7}
$$

The complementary function of (4.7) is

$$
C_2 e^{(2\lambda - 2\mu - \gamma)t} \tag{3.8}
$$

The particular integral (P.I) is given by

$$
\begin{aligned}
P.I \quad = \quad & -\frac{\nu(\lambda + \mu + 2\nu)}{(\lambda - \mu - \gamma)(\lambda - \mu)}e^{(\lambda - \mu - \gamma)t} \\
+ \quad & \frac{\nu(\lambda + \mu + 2\nu)}{(\lambda - \mu - \gamma)(2\lambda - 2\mu - \gamma)} - \frac{n_0(\lambda - \mu - 2\nu)}{(\lambda - \mu)}e^{(\lambda - \mu - \gamma)t} \\
- \quad & \frac{\nu}{(2\lambda - 2\mu - \gamma)}
\end{aligned}
\tag{3.9}
$$

Hence the general solution of (4.7) is given by

$$
\begin{aligned}
M_2(t) \quad = \quad & e^{(2\lambda - 2\mu - \gamma)t} - \frac{\nu(\lambda + \mu + 2\nu)}{(\lambda - \mu)(\lambda - \mu - \gamma)}e^{(\lambda - \mu - \gamma)t} \\
+ \quad & \frac{\nu(\gamma + \mu + 2\nu)}{(\lambda - \mu - \gamma)(2\lambda - 2\mu - \gamma)} - \frac{n_0(\lambda + \mu + 2\nu)}{(\lambda - \mu)}e^{(\lambda - \mu - \gamma)t} \\
- \quad & \frac{\nu}{2\lambda - 2\mu - \gamma}
\end{aligned}
\tag{3.10}
$$

When the process starts with n_0 individuals at time t=0, we have $M_2(0) = n_0{}^2$ Letting $t = 0$ in (4.10),we find

$$
\begin{aligned}
C_2 \quad = \quad & n_0{}^2 + \frac{\nu(\lambda + \mu + 2\nu)}{(\lambda - \mu)(\lambda - \mu - \gamma)} - \frac{\nu(\lambda + \mu + 2\nu)}{(\lambda - \mu - \gamma)(2\lambda - 2\mu - \gamma)} \\
+ \quad & n_0\frac{(\lambda + \mu + 2\nu)}{(\lambda - \mu)} + \frac{\nu}{(2\lambda - 2\mu - \gamma)}
\end{aligned}
\tag{3.11}
$$

Substituting the value of C_2 given by (4.11) in (4.10),we can able to find $M_2(t)$. We know that

$$
V[N(t)] = M_2(t) - [M(t)]^2
\tag{3.12}
$$

Hence,

$$
\begin{aligned}
V[N(t)] \quad = \quad & n_0{}^2 e^{(2\lambda - 2\mu - \gamma)t}\left[1 - e^{-\gamma t}\right] + n_0\frac{(\lambda + \mu + 2\nu)}{(\lambda - \mu)}e^{(\lambda - \mu - \gamma)t}\left[e^{(\lambda - \mu)t} - 1\right] \\
+ & \frac{2n_0\nu}{(\lambda - \mu - \gamma)}e^{(\lambda - \mu - \gamma)t}\left[1 - e^{(\lambda - \mu - \gamma)t}\right] + \frac{\nu(\lambda + \mu + 2\nu)}{(\lambda - \mu)(\lambda - \mu - \gamma)}e^{(\lambda - \mu - \gamma)t}\left[e^{(\lambda - \mu)t} - 1\right] \\
+ & \frac{\nu(\lambda + \mu + 2\nu)}{(\lambda - \mu - \gamma)(2\lambda - 2\mu - \gamma)}\left[1 - e^{(2\lambda - 2\mu - \gamma)t}\right] + \frac{\nu}{(2\lambda - 2\mu - \gamma)}\left[e^{(2\lambda - 2\mu - \gamma)t} - 1\right] \\
+ & \frac{\nu^2}{(\lambda - \mu - \gamma)^2}\left[e^{(\lambda - \mu - \gamma)t}\right]\left[1 - e^{(\lambda - \mu - \gamma)t}\right] + \frac{\nu^2}{(\lambda - \mu - \gamma)^2}\left[e^{(\lambda - \mu - \gamma)t} - 1\right]
\end{aligned}
\tag{3.13}
$$

Case II: $\lambda = \mu$

Here,

$$
M(t) = E[N(t)] = \frac{\nu}{\lambda}(1 - e^{-\gamma t}) + n_0 e^{-\gamma t}
\tag{3.14}
$$

and

$$
\begin{aligned}
V[N(t)] &= n_0{}^2 e^{-\gamma t}\left\{1 - e^{-\gamma t}\right\} + 2\,n_0\,(\lambda + \nu)t\,e^{-\gamma t} \\
&+ \quad 2\,n_0\,\frac{\nu}{\gamma}\,e^{-\gamma t}\,(e^{-\gamma t} - 1) - 2\,\frac{(\lambda + \nu)\nu}{\gamma}\,t\,e^{-\gamma t} \\
&+ \quad 2\,\frac{\nu(\lambda + \nu)}{\gamma^2}\,(1 - e^{-\gamma t}) + \frac{\nu}{\gamma}(1 - e^{-\gamma t}) \\
&+ \quad \frac{\nu^2}{\gamma^2}\,e^{-\gamma t}\,(1 - e^{-\gamma t}) + \frac{\nu^2}{\gamma^2}(e^{-\gamma t} - 1) \\
&= \quad n_0{}^2\,e^{-\gamma t}\left\{1 - e^{-\gamma t}\right\} + 2\,n_0\,(\lambda + \nu)t\,e^{-\gamma t} \\
&+ \quad 2\,n_0\,\frac{\nu}{\lambda}\,e^{-\lambda t}\,(e^{-\gamma t} - 1) - 2\,\frac{(\lambda + \nu)\nu}{\gamma}t\,e^{-\gamma t} + \frac{2\nu\lambda}{\gamma^2}(1 - e^{-\gamma t}) \\
&+ \quad \frac{\nu^2}{\gamma^2}(1 - e^{-2\gamma t}) + \frac{\nu}{\gamma}(1 - e^{-\gamma t})
\end{aligned}
\tag{3.15}
$$

4   A Simulation Study

In this section we performed a simulation study for Birth-Death-Immigration Process under the influence of total Catastrophes. Here we also compared this process with Birth-Death-Immigration Process.

Case-I: $(\lambda \neq \mu)$

From (2.15), the p.g.f. of Birth-Death-Immigration under the influence of total catastrophes is given by

$$
\begin{aligned}
\pi(z,t) &= \quad \frac{1}{\nu(z-1) - \gamma}\left[\left\{\frac{\left(\frac{\lambda z - \mu}{z - 1} - \lambda\right)e^{-(\lambda - \mu)t}}{\frac{\lambda z - \mu}{z - 1}e^{-(\lambda - \mu)t} - \lambda}\right\}^{\frac{\nu}{\lambda}} e^{-\gamma t} \right. \\
&\times \quad \left[\left\{\frac{\nu(\lambda - \mu)}{\frac{\lambda z - \mu}{z - 1}e^{-(\lambda - \mu)t} - \lambda} - \gamma\right\}\left\{\frac{\frac{\lambda z - \mu}{z - 1}e^{-(\lambda - \mu)t} - \mu}{\frac{\lambda z - \mu}{z - 1}e^{-(\lambda - \mu)t} - \lambda}\right\}^{n_0} + \gamma\right] - \gamma\right], \text{ for } \lambda \neq \mu.
\end{aligned}
$$

The p.g.f. of Birth-Death-Immigration Process is given by

$$
\pi_1(z,t) = \left[\frac{\left(\frac{\lambda z - \mu}{z - 1} - \lambda\right)e^{-(\lambda - \mu)t}}{\frac{\lambda z - \mu}{z - 1}e^{-(\lambda - \mu)t} - \lambda}\right]^{\frac{\nu}{\lambda}} \times \left[\frac{\frac{\lambda z - \mu}{z - 1}e^{-(\lambda - \mu)t} - \mu}{\frac{\lambda z - \mu}{z - 1}e^{-(\lambda - \mu)t} - \lambda}\right]^{n_0}.
$$

From Figure-1 and Figure-2, for $(\lambda < \mu)$ and $(\lambda > \mu)$ the p.g.f. of Birth-Death-Immigration Process under the influence of total Catastrophes is greater than the p.g.f. of Birth-Death-Immigration Process. i.e.,

$$
\pi(z,t) \geq \pi_1(z,t)
$$

.

The mean of Birth-Death-Immigration Process under the influence of total Catastrophes is given by

$$
M(t) = E[N(t)] = \frac{\nu}{(\lambda - \mu - \gamma)}\left[e^{(\lambda - \mu - \gamma)t} - 1\right] + n_0 e^{(\lambda - \mu - \gamma)t}.
$$

The mean of Birth-Death-Immigration Process is given by

$$
M_1(t) = E[N(t)] = \frac{\nu}{\lambda - \mu}\left(e^{(\lambda - \mu)t} - 1\right) + n_0 e^{(\lambda - \mu)t}.
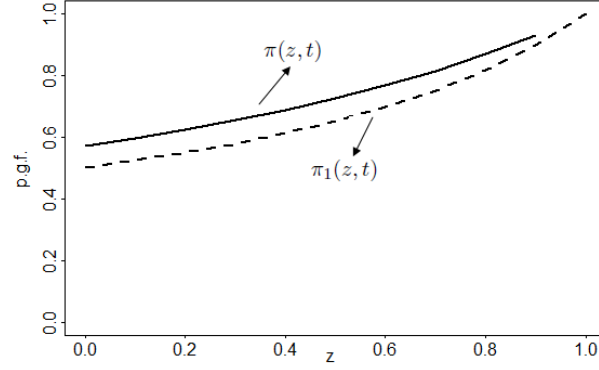$$

Figure 1: P.G.F. of Birth-Death-Immigration Process under the influence of total Catastrophes ($\lambda < \mu$).
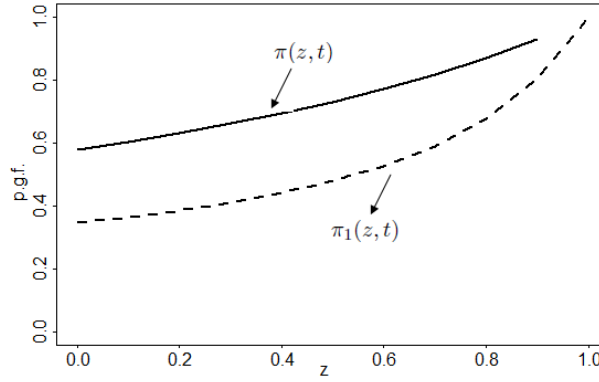

Figure 2: P.G.F. of Birth-Death-Immigration Process under the influence of total Catastrophes ($\lambda > \mu$).

From Figure-3, for $\lambda > \mu$, the expectation of Birth-Death-Immigration Process is greater than the expectation of Birth-Death-Immigration Process under the influence of total catastrophes. i.e.,

$$M(t) = \frac{\nu}{(\lambda - \mu - \gamma)} \left[e^{(\lambda - \mu - \gamma)t} - 1 \right] + n_0 e^{(\lambda - \mu - \gamma)t} < \frac{\nu}{\lambda - \mu} \left(e^{(\lambda - \mu)t} - 1 \right) + n_0 e^{(\lambda - \mu)t} = M_1(t).$$

As $\lambda > \mu$, in case of Birth-Death-Immigration Process the expected number of population increases due to birth and also arrival of new immigrants. But under the influence of catastrophes the expected number of population of the process decreases and the process will extinct after certain transition.

From Figure-4, for $\lambda < \mu$, the expectation of Birth-Death-Immigration Process is larger than the expectation of Birth-Death-Immigration Process under the influence of total catastrophes. i.e.,

$$M(t) = \frac{\nu}{(\lambda - \mu - \gamma)} \left[e^{(\lambda - \mu - \gamma)t} - 1 \right] + n_0 e^{(\lambda - \mu - \gamma)t} < \frac{\nu}{\lambda - \mu} \left(e^{(\lambda - \mu)t} - 1 \right) + n_0 e^{(\lambda - \mu)t} = M_1(t)$$

. Since $\lambda < \mu$, the expected number of population in Birth-Death-Immigration Process decreases. Under the influence of catastrophes, the expected number of population of this process also decreases. Here in both the cases process will extinct after certain transition.
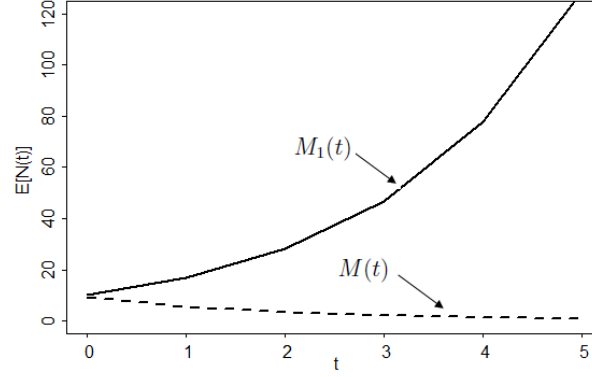
Figure 3: Mean of Birth-Death-Immigration Process under the influence of total Catastrophes ($\lambda > \mu$).
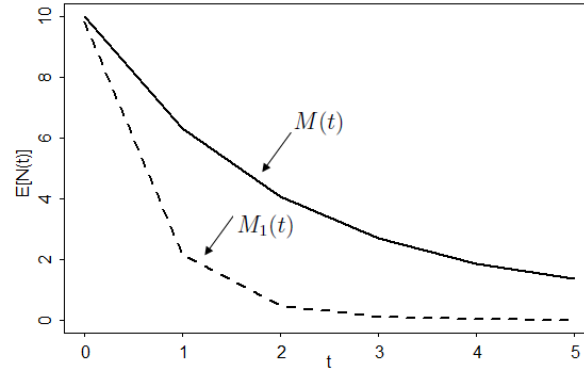

Figure 4: Mean of Birth-Death-Immigration Process under the influence of total Catastrophes ($\lambda < \mu$).

Case-II: ($\lambda = \mu$) From (2.16), the p.g.f. of Birth-Death-Immigration under the influence of total catastrophes is given by

$$\pi(z,t) = \left[\frac{\left\{ \frac{\nu(z-1)}{(1-(z-1)\lambda t)} - \gamma \right\} \left[\frac{(z-1)+\{1+(z-1)\lambda t\}}{\{1+(z-1)\lambda t\}} + \gamma \right] e^{-\lambda t}}{\{\nu(z-1) - \gamma\}} - \gamma \right] \quad \text{for } (\lambda = \mu)$$

. The p.g.f. of Birth-Death-Immigration Process is given by

$$\pi_1(z,t) = \left[\frac{1}{1 - \lambda t(z-1)} \right]^{\frac{\nu}{\lambda}} \times \left[\frac{1 - (\lambda t - 1)(z-1)}{1 - \lambda t(z-1)} \right]^{n_0}$$

.

From Figure-5, for ($\lambda = \mu$) and the p.g.f. of Birth-Death-Immigration Process under the influence of total Catastrophes is greater than the p.g.f. of Birth-Death-Immigration Process. i.e.,

$$\pi(z,t) \geq \pi_1(z,t)$$

.

The mean of Birth-Death-Immigration Process is given by
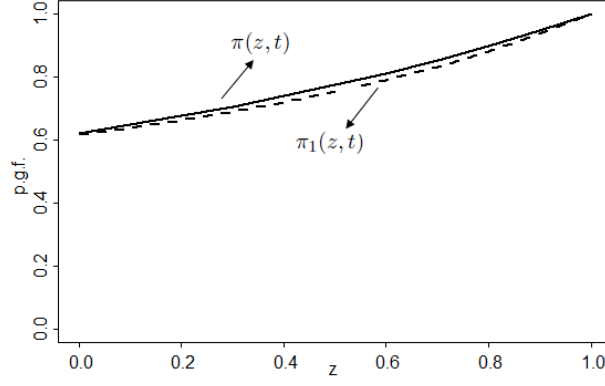
$$M_1(t) = E[N(t)] = \nu t + n_0.$$

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Figure 5: P.G.F. of Birth-Death-Immigration Process under the influence of total Catastrophes ($\lambda = \mu$).

The mean of Birth-Death-Immigration Process under the influence of total catastrophes is given by

$$M(t) = E[N(t)] = \frac{\nu}{\lambda}(1 - e^{-\gamma t}) + n_0 e^{-\gamma t}.$$
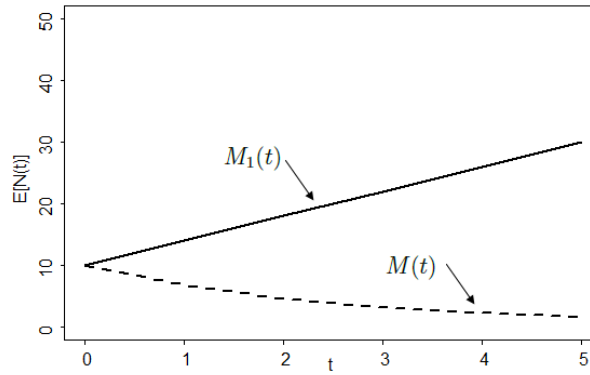

Figure 6: Mean of Birth-Death-Immigration Process under the influence of total Catastrophes ($\lambda = \mu$).

From Figure-6, the mean of Birth-Death-Immigration Process is larger than the mean of Birth-Death-Immigration Process under the influence of total catastrophes. i.e.,

$$M_1(t) = \nu t + n_0 > \frac{\nu}{\lambda}(1 - e^{-\gamma t}) + n_0 e^{-\gamma t} = M(t).$$

The expected number of population in case of Birth-Death-Immigration Process increases indefinitely due to birth and arrival of additional immigrants. But under the influence of catastrophes the expected number of population decreases. Without influence of catastrophe the process will not extinct i.e., there is a chance of population explosion. But when birth, death, immigration and catastrophes occurs simultaneously the process will extinct after certain transition i.e., the chance of extinction is nearly unity.

5   Conclusion

The process discussed in this paper is the influence of catastrophe on a simple birth-death-immigration Process, which is a generalization to several population processes involving vital processes. The probability generating function of this process is also derived. Several characteristics of this process is also studied through mean, variance and also from the probability generating function. Several other processes are also derived as particular cases with their probability generating functions. The behaviour of different processes are also analyzed and compared using a simulation study.

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