Exact analytical solution of Ivancevic options pricing model (IOPM) or Schrödinger's equation via ADM and SBA methods

Abstract

This paper is devoted to the study of the general equation of the Ivancevic option pricing model (IOPM) or Schrödinger's equation and to determine its analytical solution via the methods of numerical analysis ADM and SBA. The Ivancevic option pricing model is an adaptive wave model that is a nonlinear wave alternative to the standard Black-Scholes option pricing model, it is also a model that links quantum mechanics and financial mathematics.

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1 Introduction

The classical Black-Scholes model (BSM) is an important financial model for option pricing an valuation. In this paper, we are interested in the determination of the analytical solution of the general equation of the Ivancevic [7] or Schrödinger [12] model in quantum mechanics. It is about the equations:

$$(E):\left\{\begin{array}{l} i\dfrac{\partial w\left(t,x\right)}{\partial t}+\varepsilon\dfrac{\partial^{2}w\left(t,x\right)}{\partial x^{2}}+q\left|w\left(t,x\right)\right|^{2p}w\left(t,x\right)=0;p\in\mathbb{N}^{*}\\ w(0,x)=\beta e^{iax}\end{array}\right.$$

and

$$(F):\left\{\begin{array}{l} i\dfrac{\partial w\left(t,x\right)}{\partial t}+\varepsilon\dfrac{\partial^{2}w\left(t,x\right)}{\partial x^{2}}+\mu w\left(t,x\right)+q\left|w\left(t,x\right)\right|^{2p}w\left(t,x\right)=0;p\in\mathbb{N}^{*}\\ w(0,x)=\beta e^{iax}\end{array}\right.$$

where $\varepsilon>0,\mu>0$ and $q>0$.

2 Description of numerical Method ADM and SBA

2.1 Numerical method ADM

Consider the functional equation below :

$$Fw = f \tag{1}$$

where F is an operator defined in the Hilbert space H in H, f is a given function in H and w is the unknown function. Let us decompose as follows

$$F = L - R - N \tag{2}$$

Where L is the linear part of inverse L^{-1}, R the linear remainder and N the nonlinear part, (1) becomes :

$$Lw - Rw - Nw = f \tag{3}$$

Applying L^{-1} to (3), we get the Adomian canonical form [2]:

$$w = \theta + L^{-1}f + L^{-1}Rw + L^{-1}Nw \tag{4}$$

where

$$L\theta = 0.$$

Let us determine the solution of (1) in the form of a convergent series[3]

$$w = \sum_{n=0}^{+\infty} w_n$$

and

$$Nw = \sum_{n=0}^{+\infty} A_n < +\infty$$

where the

$$A_n = A_n\left(w_0, w_1, ..., w_n\right)$$

are Adomian polynomials [5]. We get the following Adomian algorithm[4] :

$$\begin{cases} w_0 = \theta + L^{-1}f \\ w_{n+1} = L^{-1}Rw_n + L^{-1}A_n; n \geq 0. \end{cases}$$

2.2 The Adomian polynomials

Definition :The Adomian polynomials are defined by :

$$\begin{cases} A_0 = N(w_0) \\ A_n = \dfrac{1}{n!}\dfrac{d^n}{d\lambda^n}\left[N\left(\displaystyle\sum_{k=0}^{+\infty}\lambda^k w_k\right)\right]_{\lambda=0} & : n \geq 1 \end{cases}$$

Theorem 1 *The Adomian polynomials are calculated using the formula :*

$$\left[\frac{d^n}{d\lambda^n}\sum_{k=0}^{n}\lambda^k A_k\right]_{\lambda=0} = \left[\frac{d^n}{d\lambda^n}N\left(\sum_{k=0}^{n}\lambda^k w_k\right)\right]_{\lambda=0}$$

2.3 Numerical method SBA

Consider the functional equation below :

$$Fw = f \tag{5}$$

where F is an operator defined in the Hilbert space H in H, f is a given function in H and w is the unknown function. Let us decompose as follows

$$F = L - R - N \tag{6}$$

Where L is the linear part of inverse L^{-1}, R the linear remainder and N the nonlinear part, (1) becomes :

$$Lw - Rw - Nw = f \tag{7}$$

Applying L^{-1} to (3), we get the Adomian canonical form :

$$w = \theta + L^{-1}f + L^{-1}Rw + L^{-1}Nw \tag{8}$$

where

$$L\theta = 0.$$

Equation (5) is the Adomian canonical form [1]. Using the successive approximations [7], we get:

$$w^k = \theta + L^{-1}(f) + L^{-1}(R(w^k)) + L^{-1}(N(w^{k-1}));\ k \geq 1 \tag{9}$$

This let's to the following Adomian algorithm :

$$\begin{cases} w_0^k = \theta + L^{-1}(f) + L^{-1}(N(w^{k-1})); k \geq 1 \\ w_n^k = L^{-1}(R(w_{n-1}^k)), n \geq 1 \end{cases} \tag{10}$$

The Picard principle is then applied to equation (7): let w^0 be such that $N(w^0) = 0$, for $k = 1$, we get:

$$\begin{cases} w_0^1 = \theta + L^{-1}(f\ ) + L^{-1}(N(w^0)) \\ w_n^1 = L^{-1}(R(w_{n-1}^1)), n \geq 1 \end{cases} \tag{11}$$

If the series $\left(\sum_{n=0}^{\infty} w_n^1\right)$ converges, then $w^1 = \left(\sum_{n\geq 1}^{\infty} w_n^1\right)$

For $k = 2$, we get:

$$\begin{cases} w_0^2 = \theta + L^{-1}(f\ ) + L^{-1}(N(w^1)) \\ w_n^2 = L^{-1}(R(w_{n-1}^2)), n \geq 1 \end{cases} \tag{12}$$

If the series $\left(\sum_{n=0}^{\infty} w_n^2\right)$ converges, then $w^2 = \left(\sum_{n\geq 0}^{\infty} w_n^2\right)$.

This process is repeated to k.

If the series $\left(\sum_{n=0}^{\infty} w_n^k\right)$ converges, then $w^k = \left(\sum_{n\geq 0}^{\infty} w_n^k\right)$.

Therefore $w = \lim_{k\to+\infty} w^k$ is the solution of the problem, with the following hypothese at the step $k :\ N(w^k) = 0, \forall k \geq 0$.

Theorem 2 *Consider the following Cauchy problem :*

$$(p):\begin{cases} L_t w(t,x)=\varepsilon\Delta w(t,x)+\mu w\left(t,x\right)+Nw\left(t,x\right),(t,x)\in\Omega \\ w\left(0,x\right)=h(x) \end{cases}$$

Associed to the problem (p) *, the SBA allorithm is given as :*

$$(p_{SBA}):\begin{cases} w_0^k\left(t,x\right)=h(x)+L_t^{-1}\left[N(w^{k-1}\left(t,x\right)\right]\ ;k\geq 1 \\ w_{n+1}^k\left(t,x\right)=L_t^{-1}\left[\varepsilon\Delta w_n^k(t,x)+\mu w_n^k\left(t,x\right)\right]\ ;\ n\geq 0 \end{cases}$$

(H_1)*: There is* $w^0\left(t,x\right)$ *at the step* $k=1$*,such as* $Nw^0\left(t,x\right)=0$.

(H_2): At the step $k=1,w^1\left(t,x\right)$ is the solution of :

$$\begin{cases} w_0^1\left(t,x\right)=h(x) \\ w_{n+1}^1\left(t,x\right)=L_t^{-1}\left[\varepsilon\Delta w_n^1(t,x)+\mu w_n^1\left(t,x\right)\right]\ ;\ n\geq 0. \end{cases}$$

(H_3) : At the step $k=2,Nw^1\left(t,x\right)=0$.So the algorithm :

$$(p_{SBA}):\begin{cases} w_0^k\left(t,x\right)=h(x)+L_t^{-1}\left[N(w^{k-1}\left(t,x\right)\right]\ ;k\geq 1 \\ w_{n+1}^k\left(t,x\right)=L_t^{-1}\left[\varepsilon\Delta w_n^k(t,x)+\mu w_n^k\left(t,x\right)\right]\ ;\ n\geq 0 \end{cases}$$

is convergent for $k\geq 2$ and we obtain : $w^1\left(t,x\right)=w^2(t,x)=...=w^k(t,x)$. From which the unique solution of the problem (p) is

$$w(t,x)=\lim_{k\to+\infty}w^k(t,x).$$

Proof. At step $k=1$, we have the following algorithm:

$$(p_1)\begin{cases} w_0^1\left(t,x\right)=h(x) \\ w_{n+1}^1\left(t,x\right)=L_t^{-1}\left[\varepsilon\Delta w_n^1(t,x)+\mu w_n^1\left(t,x\right)\right]\ ;\ n\geq 0 \end{cases}$$

according to hypothesis (H_1) and (H_2) ,the solution of (p_1) is $w^1\left(t,x\right)=\sum_{n=0}^{+\infty}w_n^1\left(t,x\right)$. According to the hypothesis (H_3) ,at step $k=2,Nw^1\left(t,x\right)=0$

we get the following alorthm :

$$(p_2)\begin{cases} w_0^2\left(t,x\right)=h(x) \\ w_{n+1}^2\left(t,x\right)=L_t^{-1}\left[\varepsilon\Delta w_n^2(t,x)+\mu w_n^1\left(t,x\right)\right]\ ;\ n\geq 0 \end{cases}$$

Thus, we obtain the same al

gorithm as in step $k=1$, then $w^2(t,x)=w^1\left(t,x\right)$.Thus, in a recursive way it will be for each step $k\geq 2,w^1\left(t,x\right)=w^2(t,x)=w^3\left(t,x\right)=...$

Then the solution of the problem (p) is $w(t,x)=\lim_{k\to+\infty}w^k(t,x)$.

Suppose that the problem (p) has two distinct solutions $w(t,x)\neq v(t,x)$, and consider their difference $\varphi(t,x)=w(t,x)-v(t,x)$.

For each solution, we have :

$$\begin{cases} w_0^k\left(t,x\right)=h(x)+L_t^{-1}\left[N(w^{k-1}\left(t,x\right)\right]\ ;k\geq 1 \\ w_{n+1}^k\left(t,x\right)=L_t^{-1}\left[\varepsilon\Delta w_n^k(t,x)+\mu w_n^k\left(t,x\right)\right]\ ;\ n\geq 0 \end{cases}$$

and

$$\begin{cases} v_0^k(t,x) = h(x) + L_t^{-1}\left[N(v^{k-1}(t,x)\right] ; k \geq 1 \\ v_{n+1}^k(t,x) = L_t^{-1}\left[\varepsilon\Delta v_n^k(t,x) + \mu v_n^k(t,x)\right] ; n \geq 0 \end{cases}$$

$\forall k \geq 1, N(w^{k-1}(t,x) = 0$ and $N(v^{k-1}(t,x) = 0$,so we obtain :

$$\begin{cases} w_0^k(t,x) = h(x) ; k \geq 1 \\ w_{n+1}^k(t,x) = L_t^{-1}\left[\varepsilon\Delta w_n^k(t,x) + \mu w_n^k(t,x)\right] ; n \geq 0 \end{cases}$$

and

$$\begin{cases} v_0^k(t,x) = h(x) ; k \geq 1 \\ v_{n+1}^k(t,x) = L_t^{-1}\left[\varepsilon\Delta v_n^k(t,x) + \mu v_n^k(t,x)\right] ; n \geq 0 \end{cases}$$

For the difference we get :

$$\begin{cases} \varphi_0^k(t,x) = 0 ; k \geq 1 \\ \varphi_{n+1}^k(t,x) = L_t^{-1}\left[\varepsilon\Delta \varphi_n^k(t,x) + \mu \varphi_n^k(t,x)\right] ; n \geq 0 \end{cases}$$

from which

$$\begin{cases} \varphi_0^k(t,x) = 0 \\ \varphi_1^k(t,x) = 0 \\ \ldots \\ \varphi_n^k(t,x) = 0, \forall n \geq 0 \end{cases}$$

Thus $\varphi_n^k(t,x) = \sum_{n=0}^{+\infty} \varphi_n^k(t,x) = 0$ and $w(t,x) = v(t,x)$ which contradicts our hypothesis.Therefore the problem (p)has a unique solution $w(t,x)$. ■

3 Resolution via numerical method ADM

Consider the following equation

$$(E) : \begin{cases} i\dfrac{\partial w(t,x)}{\partial t} + \varepsilon\dfrac{\partial^2 w(t,x)}{\partial x^2} + q\left|w(t,x)\right|^{2p} w(t,x) = 0 \\ w(0,x) = \beta e^{iax} \end{cases}$$

Let us determine the canonical form of Adomian, the equation

$$i\frac{\partial w(t,x)}{\partial t} + \varepsilon\frac{\partial^2 w(t,x)}{\partial x^2} + q\left|w(t,x)\right|^{2p} w(t,x) = 0; p \in \mathbb{N}^*$$

is equivalent to

$$\frac{\partial w(t,x)}{\partial t} = i\varepsilon\frac{\partial^2 w(t,x)}{\partial x^2} + iq\left|w(t,x)\right|^{2p} w(t,x)$$

from which we obtain the canonical form :

$$w(t,x) = w(0,x) + i\varepsilon\int_0^t \frac{\partial^2 w(z,x)}{\partial x^2} dz + iq\int_0^t \left|w(z,x)\right|^{2p} w(z,x)\, dz.$$

Thus, we obtain the Adomian algorithm :

$$\begin{cases} w_0(t,x) = w(0,x) \\ w_{n+1}(t,x) = i\varepsilon \int\limits_0^t \dfrac{\partial^2 w_n(z,x)}{\partial x^2} dz + iq \int\limits_0^t A_n(z,x)\, dz; n \geq 0 \end{cases}$$

Let's calculate the polynomials: : $A_0, A_1, A_2, ...$

$$\begin{cases} A_0 = |\beta|^{2p} w_0 \\ A_1 = w_1 (w_0 \overline{w_0})^p + p w_0 (w_1 \overline{w_0} + \overline{w_1} w_0)^{p-1} \\ A_2 = 2(a w_0)^p w_2 + 2p (a w_1 + b w_0)^{p-1} (w_1) + p(p-1)(2(a w_2 + b w_1 + c w_0))^{p-2} (w_0) \\ ... \end{cases}$$

Let's calculate the terms: $w_0(t,x), w_1(t,x), ...$
we obtain thus :

$$\begin{cases} w_0(t,x) = \beta e^{iax} \\ w_1(t,x) = \beta it \left(-\varepsilon a^2 + q|\beta|^{2p}\right) e^{iax} \\ w_2(t,x) = \beta \dfrac{\left[it\left(-\varepsilon a^2 + q|\beta|^{2p}\right)\right]^2}{2!} e^{iax} \\ w_3(t,x) = \beta \dfrac{\left[it\left(-\varepsilon a^2 + q|\beta|^{2p}\right)\right]^3}{3!} e^{iax} \\ ... \\ w_n(t,x) = \beta \dfrac{\left[it\left(-\varepsilon a^2 + q|\beta|^{2p}\right)\right]^n}{n!} e^{iax} \end{cases}$$

Therefore, the solution of problem (E) obtained by the ADM method is :

$$w(t,x) = \sum_{n=0}^{+\infty} w_n(t,x) = \beta \exp\left[i\left(\left(-\varepsilon a^2 + q|\beta|^{2p}\right)t + ax\right)\right].$$

Consider the following equation

$$(F) : \begin{cases} i\dfrac{\partial w(t,x)}{\partial t} + \varepsilon \dfrac{\partial^2 w(t,x)}{\partial x^2} + \mu w(t,x) + q|w(t,x)|^{2p} w(t,x) = 0 \\ w(0,x) = \beta e^{iax} \end{cases}$$

Let us determine the canonical form of Adomian, the equation

$$i\frac{\partial w(t,x)}{\partial t} + \varepsilon \frac{\partial^2 w(t,x)}{\partial x^2} + \mu w(t,x) + q|w(t,x)|^{2p} w(t,x) = 0; p \in \mathbb{N}^*$$

is equivalent to

$$\frac{\partial w(t,x)}{\partial t} = i\varepsilon \frac{\partial^2 w(t,x)}{\partial x^2} + i\mu w(t,x) + iq|w(t,x)|^{2p} w(t,x)$$

from which we obtain the canonical form :

$$w(t,x) = w(0,x) + i\varepsilon \int\limits_0^t \frac{\partial^2 w(z,x)}{\partial x^2} dz + i\mu \int\limits_0^t w(z,x)\, dz + iq \int\limits_0^t |w(z,x)|^{2p} w(z,x)\, dz.$$

Thus, we obtain the Adomian algorithm :

$$\begin{cases} w_0(t,x) = w(0,x) \\ w_{n+1}(t,x) = i\varepsilon \int\limits_0^t \dfrac{\partial^2 w_n(z,x)}{\partial x^2} dz + i\mu \int\limits_0^t w_n(z,x)\, dz + iq \int\limits_0^t A_n(z,x)\, dz; n \geq 0 \end{cases}$$

Let's calculate the polynomials : $A_0, A_1, A_2, ...$

$$\begin{cases} A_0 = |\beta|^{2p} w_0 \\ A_1 = w_1 (w_0\overline{w_0})^p + pw_0 (w_1\overline{w_0} + \overline{w_1}w_0)^{p-1} \\ A_2 = 2(\overline{w_0}w_0)^p w_2 + 2p(\overline{w_0}w_1 + \overline{w_1}w_0)^{p-1}(w_1) + p(p-1)(2(\overline{w_0}w_2 + w_1\overline{w_1} + \overline{w_2}w_0))^{p-2}(w_0) \\ ... \end{cases}$$

Let's calculate the terms : $w_0(t,x), w_1(t,x), w_2(t,x), ..$
we thus obtain : .

$$\begin{cases} w_0(t,x) = \beta e^{iax} \\ w_1(t,x) = \beta it \left(\mu - \varepsilon a^2 + q|\beta|^{2p}\right) e^{iax} \\ w_2(t,x) = \beta \dfrac{\left[it\left(\mu - \varepsilon a^2 + q|\beta|^{2p}\right)\right]^2}{2!} e^{iax} \\ w_3(t,x) = \beta \dfrac{\left[it\left(\mu - \varepsilon a^2 + q|\beta|^{2p}\right)\right]^3}{3!} e^{iax} \\ ... \\ w_n(t,x) = \beta \dfrac{\left[it\left(\mu - \varepsilon a^2 + q|\beta|^{2p}\right)\right]^n}{n!} e^{iax} \end{cases}$$

Therefore, the solution of problem (E) obtained by the ADM method is : :

$$w(t,x) = \sum_{n=0}^{+\infty} w_n(t,x) = \beta \exp\left[i\left(\left(\mu - \varepsilon a^2 + q|\beta|^{2p}\right)t + ax\right)\right].$$

4 Resolution via numerical method SBA

Consider the following equation

$$(E): \begin{cases} i\dfrac{\partial w(t,x)}{\partial t} + \varepsilon \dfrac{\partial^2 w(t,x)}{\partial x^2} + q|w(t,x)|^{2p} w(t,x) = 0 \\ w(0,x) = \beta e^{iax} \end{cases}$$

Let us determine the canonical form of Adomian, the equation

$$i\frac{\partial w(t,x)}{\partial t} + \varepsilon \frac{\partial^2 w(t,x)}{\partial x^2} + q|w(t,x)|^{2p} w(t,x) = 0 \tag{13}$$

is equivalent to

$$\frac{\partial w(t,x)}{\partial t} = i\varepsilon \frac{\partial^2 w(t,x)}{\partial x^2} + iq|w(t,x)|^{2p} w(t,x) \tag{14}$$

By putting

$$Nw\left(t,x\right) = iq\left|w\left(t,x\right)\right|^{2p} w\left(t,x\right)$$

from which we obtain the Adomian [1] canonical form :

$$w\left(t,x\right) = w\left(0,x\right) + i\varepsilon\int_0^t \frac{\partial^2 w\left(z,x\right)}{\partial x^2}dz + \int_0^t Nw\left(z,x\right)dz. \tag{15}$$

Applying to (15) the method of successive approximations [6], we obtain :

$$w^k\left(t,x\right) = w^k\left(0,x\right) + i\varepsilon\int_0^t \frac{\partial^2 w^k\left(z,x\right)}{\partial x^2}dz + \int_0^t Nw^{k-1}\left(z,x\right)dz, k \geq 1 \tag{16}$$

We thus obtain the SBA algorithm [8] :

$$\left\{\begin{array}{l} w_0^k\left(t,x\right) = w^k\left(0,x\right) + \displaystyle\int_0^t Nw^{k-1}\left(z,x\right)dz \ , \ k \geq 1 \\ w_{n+1}^k\left(t,x\right) = i\varepsilon\displaystyle\int_0^t \frac{\partial^2 w_n^k\left(z,x\right)}{\partial x^2}dz \ ; n \geq 0 \end{array}\right. \tag{17}$$

Let's apply Picard's principle[9] to (17), at step $k = 1, Nw^0\left(t,x\right) = 0$, si $w^0\left(t,x\right) = 0$, hence

$$\left\{\begin{array}{l} w_0^1\left(t,x\right) = \beta e^{iax} + \displaystyle\int_0^t Nw^0\left(z,x\right)dz \ , \ k \geq 1 \\ w_{n+1}^1\left(t,x\right) = i\varepsilon\displaystyle\int_0^t \frac{\partial^2 w_n^1\left(z,x\right)}{\partial x^2}dz \ ; n \geq 0 \end{array}\right.$$

Therefore we have

$$\left\{\begin{array}{l} w_0^1\left(t,x\right) = \beta e^{iax} \\ w_1^1\left(t,x\right) = -ia^2\beta\varepsilon t e^{iax} \\ w_2^1\left(t,x\right) = -\dfrac{1}{2}a^4t^2\beta\varepsilon^2e^{iax} \\ w_3^1\left(t,x\right) = \dfrac{1}{6}ia^6t^3\beta\varepsilon^3e^{iax} \\ \ldots \\ w_n^1\left(t,x\right) = \beta\dfrac{\left(-\varepsilon ia^2t\right)^n}{n!}e^{iax}, n \geq 0 \end{array}\right.$$

from which at step $k = 1$, we obtain :

$$w^1\left(t,x\right) = \lim_{p\to+\infty}\beta e^{iax}\sum_{p=0}^{n}\frac{\left(-\varepsilon ia^2t\right)^p}{p!} = \beta\exp\left[i\left(ax - \varepsilon a^2t\right)\right].$$

Then let's calculate $Nw^1\left(t,x\right)$

$$Nw^1\left(t,x\right) = iq\left|w^i\left(t,x\right)\right|^{2p} w^1\left(t,x\right) - iq\beta^{2p}w^1(t,x) \neq 0$$

therefore, we modify problem (E) into an equivalent problem :

$$(E):\begin{cases} i\dfrac{\partial w(t,x)}{\partial t}+\varepsilon\dfrac{\partial^2 w(t,x)}{\partial x^2}+q\left|\beta\right|^{2p}w(t,x)+\widetilde{N}w(t,x)=0 \\ w(0,x)=\beta e^{iax} \end{cases}$$

where

$$\widetilde{N}w(t,x)=q\left|w(t,x)\right|^{2p}w(t,x)-q\left|\beta\right|^{2p}w(t,x)$$

Therefore, we obtain:

$$\frac{\partial w(t,x)}{\partial t}=\varepsilon i\frac{\partial^2 w(t,x)}{\partial x^2}+qi\left|\beta\right|^{2p}w(t,x)+i\widetilde{N}w(t,x)$$

then the canonical form of Adomian[9]

$$w(t,x)=\beta e^{iax}+\varepsilon i\int_0^t\frac{\partial^2 w(z,x)}{\partial x^2}dz+qi\left|\beta\right|^{2p}\int_0^t w(z,x)\,dz+i\int_0^t\widetilde{N}w(z,x)\,dz$$

The new algorithm is then :

$$\begin{cases} w_0^k(t,x))=\beta e^{iax}+i\displaystyle\int_0^t\widetilde{N}w^{k-1}(z,x)\,dz;k\geq 1 \\ w_{n+1}^k(t,x))=\varepsilon i\displaystyle\int_0^t\frac{\partial^2 w_n^k(z,x)}{\partial x^2}dz+q\left|\beta\right|^{2p}i\int_0^t w_n^k(z,x)\,dz;n\geq 0 \end{cases}$$

Let's determine $w^1(t,x)$

$$\begin{cases} w_0^1(t,x)=\beta e^{iax} \\ w_1^1(t,x)=\beta i\left(\left(q\left|\beta\right|^{2p}-a^2\varepsilon\right)t\right)e^{iax} \\ w_2^1(t,x)=\beta\dfrac{\left(i\left(q\left|\beta\right|^{2p}-a^2\varepsilon\right)t\right)^2}{2}e^{iax} \\ w_3^1(t,x)=\beta\dfrac{\left(i\left(q\left|\beta\right|^{2p}-a^2\varepsilon\right)t\right)^3}{3!}e^{iax} \\ \ldots \\ w_n^1(t,x)=\beta\dfrac{\left(i\left(q\left|\beta\right|^{2p}-a^2\varepsilon\right)t\right)^n}{n!}e^{iax} \end{cases}$$

hence

$$w^1(t,x)=\beta e^{iax}\sum_{n=0}^{+\infty}\frac{\left(i\left(q\left|\beta\right|^{2p}-a^2\varepsilon\right)t\right)^n}{n!}=\beta\exp\left[i\left(\left(q\left|\beta\right|^{2p}-a^2\varepsilon\right)t+ax\right)\right]$$

We thus obtain $\widetilde{N}w^1(t,x)=0$
Recursively we have:

$$w^1(z,x)=w^2(z,x)=\ldots=w^k(z,x)$$

so the solution of problem (E) is :

$$w\left(t,x\right)=\beta\exp\left[i\left(\left(q\left|\beta\right|^{2p}-a^{2}\varepsilon\right)t+ax\right)\right].$$

Consider the following problem :

$$(F):\begin{cases} i\dfrac{\partial w\left(t,x\right)}{\partial t}+\varepsilon\dfrac{\partial^{2}w\left(t,x\right)}{\partial x^{2}}+\mu w\left(t,x\right)+q\left|w\left(t,x\right)\right|^{2p}w\left(t,x\right)=0\\ w(0,x)=\beta e^{iax}\end{cases}$$

We obtain the following Adomian algorithm:

$$w(t,x)=w\left(0,x\right)+i\varepsilon\int_{0}^{t}\frac{\partial^{2}w\left(z,x\right)}{\partial x^{2}}dz+i\mu\int_{0}^{t}w\left(z,x\right)dz+iq\int_{0}^{t}Nw\left(z,x\right)dz \tag{18}$$

where

$$Nw\left(t,x\right)=\left|w\left(t,x\right)\right|^{2p}w\left(t,x\right)$$

Let us apply the method of successive approximations to (18),

$$\begin{cases} w^{k}(t,x)=w^{k}\left(0,x\right)+i\varepsilon\displaystyle\int_{0}^{t}\frac{\partial^{2}w^{k}\left(z,x\right)}{\partial x^{2}}dz+\\ \qquad i\mu\displaystyle\int_{0}^{t}w^{k}\left(z,x\right)dz+iq\int_{0}^{t}Nw^{k-1}\left(z,x\right)dz,k\geq1\end{cases} \tag{19}$$

We are looking for the solution of (F) in the form of a series[10]

$$w^{k}(t,x)=\sum_{n=0}^{+\infty}w_{n}^{k}(t,x)$$

At each step $k\geq1$, we have the following algorithm [11]:

$$\begin{cases} w_{0}^{k}(t,x)=w^{k}\left(0,x\right)+iq\displaystyle\int_{0}^{t}Nw^{k-1}\left(z,x\right)dz\\ w_{n+1}^{k}(t,x)=i\varepsilon\displaystyle\int_{0}^{t}\frac{\partial^{2}w_{n}^{k}\left(z,x\right)}{\partial x^{2}}dz+i\mu\int_{0}^{t}w_{n}^{k}\left(z,x\right)dz;n\geq0\end{cases}$$

Let's calculate the terms of the series

$$w^{k}(t,x)=\sum_{n=0}^{+\infty}w_{n}^{k}(t,x)$$

At step $k=1$, for $w^{0}(t,x)=0$, we have $Nw^{0}(t,x)=0$ and we obtain ::

$$\begin{cases} w_0^1(t,x) = \beta e^{iax} \\ w_1^1(t,x) = \beta i\left(\mu - a^2\varepsilon\right) te^{iax} \\ w_2^1(t,x) = \beta \dfrac{\left(i\left(\left(\mu - a^2\varepsilon\right)\right)t\right)^2}{2!} e^{iax} \\ w_3^1(t,x) = \beta \dfrac{\left(i\left(\left(\mu - a^2\varepsilon\right)\right)t\right)^3}{3!} e^{iax} \\ \ldots \\ w_n^1(t,x) = \beta \dfrac{\left(i\left(\left(\mu - a^2\varepsilon\right)\right)t\right)^n}{n!} e^{iax} \end{cases}$$

therefore

$$w^1(t,x) = \beta e^{iax} \sum_{n=0}^{+\infty} \frac{\left(i\left(\left(\mu - a^2\varepsilon\right)\right)t\right)^n}{n!} = \beta \exp\left[i\left(\left(\mu - a^2\varepsilon\right)t + ax\right)\right]$$

Calculate $Nw^1(t,x)$, we have :

$$Nw^1(t,x) = q\left|w^1(t,x)\right|^{2p} w^1(t,x) = q\left|\beta\right|^{2p} w^1(t,x) \neq 0$$

We then modify problem (F) into an equivalent problem:

$$(F): \begin{cases} i\dfrac{\partial w(t,x)}{\partial t} + \varepsilon\dfrac{\partial^2 w(t,x)}{\partial x^2} + \mu w(t,x) + q\left|\beta\right|^{2p} w(t,x) + \widetilde{N}w(z,x) = 0 \\ w(0,x) = \beta e^{iax} \end{cases} \tag{20}$$

where

$$\widetilde{N}w(z,x) = q\left|w(t,x)\right|^{2p} w(t,x) - q\left|\beta\right|^{2p} w(t,x)$$

we have the following canonical form :

$$w(t,x) = w(0,x) + i\int_0^t \frac{\partial^2 w(z,x)}{\partial x^2}dz + i\mu\int_0^t w(z,x)\,dz + qi\left|\beta\right|^{2p}\int_0^t w(z,x)\,dz + i\int_0^t \widetilde{N}w(z,x)\,dz$$

Let's apply the method of successive approximations to (19),

$$w^k(t,x) = w^k(0,x) + i\varepsilon\int_0^t \frac{\partial^2 w^k(z,x)}{\partial x^2}dz + i\left(\mu + q\left|\beta\right|^{2p}\right)\int_0^t w^k(z,x)\,dz + i\int_0^t \widetilde{N}w^{k-1}(z,x)\,dz, k \geq 1$$

Thus, at each step $k \geq 1$,the following algorithm is obtained :

$$\begin{cases} w_0^k(t,x) = w^k(0,x) + i\displaystyle\int_0^t \widetilde{N}w^{k-1}(z,x)\,dz, k \geq 1 \\ w_{n+1}^k(t,x) = \quad i\varepsilon\displaystyle\int_0^t \frac{\partial^2 w_n^k(z,x)}{\partial x^2}dz + i\left(\mu + q\left|\beta\right|^{2p}\right)\int_0^t w_n^k(z,x)\,dz; n \geq 0 \end{cases}$$

Let's calculate $w^1(t,x)$ at step $k=1$

$$\begin{cases} w_0^1(t,x) = w^k(0,x) + i\int\limits_0^t \widetilde{N}w^0(z,x)\,dz \\ w_{n+1}^1(t,x) = \ i\varepsilon\int\limits_0^t \dfrac{\partial^2 w_n^1(z,x)}{\partial x^2}dz + i\left(\mu + q\left|\beta\right|^{2p}\right)\int\limits_0^t w_n^1(z,x)\,dz; n\geq 0 \end{cases}$$

for $w^0(t,x)=0$ we have : $Nw^0(t,x)=0$, hence

$$\begin{cases} w_0^1(t,x) = w^k(0,x) \\ w_{n+1}^1(t,x) = i\varepsilon\int\limits_0^t \dfrac{\partial^2 w_n^1(z,x)}{\partial x^2}dz + i\mu\int\limits_0^t w_n^1(z,x)\,dz + qi\left|\beta\right|^{2p}\int\limits_0^t w_n^1(z,x)\,dz; n\geq 0 \end{cases}$$

$$\begin{cases} w_0^1(t,x) = \beta e^{iax} \\ w_1^1(t,x) = \ \beta it\left(\mu - a^2\varepsilon + q\left|\beta\right|^{2p}\right)e^{iax} \\ w_2^1(t,x) = \ \beta\dfrac{\left(it\left(\mu - a^2\varepsilon + q\left|\beta\right|^{2p}\right)\right)^2}{2!}e^{iax} \\ w_3^1(t,x) = \ \beta\dfrac{\left(it\left(\mu - a^2\varepsilon + q\left|\beta\right|^{2p}\right)\right)^3}{3!}e^{iax} \\ \ldots \\ w_n^1(t,x) = \ \beta\dfrac{\left(it\left(\mu - a^2\varepsilon + q\left|\beta\right|^{2p}\right)\right)^n}{n!}e^{iax} \end{cases}$$

the solution at step $k=1$ is

$$\begin{aligned} w^1(t,x) &= \beta e^{iax}\sum_{n=0}^{+\infty}\frac{\left(it\left(\mu - a^2\varepsilon + q\left|\beta\right|^{2p}\right)\right)^n}{n!} \\ &= \beta\exp\left[i\left(\left(\mu - a^2\varepsilon + q\left|\beta\right|^{2p}\right)t + ax\right)\right] \end{aligned}$$

We thus obtain :

$$w^1(t,x) = w^2(t,x) = \ldots = w^k(t,x) = \beta\exp\left[i\left(\left(\mu - a^2\varepsilon + q\left|\beta\right|^{2p}\right)t + ax\right)\right].$$

Thus the solution of problem (F) :

$$\begin{aligned} w(t,x) &= \lim_{k\to+\infty} w^k(t,x) \\ &= \beta\exp\left[i\left(\left(\mu - a^2\varepsilon + q\left|\beta\right|^{2p}\right)t + ax\right)\right] \end{aligned}$$

4.1 Conclusion

The SBA and ADM methods have allowed us to successfully solve the Ivancevic option pricing model (IOPM) in financial mathematics, the Schrödinger model in quantum mechanics and the classical Black-Scholes equation.

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