HALF-METRIC SPACE

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Abstract. In this paper, I have introduced half-metric spaces on normed vector spaces over $\mathbb{R}$ or $\mathbb{C}$, which are similar to metric spaces by relaxing a few conditions of metric space.I have also introduced even half-metric spaces,established some properties and discussed completeness in the context of half-metric space and even half-metric space.

1. Introduction

Definition 1.1. Metric space is an ordered pair (M,d) where M is a non empty set and d is metric on M.

$$d(x, y) : MXM \to \mathbb{R} \text{ such that for any } x, y, z \in M \text{ the following holds}$$
$$1)d(x, y) \geq 0$$
$$2)d(x, y) = d(y, x)$$
$$3)d(x, z) \leq d(x, y) + d(y, z)$$
$$4)d(x, y) = 0 \iff x = y$$

Definition 1.2. A normed vector space or normed space is a vector space over the real or complex numbers, on which a norm is defined.A norm is a real-valued function defined on the vector space that is commonly denoted as $\|.\|$ which has the following properties:
for all vector x,y and scalar t

$$1)\|x\| \geq 0$$
$$2)\|x\| = 0 \iff x = 0$$
$$3)\|tx\| = |t|\|x\|$$
$$4)\|x + y\| \leq \|x\| + \|y\|$$

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Definition 1.3. A metric (M, d) on a linear space is said to be translation invariant if

$$d(x + a, y + a) = d(x, y)$$

$for\ all\ x, y, a \in M.$

Definition 1.4. A sequence $x_1, x_2, ...x_n$ is said be cauchy if for every postive real number $\epsilon > 0$ there is a positive Number N such that for all positive integers $m, n > N$, $d(x_m, x_n) < \epsilon$.

Definition 1.5. A metric space (M, d) is complete if every cauchy sequence in M converges in M.

2. MAIN RESULT

Definition 2.1. A half-metric space on a vector space equipped with $\|.\|$ over $\mathbb{R}$ or $\mathbb{C}$ is an ordered pair (M, d) ,where M is a non empty set,$\|.\|$ is the norm and d is half-metric on M,if the following holds,

$d(x, y) : MXM \to \mathbb{R}$ such that for any $x, y, z \in M$ the following holds

$$1)d(x, y) \geq 0$$

$$2)d(x, y) = d(y, x)$$

$$3)d(x, z) \leq d(x, y) + d(y, z)$$

$$4)d(0, y) = 0 \iff y = 0$$

It can be noted that all metric spaces over normed vector space are half-metric spaces.

Definition 2.2. A half-metric space is said to translation invariant if

$$d(x + a, y + a) = d(x, y)$$

$for\ all\ x, y, a \in M.$

2.1. **Note.** For most of the examples, I have considered set of real numbers over real field for simplicity.

2.2. **Example.** $(\mathbb{R}, d)$ where,$d(x, y) = |x^2 - y^2|$, clearly d is half-metric on $\mathbb{R}$.

Definition 2.3. A half-metric space is said to be even if for all x,y in M

$$d(x, y) = d(x, -y)$$

2.3. **Note.** Metric spaces cannot be even half-metric spaces because $d(x, x) = d(x, -x)$ only for $x = 0$

2.4. **Example.** $(\mathbb{R}, d)$, where ,$d(x,y) = |x^2 - y^2|$, consider $d(x,-y) = |x^2 - (-y)^2|$, $d(x,-y) = |x^2 - y^2| \implies d(x,-y) = d(x,y)$ for all x,y in $\mathbb{R}$.

Definition 2.4. A sequence $x_1, x_2, ...x_n$ is said be cauchy if for every postive real number $\epsilon > 0$ there is a positive Number N such that for all positive integers $m, n > N$, $d(x_m, x_n) < \epsilon$,where d is half-metric.

Definition 2.5. A half-metric space (M, d) is complete if every cauchy sequence in M converges in M.

Proposition 2.6. *If a half-metric space is even then it cannot be translation invariant.*
Alternatively if a half-metric space is translation invariant then it cannot be even.

Proof. let (M, d) be even half-metric space. then for all x,y in M

$$d(x,y) = d(x,-y)$$

assume d is translation invariant,then for all x,y,c in M

$$d(x+c, y+c) = d(x,y)$$

$$d(x+c, -y+c) = d(x,-y)$$

implies

$$d(x+c, y+c) = d(x+c, -y+c)$$

for all c in M, let $x = -c, y = c$

$$d(0, 2c) = d(0,0)$$

$$d(0, 2c) = 0$$

using properties of half-metric

$$2c = 0$$

$$c = 0$$

a contradiction because c was arbitrary, therefore even half-metric spaces cannot be translation invariant.

□

2.5. **Example.** $(\mathbb{R}, d)$ *be a metric space* such that $d(x,y) = |x|+|y|$, clearly it is an even half-metric space.
consider $d(1+2, 2+2) = d(3,4) = |3| + |4| = 7$
and $d(1,2) = |1| + |2| = 3$
$d(1+2, 2+2) \neq d(1,2)$ therefore d is not translation invariant

2.6. **Example.** $(\mathbb{R}, d)$ *be a metric space* such that, $d(x,y) = |x|+|y|+|x||y|$ clearly it is an even half-metric space on $\mathbb{R}$
therefore,it is not translation invariant.

2.7. **Example.** $(\mathbb{R}, d)$, $d = |x-y|$,clearly it is translation invariant,therefore it is not an even half-metric space.

Proposition 2.7. *An even half-metric space is complete* $\iff$ *all cauchy sequence in that space converge to zero.*

Proof. Let (M,d) be complete even half-metric space, let x_n be a cauchy sequence in M then

$$\lim_{n\to\infty} d(x_n, x) = 0$$

because M is complete, $\implies x \in M$
then by definition of even half-metric space

$$\lim_{n\to\infty} d(x_n, -x) = 0$$

$\implies x_n \to x$ as $n \to \infty$ and $x_n \to -x$ as $n \to \infty$

$$\implies x = 0$$

because limits are unique if they exist.
Conversely
Let all cauchy sequence x_n converge to zero in (M, d)
since M is even half-metric space, by definition of vector space, zero is in M.
hence (M, d) is complete.

2.8. **Example.** $(\mathbb{R}, d)$ such that

$$d(x, y) = |x| + |y|$$

and x_n be a cauchy sequence in $\mathbb{R}$
i.e

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0$$

$\implies$

$$\lim_{n,m\to\infty} (|x_n| + |x_m|) = 0$$

$\implies$

$$x_n \to 0, x_m \to 0$$

as n,m $\to \infty$ therefore $(\mathbb{R}, d)$ is complete even half-metric space.

2.9. **Example.** $(\mathbb{R}, d)$, $d(x,y) = |x^2 - y^2|$, consider

$$x_n = \left(1 + \frac{1}{n}\right)^n$$

x_n is cauchy in $\mathbb{R}$ because

$$d(\left(1 + \frac{1}{n}\right)^n, e) < \epsilon$$

for $n > N$, since d is even half metric

$$d(\left(1 + \frac{1}{n}\right)^n, -e) < \epsilon$$

for $n > N$

$$\implies x_n \to e \; and \; x_n \to -e$$

which is not possible,therefore x_n is not convergent. $\implies$ (R, d) is not complete.

□

2.10. **Conclusion.** Vector spaces can be associated with a particular real valued function that satisfy most of the properties of metric spaces,furthermore vector spaces can be associated with one more strong condition on half-metric space which leads to some interesting analysis.It is also clear that metric space on vector spaces is half-metric space and metric space on vectors spaces is not even half-metric space.

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