Original Research Article

Bound Estimation of Some Special Functions

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Abstract

This article estimates bounds of several special functions. It also gives mathematical proofs and graphs of the corresponding functions. The results are applicable in aspect of inequalities.

Keywords: Estimation, bound, special function, inequalities

1. Introduction

T_3 tree, which was introduced in article [1], brought some interesting inequalities and bound estimation for special functions, as shown in [2,3,4,5]. During the study of the tree, several new special functions are found. They are $f(\alpha)=\frac{\sqrt{\alpha}}{2-\sqrt{\alpha}}-(\frac{\alpha+1}{2})^2$, $f(\sigma)=\sqrt{\frac{2^\sigma\sqrt{2}}{2^\sigma-1}}$, $f(\sigma)=\sqrt{\frac{2^\sigma}{(2^\sigma-1)\sqrt{2}}}$, $f(\sigma)=\sqrt{\frac{2^{\sigma+1}}{2^\sigma+\sqrt{2}}}$, $f(\sigma)=\sqrt{\frac{2^{\sigma+1}}{2^\sigma+1}}$ and $f(\sigma)=\sqrt{\frac{2^\sigma\sqrt{2}}{(2^\sigma+1)}}$. It also needs to estimate the bounds of them. Because there are no corresponding answers on the Internet and related reference handbooks [6,7,8]. This paper presents the required answers. Results of this paper are helpful for further study of T_3 tree.

2. Main Results and Proofs

Theorem 1. Let α be a real number with $\alpha\in(1,4)\cup(4,\infty)$;then

$$f(\alpha)=\frac{\sqrt{\alpha}}{2-\sqrt{\alpha}}-(\frac{\alpha+1}{2})^2$$

thus

$$0<f(\alpha)<\infty,\ \alpha\in(1,4)$$

and

$$-\infty<f(\alpha)<(-17.345),\ \alpha\in(4,6)$$

Proof. Direct calculation shows

$$f'(\alpha)=\frac{\frac{1}{2}\alpha^{-\frac{1}{2}}(2-\sqrt{\alpha})-\sqrt{\alpha}(-\frac{1}{2}\alpha^{-\frac{1}{2}})}{(2-\sqrt{\alpha})^2}-\frac{1}{2}\alpha-\frac{1}{2}$$
$$=\frac{1}{\sqrt{\alpha}(2-\sqrt{\alpha})^2}-\frac{\alpha}{2}-\frac{1}{2}.$$

The original function can be changed to

$$.\qquad \frac{8\sqrt{\alpha}+4\alpha-(4-\alpha)(\alpha+1)^2}{4(4-\alpha)}>0 \tag{1}$$

Since $1<\alpha<4$， $4(4-\alpha)>0$. Next is to show

$$8\sqrt{\alpha}+4\alpha-(4-\alpha)(\alpha+1)^2>0$$

Let $h(\alpha)=8\sqrt{\alpha}+4\alpha-(4-\alpha)(\alpha+1)^2$; then

$$h(\alpha)=8\sqrt{\alpha}+4\alpha-4(\alpha+1)^2+\alpha(\alpha+1)^2$$
$$=8\sqrt{\alpha+1-1}+4(\alpha+1-1)-4(\alpha+1)^2+(\alpha+1-1)(\alpha+1)^2.$$

Assume $\alpha+1=t$;then $2<t<5$ and

$$f(t)=8\sqrt{t-1}+4(t-1)-4t^2+(t-1)t^2$$
$$=t^3-5t^2+4t+8\sqrt{t-1}-4.$$

Direct calculation yields

$$f'(t)=3t^2-10t+4+\frac{4}{\sqrt{t-1}}.$$

$$f''(t)=6t-\frac{2}{\sqrt{(t-1)^3}}-10$$
$$=6t-\frac{2}{(t-1)\sqrt{t-1}}-10.$$

Note that, $\frac{1}{4}<\frac{2}{(t-1)\sqrt{t-1}}<2$ and when $f''(t)>0$ .This means $f'(t)$ is monotonically increasing. Since $f'(t)>0$ when $t>2$, $f(t)$ is monotonically increasing. Considering $f(2)=0$, it is obtained

$$f(\alpha)>0$$

Since $f(4-0)=+\infty$ and $f(\alpha)>0$， it holds

$$0<f(\alpha)<+\infty \quad \alpha\in(1,4)$$

Since $\alpha_0=\left(\frac{2+(35+3\sqrt{129})^{\frac{1}{3}}+4(35+3\sqrt{129})^{-\frac{1}{3}}}{3}\right)^2$， it holds $f'(\alpha_0)=0$

and consequently

$$f''(\alpha_0)=\frac{-(\frac{2}{\sqrt{\alpha_0}}+\frac{3\sqrt{\alpha_0}}{2}-4)}{\alpha_0(2-\sqrt{\alpha_0})^4}-\frac{1}{2}\approx -4.668<0$$

It is obvious that $f(\alpha_0)=-17.34544147$ when $4<\alpha<6.$

Since $f(4+0)=-\infty$ and $f(\alpha_0)=-17.34544147$, it holds

$$-\infty<f(\alpha)<(-17.345),\ \alpha\in(4,6)$$

□

Using Maple software to draw Figure 1, showing the graph of the function $f(x)=\frac{\sqrt{x}}{2-\sqrt{x}}-(\frac{x+1}{2})^2$ with $a\in(1,4)\cup(4,\infty)$, which is just what Theorem 1 states. The Maple commands are the follows.

$af := proc(x)$

$\frac{\sqrt{x}}{2-\sqrt{x}}-(\frac{x+1}{2})^2;$

end proc

plot([af(x)], x=1..6)

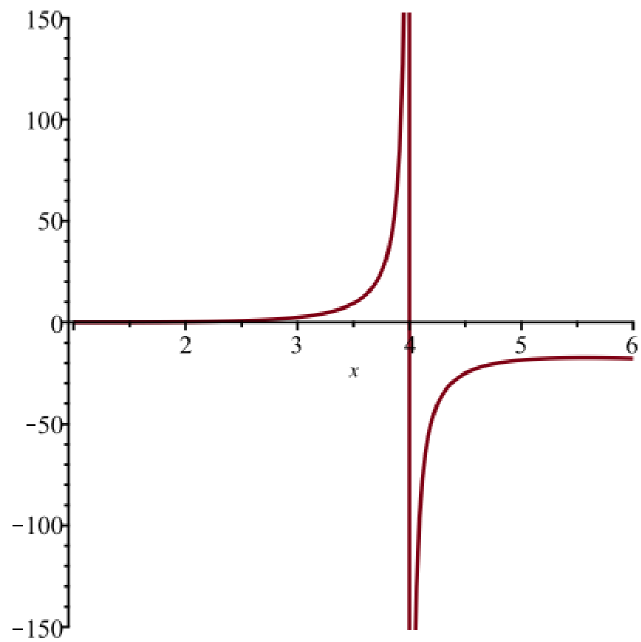

Figure.1 The graph of the function

Theorem 2. Let $f(\sigma)=\sqrt{\frac{a2^\sigma}{2^\sigma-1}}$ with $\sigma>1$ and $a>0$;then $f'(\sigma)<0$ and $\sqrt{a}\le f(\sigma)<\sqrt{2a}$.

Proof. Let $g(\sigma)=\frac{a2^\sigma}{2^\sigma-1}$

Direct calculation shows

$$g'(\sigma) = \frac{a2^{\sigma}\ln 2(2^{\sigma}-1) - a2^{\sigma}(2^{\sigma}\ln 2)}{(2^{\sigma}-1)^2} = \frac{-a2^{\sigma}\ln 2}{(2^{\sigma}-1)^2}. \quad (2)$$

The conditions $a > 0$ and $\sigma > 1$ mean $-a2^{\sigma}\ln 2 < 0$; thus

$$g'(\sigma) < 0 \quad (3)$$

Obviously, (2) and (3) result in $f'(\sigma) < 0$. Since $\sigma = 1$, it can be $f(1) = \sqrt{2a}$. When σ is to infinity, the value of $f(\sigma)$ is equal to $\sqrt{a}$. It leads to $\sqrt{a} \le f(\sigma) < \sqrt{2a}$.
□

Corollary 1. Let $f(\sigma) = \sqrt{\frac{2^{\sigma}\sqrt{2}}{2^{\sigma}-1}}$ with $\sigma > 1$;then $1.189 \le f(\sigma) < 1.682$.

Proof. According to Theorem 2, since $a = \sqrt{2}$, it can obtain $1.189 \le f(\sigma) < 1.682$ when $\sigma > 1$.

Using Maple software to draw Figure 2, displaying the graph of the function $f(x) = \sqrt{\frac{2^{x}\sqrt{2}}{2^{x}-1}}$ when $x > 1$,which is just what Corollary 1 states. The Maple commands are the follows.

$bf := proc(x)$

$\sqrt{\frac{2^{x}\sqrt{2}}{2^{x}-1}};$

end proc

plot([bf(x)], x=1..10)

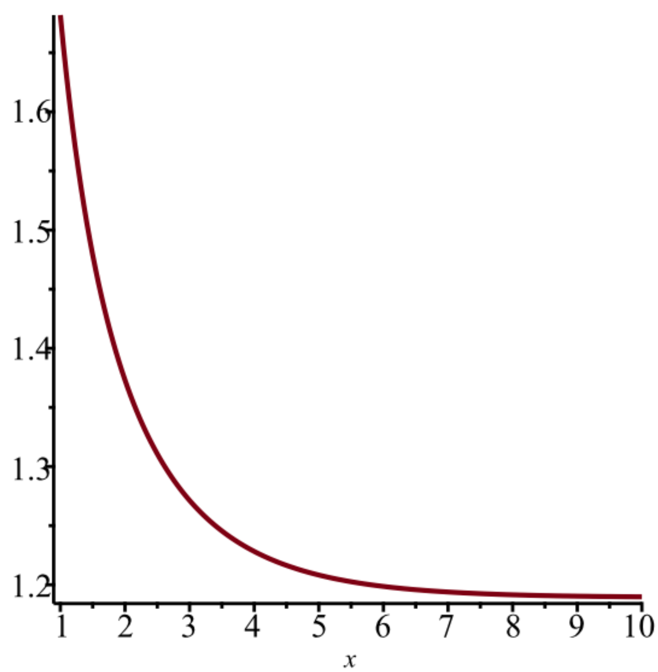

Figure.2 The graph of the function

Corollary 2. Let $h(\sigma)=\sqrt{\frac{2^{\sigma}}{(2^{\sigma}-1)\sqrt{2}}}$ with $\sigma>1$;then $0.841\le h(\sigma)<1.189$

Proof. By Theorem 2, Since $\sigma>1$ and $a>0$,it holds

$$\sqrt{a}\le\sqrt{\frac{a2^{\sigma}}{2^{\sigma}-1}}<\sqrt{2a}$$

If $a=\frac{\sqrt{2}}{2}$, it can get $0.841\le h(\sigma)<1.189$.

□

Using Maple software to draw Figure 3, displaying the graph of the function $h(x)=\sqrt{\frac{2^{x}}{(2^{x}-1)\sqrt{2}}}$ when $x>1$,which is just what Corollary 2 states. The Maple commands are the follows.

$$cf := proc(x)$$
$$\sqrt{\frac{(2)^{x}}{((2)^{x}-1)\sqrt{2}}};$$
$$end\ proc$$
$$plot([cf(x)], x=1..10)$$

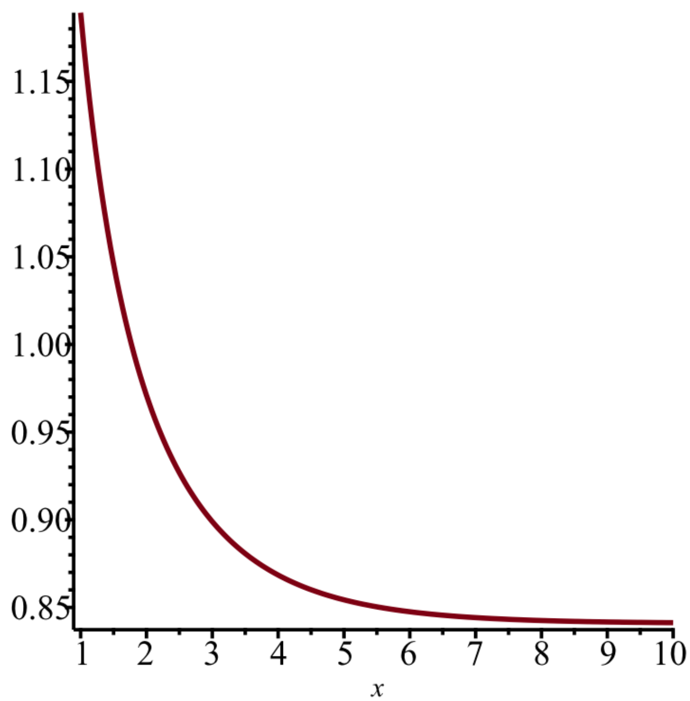

Figure.3 The graph of the function

Theorem 3. Let $f(\sigma)=\sqrt{\frac{2^{\sigma}}{2^{\sigma}+a}}$ with $\sigma>1$and $a>0$;then $f'(\sigma)>0$ and $\sqrt{\frac{2}{2+a}}<f(\sigma)<1$.

Proof. Let $g(\sigma)=\frac{2^{\sigma}}{2^{\sigma}+a}$;

Direct calculation yields

$$g'(\sigma) = \frac{2^{\sigma} a \ln 2}{(2^{\sigma} + a)^2}$$

When $\sigma > 1$ and $a > 0$, it is obviously $g(\sigma)' > 0$ thus

$$f'(\sigma) > 0.$$

Then by $f(1) = \sqrt{\frac{2}{2+a}}$ and $\lim_{\sigma \to \infty} = \sqrt{\frac{2^{\sigma}}{2^{\sigma}+a}} = 1$ thus $\sqrt{\frac{2}{2+a}} \le f(\sigma) < 1$.

□

Corollary 3. Let $f(\sigma) = \sqrt{\frac{2^{\sigma+1}}{2^{\sigma}+\sqrt{2}}}$ with $\sigma > 1$;then $1.082 < \sqrt{\frac{2^{\sigma+1}}{2^{\sigma}+\sqrt{2}}} \le 1.414$.

Proof. According to theorem 3, Since $f(1) = \sqrt{\frac{2^2}{2+\sqrt{2}}} = 1.082$ and $\lim_{\sigma \to \infty} = \sqrt{2} = 1.414$ hence $1.082 \le f(\sigma) < 1.414$ when $\sigma > 1$.

□

Using Maple software to draw Figure 4, showing the graph of the function $f(x) = \sqrt{\frac{2^{x+1}}{2^{x}+\sqrt{2}}}$. It is seen $1.082 < f(x) \le 1.414$ when $x > 1$,which is just what Corollary 3 states. The Maple commands are the follows.

$$\begin{aligned} &df := proc(x) \\ &\sqrt{\frac{(2)^{x+1}}{(2)^{x}+\sqrt{2}}}; \\ &end\ proc \\ &plot([df(x)], x = 1..10) \end{aligned}$$

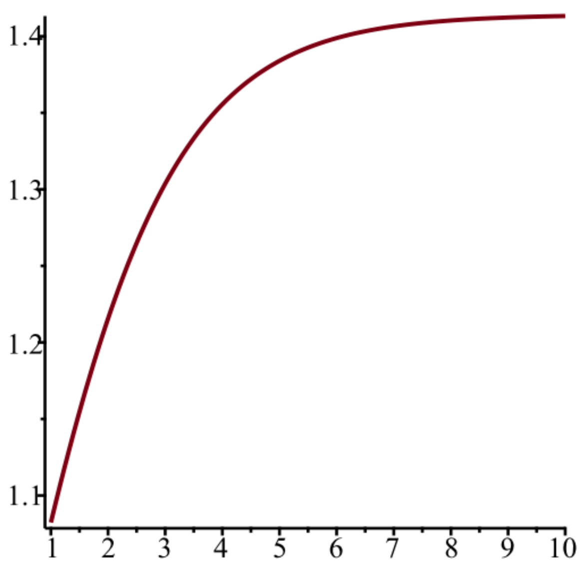

Figure.4 The graph of the function

Corollary 4. Let $f(\sigma)=\sqrt{\frac{2^{\sigma+1}}{2^{\sigma}+1}}$ with $\sigma>1$;then $1.155<\sqrt{\frac{2^{\sigma+1}}{2^{\sigma}+1}}\le 1.414$.

Proof. According to the analysis of Theorem 3, since $f(1)=\sqrt{\frac{2^{2}}{2+1}}=1.155$ and $\lim_{\sigma\to\infty}=\sqrt{\frac{2^{\sigma+1}}{2^{\sigma}+1}}=1.414$, so it is directly obtained $1.155\le f(\sigma)<1.414$ when $\sigma>1$.

□

Using Maple software to draw Figure 5, showing the graph of the function $f(x)=\sqrt{\frac{2^{x+1}}{2^{x}+1}}$ when $x>1$. The Maple commands are the follows.

$$ef := proc(x)$$
$$\sqrt{\frac{(2)^{x+1}}{(2)^{x}+1}};$$
$$end\ proc$$
$$plot([ef(x)], x=1..10)$$

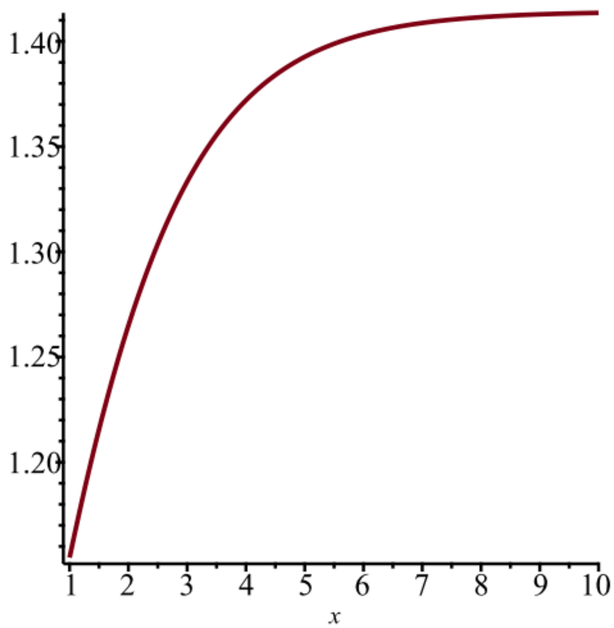

Figure.5 The graph of the function

Theorem 4. Suppose $f(\sigma)=\sqrt{\frac{a2^{\sigma}}{2^{\sigma}+1}}$ with $\sigma>1$ and $a>0$;then $f'(\sigma)>0$ and $\sqrt{\frac{2a}{3}}<\sqrt{\frac{2^{\sigma}}{2^{\sigma}+a}}<\sqrt{a}$.

Proof. Let $g(\sigma)=\frac{a2^{\sigma}}{2^{\sigma}+1}$;then

$$g'(\sigma)=\frac{2^{\sigma}a\ln 2}{(2^{\sigma}+1)^{2}}.$$

When $\sigma>1$ and $a>0$, this is obviously $g(\sigma)'>0$ thus

$$f'(\sigma) > 0 \,.$$

Since $\sigma = 1$, it can be $f(1) = \sqrt{\frac{2a}{3}}$ .When σ is to infinity， the value of $f(\sigma)$ is equal to $\sqrt{a}$ .Consequently $\sqrt{\frac{2a}{3}} \le f(\sigma) < \sqrt{a}$.

□

Corollary 5. Assume $f(\sigma) = \sqrt{\frac{2^{\sigma}\sqrt{2}}{(2^{\sigma}+1)}}$ with $\sigma > 1$;then $f'(\sigma) > 0$ and $0.971 < \sqrt{\frac{2^{\sigma}\sqrt{2}}{(2^{\sigma}+1)}} \le 1.189$.

Proof. According to theorem 4, It yields $0.971 \le f(\sigma) < 1.189$ when $\sigma > 1$.

□

Using Maple software to draw Figure 6, showing the graph of the function $f(x) = \sqrt{\frac{2^{x}\sqrt{2}}{(2^{x}+1)}}$. It is seen $0.971 < f(x) \le 1.189$ when $x > 1$. This is exactly what Corollary 5 says. The Maple commands are the follows.

$$gf := proc(x)$$
$$\sqrt{\frac{2^{x}\sqrt{2}}{(2^{x}+1)}};$$
$$plot([gf(x)],\ x=1..10)$$

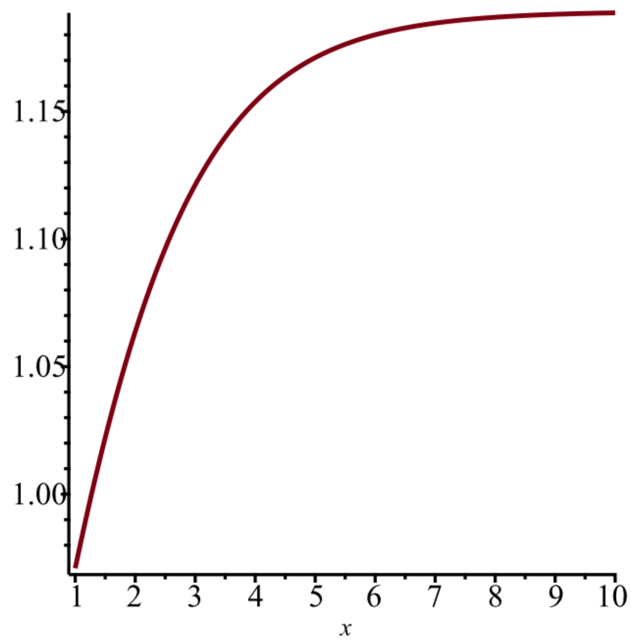

Figure 6: The graph of the function

3. Conclusion

This paper proves the estimation of several special function boundaries encountered in the research process, provides a mathematical foundation and solution ideas for estimating function boundaries, so that it can be applied to engineering practice, and it is hoped that it will be helpful to researchers.

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Competing Interests

Authors have declared that no competing interests exist.

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