Integro-differential equations for a class of delayed renewal risk processes with dependence

Abstract: The Gerber-Shiu discounted penalty function is considered for a class of delayed renewal risk processes. In (Willmot 2004), special cases of the model include the stationary renewal risk model and the situation where the time until the first claim is exponentially distributed. In this paper, we consider a class of delayed and perturbed risk model with dependence between interclaim arrivals and claim sizes. The integro-differential equations for the Gerber-Shiu discounted penalty functions are derived.
Keywords: Gerber-Shiu discounted penalty function; delayed renewal risk process; multi-dependence events; Integro-differential equation

1 Introduction

As a valuable analytical tool for analyzing the ruin probability, the Gerber-Shiu function is often used to predict and study the ruin risk that some insurance companies may cause due to some serious economic claims. To this end, we apply some models to predict these ruin probabilities and ruin time. For many years, a large number of scholars have studied the compound Poisson risk model perturbed by a diffusion process. However these models all assume that interval and claim amount are independent. [1] add diffusion to the dependent risk model of [2] and study the ruin probabilities by using a potential measure. In addition, [3] consider a compound Poisson risk model perturbed by a Brownian motion. Therefore, the Gerber-Shiu function in the delayed renewal model has also been extended. For instance, [4] considered a case where the first inter-claim time is supposed to follow a different density rather than the common density of the subsequent inter-claims times. Most recently, [5] consider the risk model perturbed by a diffusion process with a time delay in the arrival of the first two claims and take into account dependence between claim sizes and the inter-claims times. This paper is based on [5] and applies the dependent risk model of [2]. Then the integro-differential equations of the Gerber-Shiu discounted penalty functions are given.

The paper is structured as follows: in Section 2, we describe the risk model. In Section 3, we derive dependence structure and obtain Lundberg-type equation. In Section 4, we derive the integro-differential Equations that satisfy the Gerber–Shiu functions. Then the concluding remarks drive in Section 5.

2 Risk model

We consider the following compound Poisson risk model that is perturbed by a Brownian motion

$$U(t) = u + ct - S(t) + \sigma B(t), \tag{2.1}$$

where $u \geq 0$ is the initial surplus and $c > 0$ is the premium rate. The aggregate claims $S(t) = \sum_{i=1}^{N(t)} X_i$ is a compound Poisson process where $\{N(t), t \geq 0\}$ is a Poisson process denoting the number of claims up to time t, $B(t)$ independent of the aggregate claims process is a standard Brownian motion starting from zero, and $\sigma > 0$ is the diffusion volatility.
• $\{X_i, i \geq 1\}$ is a sequence of strictly positive random variables representing the individual claim sizes. $\{X_i\}_{i=1}^{\infty}$ are independent, $\{X_i\}_{i=3}^{\infty}$ are independent and distributed as the generic X;
• The interclaim times $\{W_i, i \geq 1\}$ is a sequence of exponential random variables. We denote by W_i the time between the $(i-1)$th and the ith claim for $i = 2, 3, \cdots$ W_1 and W_2 are exponentially distributed with parameter λ_1 and λ_2 respectively; $\{W_i\}_{i=3}^{\infty}$ are exponentially distributed with parameters λ;
• X_i and W_i are dependent with β_i which β_i i.e. the common exponentially decreasing rate;
• We assume that the bivariate random vectors (W_j, X_j) for $j \in \mathbb{N}^+$ are mutually independent but that the r.v.'s W_j and X_j are no longer independent;
• We assume the time arrival of the first claim W_1 has density function given by

$$f_{W_1}(t) = q\lambda_1 e^{-\lambda_1 t} + (1-q)\frac{e^{-\lambda_1 t}\int_t^{\infty} f_{W_2}(y)dy}{\int_0^{\infty} e^{-\lambda_1 y}\bar{F}_{W_2}(y)dy} = qf_{V_1}(t) + (1-q)f_{V_2}(t), \tag{2.2}$$

where $0 \leq q \leq 1$, $\lambda_1 > 0$, and the inter-occurrence time from the second claim W_2 has the density function f_{W_2} with survival function $\bar{F}_{W_2}$. When $q = 0$, f_{W_1} is a generalized equilibrium distribution, and when $q = 1$, W_1 is exponentially distributed, which is an intriguing choice for the time until the first claim occurs. The time from the second claim W_2 has density function given by $f_{W_2}(t) = \lambda_2 e^{-\lambda_2 t}$, and the subsequent claims inter-occurrence times $\{W_i\}_{i=3}^{\infty}$ are exponentially distributed with parameter λ, $W_i, i = 1, 2, 3, \cdots$ are independent. Since we assume that W_2 is exponentially distributed with parameter λ_2, the distribution of V_2 which defined in Equation (2.2) becomes

$$f_{V_2}(t) = (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)t}, t \geq 0;$$

• We consider a dependence structure between the claim amount and the interclaim time r.v.'s X_k and W_k that is mathematically tractable. We suppose the density of $X_k|W_k$ to be defined as a special mixture of two arbitrary density function f_1 and f_2 (with respective means (θ_1 and θ_2),i.e.

$$f_{X_k|W_k}(x) = e^{-\beta_k W_k} f_1(x) + (1 - e^{-\beta_k W_k}) f_2(x), x \geq 0, k = 1, 2, \cdots \tag{2.3}$$

from (2.3), the weight assigned to the c.d.f. F_1 is an exponentially decreasing function (at rate β_k) of the time elapsed since the last claim W_k. The resulting marginal distribution of X_k is

$$f_{X_k}(x) = \frac{\lambda_k}{\lambda_k + \beta_k} f_1(x) + \frac{\beta_k}{\lambda_k + \beta_k} f_2(x), k = 1, 2, \cdots \tag{2.4}$$

for $\beta_i = \beta, i = 3, 4, \cdots$

Let $\tau = inf_{t\geq 0}\{t, U(t) < 0\}$ be the time of ruin with $\tau = \infty$ if $U(t) \geq 0$ for all $t \geq 0$ (i.e. ruin does not occur). We denote the claim arrival times $\{T_j, j \in \mathbb{N}^+\}$ by $T_j = W_1 + W_2 + \cdots + W_j$. The deficit at ruin and the surplus just prior to ruin are respectively denoted by $|U_\tau|$ and $U_{\tau-}$. The Gerber-Shiu function $m^*(u)$ is defined as

$$m^*(u) = E[e^{-\delta\tau}\omega(U(\tau-), |U(\tau)|)I(\tau < \infty)|U(0) = u], \tag{2.5}$$

where $\delta \geq 0$ is the force of interest, $I(\cdot)$ is the indicator function, $\omega(x_1, x_2)$ is a nonnegative function of the surplus before ruin $U(\tau-)$ and the deficit at ruin $|U(\tau)|$. By observing the sample paths of $U(t)$, we know that ruin can be caused either by the oscillation of the Brownian motion or a downward jump. We decompose the Gerber-Shiu function as follows

$$m^*(u) = \phi_d^*(u) + \psi_d^*(u), \tag{2.6}$$

where

$$\phi_d^*(u) = E[e^{-\delta\tau}\omega(U(\tau-), |U(\tau)|)I(\tau < \infty, U(\tau) < 0)|U(0) = u],$$

is the Gerber-Shiu function when ruin is caused by a claim, and

$$\begin{aligned}\psi_d^*(u) &= E[e^{-\delta\tau}\omega(U(\tau-), |U(\tau)|)I(\tau < \infty, U(\tau) = 0)|U(0) = u] \\ &= \omega(0,0)E[e^{-\delta\tau}I(\tau < \infty, U(\tau) = 0)|U(0) = u],\end{aligned}$$

is the Gerber-Shiu function when ruin is caused by oscillation. We assume that $\omega(0, 0) = 1$. To guarantee that ruin is not a certain event, we assume that the following net profit condition holds

$$E[cW - X] > 0, \tag{2.7}$$

then (2.7) is equivalent to

$$\frac{c}{\lambda_k} - \frac{\lambda_k\theta_1 + \beta_k\theta_2}{\beta_k + \lambda_k} > 0, \tag{2.8}$$

it is clear that the increments $(X_j - cW_j), j \in \mathbb{N}^+$ of the surplus process are still independent.

A special setting of $\delta = 0$ and $\omega \equiv 1$ bring $\phi_d^*(u)$ and $\psi_d^*(u)$ to the ruin probabilities $\phi_w^*(u)$ and $\psi_w^*(u)$.

3 Lundberg-type equation

We analyze the roots of a Lundberg-type equation associated with the risk process. Let $U_0 = 0$, and for $n \in \mathbb{N}^+$, denote by U_n the surplus immediately after the nth claim, i.e.

$$\begin{aligned} U_n &= u + \sum_{i=1}^{n} (cW_i - X_i) + \sigma B\left(\sum_{i=1}^{n} W_i\right) \\ &= u + \sum_{i=1}^{n} (cW_i - X_i + \sigma B(W_i))\,. \end{aligned}$$

We seek for a number s such that the process

$$\{e^{-\delta T_n + sU_n}\}_{n=1}^{\infty},$$

is a martingale. Here the martingale condition which is called the Lundberg-type equation is

$$L(s) = E[e^{-\delta W_{n+1} + s(cW_{n+1} + \sigma B_{W_{n+1}} - X_{n+1})}] = 1. \tag{3.1}$$

By (2.5), we can calculate $L(s)$ as

$$\begin{aligned} L(s) &= E\{E(e^{-\delta W + s(cW + \sigma B_w - X)})|(W, X)\} \\ &= \int_0^{+\infty}\int_0^{+\infty} f_{W,X}(x,t)E(e^{-\delta t + s(ct + \sigma B_t - x)}|W = t, X = x)dxdt \\ &= \frac{\lambda \widetilde{f_1}(s)}{\lambda + \beta + \delta - sc - \frac{\sigma^2 s^2}{2}} + \frac{\lambda \widetilde{f_2}(s)}{\lambda + \delta - sc - \frac{\sigma^2 s^2}{2}} - \frac{\lambda \widetilde{f_2}(s)}{\lambda + \beta + \delta - sc - \frac{\sigma^2 s^2}{2}} \\ &= -\frac{2\lambda(\widetilde{f_1}(s) - \widetilde{f_2}(s))}{\sigma^2 \mathcal{A}_1(s)} - \frac{2\lambda \widetilde{f_2}(s)}{\sigma^2 \mathcal{A}_2(s)}, \end{aligned}$$

where

$$\mathcal{A}_1(s) = s^2 + \frac{2c}{\sigma^2} - \frac{2(\lambda + \delta + \beta)}{\sigma^2},$$

$$\mathcal{A}_2(s) = s^2 + \frac{2c}{\sigma^2} - \frac{2(\lambda + \delta)}{\sigma^2}.$$

Where $\widetilde{f_1}(s)$ and $\widetilde{f_2}(s)$ are the Laplace transforms of $f_1(s)$ and $f_2(s)$, i.e. $\widetilde{f_i}(s) = \int_0^\infty e^{-sx} f_i(x)dx$, (i=1,2). Then the Lundberg-type equation (2.1) reduces to

$$L(s) = \frac{\lambda(\lambda + \delta - sc - \frac{\sigma^2 s^2}{2})\widetilde{f_1}(s) + \lambda\beta\widetilde{f_2}(s)}{(\lambda + \beta + \delta - sc - \frac{\sigma^2 s^2}{2})(\lambda + \delta - sc - \frac{\sigma^2 s^2}{2})}. \tag{3.2}$$

When $\delta > 0$, (3.2) has exactly two roots, say $\rho_1(\delta)$, $\rho_2(\delta)$ with $Re(\rho_i(\delta)) > 0$ for $i = 1, 2$. And when $\delta = 0$, (3.2) has exactly one root, say $\rho_1(0)$, with $Re(\rho_1(0)) > 0$ and the second root $\rho_2(0) = 0$.

4 Integro-differential equations

In this section, we derive the integro-differential Equations satisfied by the Gerber–Shiu functions when ruin is caused by claims and by oscillations respectively. For these two kinds of delayed and perturbed risk model, we will discuss separately. The first-order delayed and perturbed risk model (Type I), a model such that after the first claim the process becomes ordinary. In this case,the occurrence time of the first claim is exponentially distributed with parameterλ_1, and the process becomes ordinary with claim inter-occurrence time following exponential distribution with parameter λ. And the second-order delayed and perturbed risk model(Type II) that the time occurrence of the first claim follows the distribution of Equation (2.2) and the time until the second claim is exponentially distributed with parameter λ_2.

Now we introduce some preliminary results. Let $Z(t) = -ct - \sigma B(t)$, which is a Brownian motion starting from zero with drift $-c$ and variance σ^2, note $\bar{Z(t)} = sup_{\{}0 \geq s \geq t\} Z(s)$. The first hitting time of the value $u > 0$ is defined by $\tau_u = inf\{t \geq 0 : Z(t) = u\}$. For $\delta \geq 0$, by [13], we have

$$E[e^{-\delta\tau_u}] = e^{-\eta u}, \tag{4.1}$$

when $\eta = \frac{c}{\sigma^2} + \sqrt{\frac{2\delta}{\sigma^2} + \frac{c^2}{\sigma^4}}$.

By [14],

$$E[e^{-sZ_{e_q}}] = \int_0^\infty E[e^{-sZ_t}] f_{e_q}(t)dt = \frac{q}{q - cs - \frac{\sigma^2 s^2}{2}},$$

and $\boldsymbol{e}_q$ is the qth unit column vector.

The roots of $q - cs - \frac{\sigma^2 s^2}{2}$ are $-v_1$ and v_2 which are $v_1 = \frac{c}{\sigma^2} + \sqrt{\frac{2q}{\sigma^2} + \frac{c^2}{\sigma^4}}, v_2 = -\frac{c}{\sigma^2} + \sqrt{\frac{2q}{\sigma^2} + \frac{c^2}{\sigma^4}}$.

We define the following potential measure for $\delta \geq 0$,

$$\mathcal{P}(u, dy, dx) = E[e^{-\delta W} I(Z(\bar{W}) < u, Z(W) \in dy, X \in dx)], u > 0, u > y, \tag{4.2}$$

which can be obtained by the following Lemma in applied probability.

By Lemma 2 of [15], for $0 \leq y < u$, the measure $P(u, dy, dx)$ has a density given by

$$\begin{aligned}\mathcal{P}(u, y, x) =& \frac{\lambda_2 \eta_1 \eta_2}{(\lambda_2 + \delta + \beta_2)(\eta_1 + \eta_2)}(e^{-\eta_1 y} - e^{-(\eta_1+\eta_2)u+\eta_2 y})(f_1(x) - f_2(x)) \\ &+ \frac{\lambda_2 \omega_1 \omega_2}{(\lambda_2 + \delta)(\omega_1 + \omega_2)}(e^{-\omega_1 y} - e^{-(\omega_1+\omega_2)u+\omega_2 y}) f_2(x),\end{aligned} \tag{4.3}$$

for $0 \leq y < u$, and

$$\mathcal{P}(u,y,x) = \frac{\lambda_2\eta_1\eta_2}{(\lambda_2+\delta+\beta_2)(\eta_1+\eta_2)}(e^{\eta_2 y} - e^{-(\eta_1+\eta_2)u+\eta_2 y})(f_1(x)-f_2(x)) + \frac{\lambda_2\omega_1\omega_2}{(\lambda_2+\delta)(\omega_1+\omega_2)}(e^{\omega_2 y} - e^{-(\omega_1+\omega_2)u+\omega_2 y})f_2(x), \tag{4.4}$$

for $y < 0$, where
$\eta_1 = \frac{c}{\sigma^2} + \sqrt{\frac{2(\lambda_2+\delta+\beta_2)}{\sigma^2} + \frac{c^2}{\sigma^4}}$, $\eta_2 = -\frac{c}{\sigma^2} + \sqrt{\frac{2(\lambda_2+\delta+\beta_2)}{\sigma^2} + \frac{c^2}{\sigma^4}}$,
$\omega_1 = \frac{c}{\sigma^2} + \sqrt{\frac{2(\lambda_2+\delta)}{\sigma^2} + \frac{c^2}{\sigma^4}}$, $\omega_2 = -\frac{c}{\sigma^2} + \sqrt{\frac{2(\lambda_2+\delta)}{\sigma^2} + \frac{c^2}{\sigma^4}}$.

Setting $\mathcal{D} := \frac{d}{du}(\cdot), \mathcal{D}^2 := \frac{d^2}{du^2}(\cdot)$, $\mathcal{I}$ the identity operator, we define the following differentiation operators:

$$\mathcal{P}_1(\mathcal{D}) = \mathcal{D}^2 + \frac{2c}{\sigma^2}\mathcal{D} - \frac{2(\lambda_2+\delta+\beta_2)}{\sigma^2}\mathcal{I} = (\mathcal{D}+\eta_1\mathcal{I})(\mathcal{D}-\eta_2\mathcal{I}),$$
$$\mathcal{P}_2(\mathcal{D}) = \mathcal{D}^2 + \frac{2c}{\sigma^2}\mathcal{D} - \frac{2(\lambda_2+\delta)}{\sigma^2}\mathcal{I} = (\mathcal{D}+\omega_1\mathcal{I})(\mathcal{D}-\omega_2\mathcal{I}),$$

$$\mathcal{A}_1(\mathcal{D}) = lim_{\lambda_2\to\lambda}\mathcal{P}_1(\mathcal{D})\mathcal{A}_2(\mathcal{D}) = lim_{\lambda_2\to\lambda}\mathcal{P}_2(\mathcal{D}).$$

Theorem 1. Under the assumptions of the first-order delayed and perturbed risk model (Type I) defined in Equation (2.1), the Gerber–Shiu function ϕ_d when the ruin is caused by claims satisfies the following integro-differential equation.

$$\mathcal{P}_1(\mathcal{D})\mathcal{P}_2(\mathcal{D})\phi_d(u) = -\frac{2\lambda_2}{\sigma^2}\mathcal{P}_2(\mathcal{D})(\sigma_{\omega,1}(u) - \sigma_{\omega,2}(u)) - \frac{2\lambda_2}{\sigma^2}\mathcal{P}_1(\mathcal{D})\sigma_{\omega,2}(u), \tag{4.5}$$

with the boundary conditions

$$\phi_d(0) = 0, \tag{4.6}$$

$$\phi_d^{''}(0) + \frac{2c}{\sigma^2}\phi_d^{'}(0) = -\frac{2\lambda_2}{\sigma^2}\omega_1(0), \tag{4.7}$$

where

$$\sigma_{\omega,1}(u) = \int_0^u \phi(u-x)f_1(x)dx + \omega_1(u); \sigma_{\omega,2}(u) = \int_0^u \phi(u-x)f_2(x)dx + \omega_2(u);$$
$$\omega_1(u) = \int_u^\infty \omega(u,x-u)f_1(x)dx; \omega_2(u) = \int_u^\infty \omega(u,x-u)f_2(x)dx.$$

The ordinary Gerber-Shiu function ϕ which satisfies

$$\mathcal{A}_1(\mathcal{D})\mathcal{A}_2(\mathcal{D})\phi(u) = -\frac{2\lambda}{\sigma^2}\mathcal{A}_2(\mathcal{D})(\sigma_{\omega,1}(u) - \sigma_2(u)) - \frac{2\lambda}{\sigma^2}\mathcal{A}_1(\mathcal{D})\sigma_{\omega,2}(u).$$

Proof. By conditioning on the time and amount of the first claim and recalling the definition of $p(u,y,x)$, we have

$$\begin{aligned}\phi_d(u) = &\int_0^\infty \int_{-\infty}^{u} \int_0^{u-y} e^{-\delta t} Pr(\bar{Z}(t) < u, Z(t) \in dy)\phi(u-y-x) f_{X_2,W_2}(x,t)dxdt \\ &+ \int_0^\infty \int_{-\infty}^{u} \int_{u-y}^{\infty} e^{-\delta t} Pr(\bar{Z}(t) < u, Z(t) \in dy)\omega(u-y, x-(u-y)) f_{X_2,W_2}(x,t)dxdt,\end{aligned} \tag{4.8}$$

by (4.3) (4.4), we can rewrite (4.9)as

$$\begin{aligned}\phi_d(u) =&\frac{\lambda_2\eta_1\eta_2}{(\lambda_2+\delta+\beta_2)(\eta_1+\eta_2)} \int_0^u (e^{-\eta_1 y} - e^{-(\eta_1+\eta_2)u+\eta_2 y})(\sigma_{\omega,1}(u-y) - \sigma_{\omega,2}(u-y))dy \\ &+ \frac{\lambda_2\omega_1\omega_2}{(\lambda_2+\delta)(\omega_1+\omega_2)} \int_0^u (e^{-\omega_1 y} - e^{-(\omega_1+\omega_2)u+\omega_2 y})\sigma_{\omega,2}(u-y)dy \\ &+ \frac{\lambda_2\eta_1\eta_2}{(\lambda_2+\delta+\beta_2)(\eta_1+\eta_2)} \int_{-\infty}^0 (e^{\eta_2 y} - e^{-(\eta_1+\eta_2)u+\eta_2 y})(\sigma_{\omega,1}(u-y) - \sigma_{\omega,2}(u-y))dy \\ &+ \frac{\lambda_2\omega_1\omega_2}{(\lambda_2+\delta)(\omega_1+\omega_2)} \int_{-\infty}^0 (e^{\omega_2 y} - e^{-(\omega_1+\omega_2)u+\omega_2 y})\sigma_{\omega,2}(u-y)dy,\end{aligned} \tag{4.9}$$

where

$$\sigma_{\omega,1}(u) = \int_0^u \phi(u-x)f_1(x)dx + \omega_1(u); \sigma_{\omega,2}(u) = \int_0^u \phi(u-x)f_2(x)dx + \omega_2(u);$$
$$\omega_1(u) = \int_u^\infty \omega(u, x-u)f_1(x)dx; \omega_2(u) = \int_u^\infty \omega(u, x-u)f_2(x)dx.$$

Let $s = u - y$ in (4.10), and we have

$$\begin{aligned}\phi_d(u) =&\frac{\lambda_2\eta_1\eta_2}{(\lambda_2+\delta+\beta_2)(\eta_1+\eta_2)} (\int_0^u (e^{-\eta_1(u-s)}(\sigma_{\omega,1}(s) - \sigma_{\omega,2}(s))ds + \int_u^\infty e^{\eta_2(u-s)}(\sigma_{\omega,1}(s) - \sigma_{\omega,2}(s))ds \\ &- \int_0^\infty e^{-\eta_1 u - \eta_2 s}(\sigma_{\omega,1}(s) - \sigma_{\omega,2}(s))ds) \\ &+ \frac{\lambda_2\omega_1\omega_2}{(\lambda_2+\delta)(\omega_1+\omega_2)} \left(\int_0^u e^{-\omega_1(u-s)}\sigma_{\omega,2}(s)ds + \int_u^\infty e^{\omega_2(u-s)}\sigma_{\omega,2}(s)ds - \int_0^\infty e^{-\omega_1 u - \omega_2 s}\sigma_{\omega,2}(s)ds \right),\end{aligned} \tag{4.10}$$

then setting $u = 0$ in (4.11) gives the boundary condition (4.7).

Applying the operator $\mathcal{P}_1(\mathcal{D})\mathcal{P}_2(\mathcal{D})$ to both sides of (4.11), we can obtain the integro-differential equation (4.6).

Next we differentiate the integral equation (4.11) w.r.t. u and setting $u = 0$, we can get

$$\phi_d'(0) = \frac{2\lambda_2}{\sigma^2} \int_0^\infty e^{-\eta_2 s}(\sigma_{\omega,1}(s) - \sigma_{\omega,2}(s))ds + \frac{2\lambda_2}{\sigma^2} \int_0^\infty e^{-\omega_2 s}\sigma_{\omega,2}(s)ds. \tag{4.11}$$

Differentiating (4.11) again and then setting $u = 0$, we can get

$$\phi_d''(0) = -\frac{2\lambda_2}{\sigma^2}\omega_1(0) - \frac{4\lambda_2 c}{\sigma^4}\int_0^\infty e^{-\eta_2 s}(\sigma_{\omega,1}(s) - \sigma_{\omega,2}(s))ds - \frac{4\lambda_2 c}{\sigma^4}\int_0^\infty e^{-\omega_2 s}\sigma_{\omega,2}(s)ds. \quad (4.12)$$

Finally, comparing (4.12) and (4.13) gives the boundary condition (4.8).

Theorem 2. Under the assumptions of the first-order delayed and perturbed risk model (Type I) defined in Equation (2.1), the Gerber–Shiu function ψ_d when the ruin is caused by oscillation satisfies the following integro-differential equation.

$$\mathcal{P}_1(\mathcal{D})\mathcal{P}_2(\mathcal{D})\psi_d(u) = -\frac{2\lambda_2}{\sigma^2}\mathcal{P}_2(\mathcal{D})(\sigma_{d,1}(u) - \sigma_{d,2}(u)) - \frac{2\lambda_2}{\sigma^2}\mathcal{P}_1(\mathcal{D})\sigma_{d,2}(u), \quad (4.13)$$

with the boundary conditions

$$\psi_d(0) = 1, \quad (4.14)$$

$$\psi_d''(0) + \frac{2c}{\sigma^2}\psi_d'(0) = \frac{2(\lambda_2 + \delta)}{\sigma^2}, \quad (4.15)$$

where

$$\sigma_{d,1}(u) = \int_0^u \psi(u-x)f_1(x)dx;$$
$$\sigma_{d,2}(u) = \int_0^u \psi(u-x)f_2(x)dx.$$

The ordinary Gerber-Shiu function ψ which satisfies

$$\mathcal{A}_1(\mathcal{D})\mathcal{A}_2(\mathcal{D})\psi(u) = -\frac{2\lambda}{\sigma^2}\mathcal{A}_2(\mathcal{D})(\sigma_{d,1}(u) - \sigma_2(u)) - \frac{2\lambda}{\sigma^2}\mathcal{A}_1(\mathcal{D})\sigma_{d,2}(u).$$

Proof. Let $\tau_u = inf\{t \geq 0 : Z(t) = u\}$, we have

$$E[e^{-\delta\tau_u}1(\tau_u < W_2)] = E[e^{-\delta\tau_u}E[1(\tau_u < W_2)]|Z(t)] = E[e^{-(\lambda_2+\delta)\tau_u}] = e^{-\omega_1 u},$$

where

$$\omega_1 = \frac{c}{\sigma^2} + \sqrt{\frac{2(\lambda_2+\delta)}{\sigma^2} + \frac{c^2}{\sigma^4}}.$$

Because of that ruin caused by oscillation may occur or not before the first claim, we have

$$\begin{aligned}\psi_d(u) =& \int_0^\infty \int_{-\infty}^u \int_0^{u-y} e^{-\delta t} Pr(\bar{Z}(t) < u, Z(t) \in dy)\psi(u-y-x)f_{X_2,W_2}(x,t)dxdt \\ &+ E[e^{-\delta\tau_u}I(\tau_u < W_2)] \\ =& e^{-\omega_1 u} + \int_{-\infty}^u \int_0^{u-y} \psi(u-y-x)\mathcal{P}(u,y,x)dxdy, \quad (4.16)\end{aligned}$$

when $y<0$,

$$\mathcal{P}(u,y,x)=\frac{\lambda_2\eta_1\eta_2}{(\lambda_2+\delta+\beta_2)(\eta_1+\eta_2)}(e^{\eta_2 y}-e^{-(\eta_1+\eta_2)u+\eta_2 y})(f_1(x)-f_2(x))$$
$$+\frac{\lambda_2\omega_1\omega_2}{(\lambda_2+\delta)(\omega_1+\omega_2)}(e^{\omega_2 y}-e^{-(\omega_1+\omega_2)u+\omega_2 y})f_2(x).$$

Then we can rewrite (4.17) and setting $s=u-y$, as

$$\psi_d(u)=e^{-\omega_1 u}+\frac{\lambda_2\eta_1\eta_2}{(\lambda_2+\delta+\beta_2)(\eta_1+\eta_2)}\Big[\int_u^{\infty}e^{\eta_2(u-s)}(\sigma_{d,1}(s)-\sigma_{d,2}(s))ds$$
$$-\int_0^{\infty}e^{-\eta_1 u-\eta_2 s}(\sigma_{d,1}(s)-\sigma_{d,2}(s))ds)\Big]$$
$$+\frac{\lambda_2\omega_1\omega_2}{(\lambda_2+\delta)(\omega_1+\omega_2)}\Big[\int_u^{\infty}e^{\omega_2(u-s)}\sigma_{d,2}(s)ds-\int_0^{\infty}e^{-\omega_1 u-\omega_2 s}\sigma_{d,2}(s)ds\Big],\tag{4.17}$$

then setting $u=0$ in (4.18) gives the boundary condition $\psi_d(0)=1$.

Applying the operator $\mathcal{P}_1(\mathcal{D})\mathcal{P}_2(\mathcal{D})$ to both sides of (4.18),we can obtain the equation (4.14).

We differentiate the integral equation (4.18) and setting $u=0$, we can get

$$\psi_d^{'}(0)=-\frac{c}{\sigma^2}-\sqrt{\frac{2(\lambda_2+\delta)}{\sigma^2}+\frac{c^2}{\sigma^4}}+\frac{2\lambda_2}{\sigma^2}\int_0^{\infty}e^{-\eta_2 s}(\sigma_{d,1}(s)-\sigma_{d,2}(s))ds+\frac{2\lambda_2}{\sigma^2}\int_0^{\infty}e^{-\omega_2 s}\sigma_{d,2}(s)ds.\tag{4.18}$$

Next, differentiating again and setting $u=0$, we can get

$$\psi_d^{''}(0)=\omega_1^2-\frac{4\lambda_2 c}{\sigma^4}\int_0^{\infty}e^{-\eta_2 s}(\sigma_{d,1}(s)-\sigma_{d,2}(s))ds-\frac{4\lambda_2 c}{\sigma^4}\int_0^{\infty}e^{-\omega_2 s}\sigma_{d,2}(s)ds.\tag{4.19}$$

Finally, comparing (4.19) and (4.20) gives the boundary condition (4.16).

Theorem 3. Under the assumptions of the second-order delayed and perturbed risk model (Type II) defined in Equation (2.1), the Gerber–Shiu function ϕ_d^* when the ruin is caused by claims satisfies the following integro-differential equation.

$$\mathcal{B}_1(\mathcal{D})\mathcal{B}_2(\mathcal{D})\mathcal{B}_{1e}(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})\phi_d^*(u)=$$
$$-\Big(q\frac{2\lambda_1}{\sigma^2}\mathcal{B}_2(\mathcal{D})\mathcal{B}_{1e}(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})+(1-q)\frac{2(\lambda_1+\lambda_2)}{\sigma^2}\mathcal{B}_1(\mathcal{D})\mathcal{B}_2(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})\Big)\Big(\sigma_{\omega,1}(u)-\sigma_{\omega,2}(u)\Big)$$
$$-\Big(q\frac{2\lambda_1}{\sigma^2}\mathcal{B}_1(\mathcal{D})\mathcal{B}_{1e}(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})+(1-q)\frac{2(\lambda_1+\lambda_2)}{\sigma^2}\mathcal{B}_1(\mathcal{D})\mathcal{B}_2(\mathcal{D})\mathcal{B}_{1e}(\mathcal{D})\Big)\sigma_{\omega,2}(u),\tag{4.20}$$

with the boundary conditions

$$\phi_d^*(0) = 0, \tag{4.21}$$

$$\phi_d^{*''}(0) + \frac{2c}{\sigma^2}\phi_d^{*'}(0) = -\Big(q\frac{2\lambda_1}{\sigma^2} + (1-q)\frac{2(\lambda_1+\lambda_2)}{\sigma^2}\Big)\omega_1(0), \tag{4.22}$$

where

$$\sigma_{\omega,1}(u) = \int_0^u \phi_d(u-x)f_1(x)dx + \omega_1(u); \sigma_{\omega,2}(u) = \int_0^u \phi_d(u-x)f_2(x)dx + \omega_2(u);$$
$$\omega_1(u) = \int_u^\infty \omega(u, x-u)f_1(x)dx; \omega_2(u) = \int_u^\infty \omega(u, x-u)f_2(x)dx.$$

Proof. By conditioning on the time and amount of the first claim and recalling the definition of $p(u, y, x)$, we have

$$\begin{aligned}
\phi_d^*(u) &= E\Big[e^{-\delta W_1}E[w(u-Z_{w1}, X_1-u+Z_{w1})1(X_1>u-Z_{w1}, \bar{Z}_{W1}<u)|(W_1,X_1)]\Big] \\
&+ E\Big[e^{-\delta W_1}E[\phi_d(u-Z_{w1}-X_1)1(X_1<u-Z_{w1}, \bar{Z}_{W1}<u)|(W_1,X_1)]\Big] \\
&= qE\Big[e^{-\delta V_1}E[w(u-Z_{V1}, X_1-u+Z_{V1})1(X_1>u-Z_{V1}, \bar{Z}_{V1}<u)|(V_1,X_1)]\Big] \\
&+ (1-q)E\Big[e^{-\delta V_2}E[w(u-Z_{V2}, X_1-u+Z_{V2})1(X_1>u-Z_{V2}, \bar{Z}_{V2}<u)|(V_2,X_1)]\Big] \\
&+ qE\Big[e^{-\delta V_1}E[\phi_d(u-Z_{V1}-X_1)1(X_1<u-Z_{V1}, \bar{Z}_{V1}<u)|(V_1,X_1)]\Big] \\
&+ (1-q)E\Big[e^{-\delta V_2}E[\phi_d(u-Z_{V2}-X_1)1(X_1<u-Z_{V2}, \bar{Z}_{V2}<u)|(V_2,X_1)]\Big] \\
&= q\phi_1(u) + (1-q)\phi_2(u).
\end{aligned} \tag{4.23}$$

Then,by (4.3) and (4.4), we have

$$\begin{aligned}
\phi_1(u) &= \int_{-\infty}^u \int_{u-y}^\infty \omega(u-y, x-(u-y))\mathcal{P}(u,y,x|\lambda_1,\beta_1)dxdy \\
&+ \int_{-\infty}^u \int_0^{u-y} \phi_d(u-y-x)\mathcal{P}(u,y,x|\lambda_1,\beta_1)dxdy \\
&= \frac{\lambda_1\xi_1\xi_2}{(\lambda_1+\delta+\beta_1)(\xi_1+\xi_2)}\int_0^u (e^{-\xi_1 y} - e^{-(\xi_1+\xi_2)u+\xi_2 y})(\sigma_{\omega,1}(u-y) - \sigma_{\omega,2}(u-y))dy \\
&+ \frac{\lambda_1\zeta_1\zeta_2}{(\lambda_1+\delta)(\zeta_1+\zeta_2)}\int_0^u (e^{-\zeta_1 y} - e^{-(\zeta_1+\zeta_2)u+\zeta_2 y})\sigma_{\omega,2}(u-y)dy \\
&+ \frac{\lambda_1\xi_1\xi_2}{(\lambda_1+\delta+\beta_1)(\xi_1+\xi_2)}\int_{-\infty}^0 (e^{\xi_2 y} - e^{-(\xi_1+\xi_2)u+\xi_2 y})(\sigma_{\omega,1}(u-y) - \sigma_{\omega,2}(u-y))dy \\
&+ \frac{\lambda_1\zeta_1\zeta_2}{(\lambda_1+\delta)(\zeta_1+\zeta_2)}\int_{-\infty}^0 (e^{\zeta_2 y} - e^{-(\zeta_1+\zeta_2)u+\zeta_2 y})\sigma_{\omega,2}(u-y)dy,
\end{aligned} \tag{4.24}$$

$$
\begin{aligned}
\phi_2(u) &= \int_{-\infty}^{u}\int_{u-y}^{\infty}\omega(u-y,x-(u-y))\mathcal{P}(u,y,x|\lambda_1+\lambda_2,\beta_1)dxdy \\
&+ \int_{-\infty}^{u}\int_{0}^{u-y}\phi_d(u-y-x)\mathcal{P}(u,y,x|\lambda_1+\lambda_2,\beta_1)dxdy \\
&= \frac{(\lambda_1+\lambda_2)\xi_1'\xi_2'}{(\lambda_1+\lambda_2+\delta+\beta_1)(\xi_1'+\xi_2')}\int_0^u (e^{-\xi_1' y}-e^{-(\xi_1'+\xi_2')u+\xi_2' y})(\sigma_{\omega,1}(u-y)-\sigma_{\omega,2}(u-y))dy \\
&+ \frac{(\lambda_1+\lambda_2)\zeta_1'\zeta_2'}{(\lambda_1+\lambda_2+\delta)(\omega_1'+\omega_2')}\int_0^u (e^{-\zeta_1' y}-e^{-(\zeta_1'+\zeta_2')u+\zeta_2' y})\sigma_{\omega,2}(u-y)dy \\
&+ \frac{(\lambda_1+\lambda_2)\xi_1'\xi_2'}{(\lambda_1+\lambda_2+\delta+\beta_1)(\xi_1'+\xi_2')}\int_{-\infty}^0 (e^{\xi_2' y}-e^{-(\xi_1'+\xi_2')u+\xi_2' y})(\sigma_{\omega,1}(u-y)-\sigma_{\omega,2}(u-y))dy \\
&+ \frac{(\lambda_1+\lambda_2)\zeta_1'\zeta_2'}{(\lambda_1+\lambda_2+\delta)(\zeta_1'+\zeta_2')}\int_{-\infty}^0 (e^{\zeta_2' y}-e^{-(\zeta_1'+\zeta_2')u+\zeta_2' y})\sigma_{\omega,2}(u-y)dy, \qquad (4.25)
\end{aligned}
$$

where
$\xi_1=\frac{c}{\sigma^2}+\sqrt{\frac{2(\lambda_1+\delta+\beta_1)}{\sigma^2}+\frac{c^2}{\sigma^4}}$, $\xi_2=-\frac{c}{\sigma^2}+\sqrt{\frac{2(\lambda_1+\delta+\beta_1)}{\sigma^2}+\frac{c^2}{\sigma^4}}$,
$\zeta_1=\frac{c}{\sigma^2}+\sqrt{\frac{2(\lambda_1+\delta)}{\sigma^2}+\frac{c^2}{\sigma^4}}$, $\zeta_2=-\frac{c}{\sigma^2}+\sqrt{\frac{2(\lambda_1+\delta)}{\sigma^2}+\frac{c^2}{\sigma^4}}$,
$\xi_1'=\frac{c}{\sigma^2}+\sqrt{\frac{2(\lambda_1+\lambda_2+\delta+\beta_1)}{\sigma^2}+\frac{c^2}{\sigma^4}}$, $\xi_2'=-\frac{c}{\sigma^2}+\sqrt{\frac{2(\lambda_1+\lambda_2+\delta+\beta_1)}{\sigma^2}+\frac{c^2}{\sigma^4}}$,
$\zeta_1'=\frac{c}{\sigma^2}+\sqrt{\frac{2(\lambda_1+\lambda_2+\delta)}{\sigma^2}+\frac{c^2}{\sigma^4}}$, $\zeta_2'=-\frac{c}{\sigma^2}+\sqrt{\frac{2(\lambda_1+\lambda_2+\delta)}{\sigma^2}+\frac{c^2}{\sigma^4}}$,

$$
\sigma_{\omega,1}(u)=\int_0^u\phi_d(u-x)f_1(x)dx+\omega_1(u), \sigma_{\omega,2}(u)=\int_0^u\phi_d(u-x)f_2(x)dx+\omega_2(u),
$$

$$
\omega_1(u)=\int_u^\infty\omega(u,x-u)f_1(x)dx, \omega_2(u)=\int_u^\infty\omega(u,x-u)f_2(x)dx.
$$

Setting $s=u-y$ in (4.25) and (4.26), we have

$$
\begin{aligned}
\phi_1(u) =&\frac{\lambda_1\xi_1\xi_2}{(\lambda_1+\delta+\beta_1)(\xi_1+\xi_2)}\Big(\int_0^u e^{-\xi_1(u-s)}(\sigma_{\omega,1}(s)-\sigma_{\omega,2}(s))ds+\int_u^\infty e^{\xi_2(u-s)}(\sigma_{\omega,1}(s)-\sigma_{\omega,2}(s))ds \\
&-\int_0^\infty e^{-\xi_1 u-\xi_2 s}(\sigma_{\omega,1}(s)-\sigma_{\omega,2}(s))ds)\Big) \\
&+\frac{\lambda_1\zeta_1\zeta_2}{(\lambda_1+\delta)(\zeta_1+\zeta_2)}\Big(\int_0^u e^{-\zeta_1(u-s)}\sigma_{\omega,2}(s)ds+\int_u^\infty e^{\zeta_2(u-s)}\sigma_{\omega,2}(s)ds-\int_0^\infty e^{-\zeta_1 u-\zeta_2 s}\sigma_{\omega,2}(s)ds\Big), \qquad (4.26)
\end{aligned}
$$

$$
\begin{aligned}
\phi_2(u) =&\frac{(\lambda_1+\lambda_2)\xi_1'\xi_2'}{(\lambda_1+\lambda_2+\delta+\beta_1)(\xi_1'+\xi_2')}\Big(\int_0^u e^{-\xi_1'(u-s)}(\sigma_{\omega,1}(s)-\sigma_{\omega,2}(s))ds+\int_u^\infty e^{\xi_2'(u-s)}(\sigma_{\omega,1}(s)-\sigma_{\omega,2}(s))ds \\
&-\int_0^\infty e^{-\xi_1' u-\xi_2' s}(\sigma_{\omega,1}(s)-\sigma_{\omega,2}(s))ds)\Big) \\
&+\frac{(\lambda_1+\lambda_2)\zeta_1'\zeta_2'}{(\lambda_1+\lambda_2+\delta)(\zeta_1'+\zeta_2')}\Big(\int_0^u e^{-\zeta_1'(u-s)}\sigma_{\omega,2}(s)ds+\int_u^\infty e^{\zeta_2'(u-s)}\sigma_{\omega,2}(s)ds-\int_0^\infty e^{-\zeta_1' u-\zeta_2' s}\sigma_{\omega,2}(s)ds\Big), \qquad (4.27)
\end{aligned}
$$

then setting $u=0$ in (4.27) and (4.28), we can get $\phi_1(0)=0$, $\phi_2(0)=0$, it's easy to get the boundary condition $\phi_d^*(0)=0$.

Applying the operator $\mathcal{B}_1(\mathcal{D})\mathcal{B}_2(\mathcal{D})$ to both sides of (4.27), and put the operator $\mathcal{B}_{1e}(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})$ to both sides of (4.28), we can obtain the follows

$$\mathcal{B}_1(\mathcal{D})\mathcal{B}_2(\mathcal{D})\phi_1(u)=-\frac{2\lambda_1}{\sigma^2}\mathcal{B}_2(\mathcal{D})(\sigma_{\omega,1}(u)-\sigma_{\omega,2}(u))-\frac{2\lambda_1}{\sigma^2}\mathcal{B}_1(\mathcal{D})\sigma_{\omega,2}(u), \tag{4.28}$$

$$\mathcal{B}_{1e}(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})\phi_2(u)=-\frac{2(\lambda_1+\lambda_2)}{\sigma^2}\mathcal{B}_{2e}(\mathcal{D})(\sigma_{\omega,1}(u)-\sigma_{\omega,2}(u))-\frac{2(\lambda_1+\lambda_2)}{\sigma^2}\mathcal{B}_{1e}(\mathcal{D})\sigma_{\omega,2}(u). \tag{4.29}$$

Applying the operator $q\mathcal{B}_{1e}(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})$ to equation (4.29), and added the operator $(1-q)\mathcal{B}_1(\mathcal{D})\mathcal{B}_2(\mathcal{D})$ to equation (4.30) drives the equation (4.21).
Similarly as Theorem 1 we differentiate the integral equation (4.27) and (4.28) respectively and set $u=0$, the results as follows

$$\phi_1^{'}(0)=\frac{2\lambda_1}{\sigma^2}\int_0^\infty e^{-\xi_2 s}(\sigma_{\omega,1}(s)-\sigma_{\omega,2}(s))ds+\frac{2\lambda_1}{\sigma^2}\int_0^\infty e^{-\zeta_2 s}\sigma_{\omega,2}(s)ds, \tag{4.30}$$

$$\phi_2^{'}(0)=\frac{2(\lambda_1+\lambda_2)}{\sigma^2}\int_0^\infty e^{-\xi_2^{'} s}(\sigma_{\omega,1}(s)-\sigma_{\omega,2}(s))ds+\frac{2(\lambda_1+\lambda_2)}{\sigma^2}\int_0^\infty e^{-\zeta_2^{'} s}\sigma_{\omega,2}(s)ds, \tag{4.31}$$

$$\phi_1^{''}(0)=-\frac{2\lambda_1}{\sigma^2}\zeta_1(0)-\frac{4\lambda_1 c}{\sigma^4}\int_0^\infty e^{-\xi_2 s}(\sigma_{\omega,1}(s)-\sigma_{\omega,2}(s))ds-\frac{4\lambda_1 c}{\sigma^4}\int_0^\infty e^{-\zeta_2 s}\sigma_{\omega,2}(s)ds, \tag{4.32}$$

$$\phi_2^{''}(0)=-\frac{2(\lambda_1+\lambda_2)}{\sigma^2}\zeta_1^{'}(0)-\frac{4(\lambda_1+\lambda_2)c}{\sigma^4}\int_0^\infty e^{-\xi_2^{'} s}(\sigma_{\omega,1}(s)-\sigma_{\omega,2}(s))ds-\frac{4(\lambda_1+\lambda_2)c}{\sigma^4}\int_0^\infty e^{-\zeta_2^{'} s}\sigma_{\omega,2}(s)ds. \tag{4.33}$$

Finally, comparing (4.31)-(4.34) gives the boundary condition (4.23).

Theorem 4. Under the assumptions of the second-order delayed and perturbed risk model (Type II) defined in Equation (2.1), the Gerber–Shiu function ψ_d^* when the ruin is caused by oscillation satisfies the following integro-differential equation.

$$\begin{aligned}&\mathcal{B}_1(\mathcal{D})\mathcal{B}_2(\mathcal{D})\mathcal{B}_{1e}(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})\psi_d^*(u)=\\&-\Big(q\frac{2\lambda_1}{\sigma^2}\mathcal{B}_2(\mathcal{D})\mathcal{B}_{1e}(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})+(1-q)\frac{2(\lambda_1+\lambda_2)}{\sigma^2}\mathcal{B}_1(\mathcal{D})\mathcal{B}_2(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})\Big)\Big(\sigma_{d,1}(u)-\sigma_{d,2}(u)\Big)\\&-\Big(q\frac{2\lambda_1}{\sigma^2}\mathcal{B}_1(\mathcal{D})\mathcal{B}_{1e}(\mathcal{D})\mathcal{B}_{2e}(\mathcal{D})+(1-q)\frac{2(\lambda_1+\lambda_2)}{\sigma^2}\mathcal{B}_1(\mathcal{D})\mathcal{B}_2(\mathcal{D})\mathcal{B}_{1e}(\mathcal{D})\Big)\sigma_{d,2}(u),\end{aligned} \tag{4.34}$$

with the boundary conditions

$$\psi_d^*(0)=1, \tag{4.35}$$

$$\psi_d^{*''}(0) + \frac{2c}{\sigma^2}\psi_d^{*'}(0) = q\frac{2(\lambda_1+\delta)}{\sigma^2} + (1-q)\frac{2(\lambda_1+\lambda_2+\delta)}{\sigma^2}, \tag{4.36}$$

where

$$\sigma_{d,1}(u) = \int_0^u \psi_d(u-x) f_1(x)dx,$$
$$\sigma_{d,2}(u) = \int_0^u \psi_d(u-x) f_2(x)dx.$$

Proof.Let $\tau_u = inf\{t \geq 0 : Z(t) = u\}$, we have

$$\begin{aligned} E[e^{-\delta\tau_u}1(\tau_u < W_2)] &= qE[e^{-\delta\tau_u}E[1(\tau_u < V_1)]|Z(t)] + (1-q)E[e^{-\delta\tau_u}E[1(\tau_u < V_2)]|Z(t)] \\ &= qE[e^{-(\lambda_1+\delta)\tau_u}] + (1-q)E[e^{-(\lambda_1+\lambda_2+\delta)\tau_u}] \\ &= qe^{-\zeta_1 u} + (1-q)e^{-\zeta_1' u}, \end{aligned} \tag{4.37}$$

where

$$\zeta_1 = \frac{c}{\sigma^2} + \sqrt{\frac{2(\lambda_1+\delta)}{\sigma^2} + \frac{c^2}{\sigma^4}}$$

$$\zeta_1' = \frac{c}{\sigma^2} + \sqrt{\frac{2(\lambda_1+\lambda_2+\delta)}{\sigma^2} + \frac{c^2}{\sigma^4}}.$$

From Theorem 2, we have

$$\begin{aligned} \psi_d^*(u) &= E[e^{-\delta\tau_u}1(\tau < W_1)] + E\Big[e^{-\delta W_1}E[\psi_d(u - Z_{W1} - X_1)1(X_1 < u - Z_{w1}, \bar{Z}_{w1} < u)|(W_1, X_1)]\Big] \\ &= q\Big[e^{-\omega_1 u} + \int_{-\infty}^u \int_0^{u-y} \psi_d(u-y-x)\mathcal{P}(u,y,x|\lambda_1,\beta_1)dxdy\Big] \\ &+ (1-q)\Big[e^{-\omega_1' u} + \int_{-\infty}^u \int_0^{u-y} \psi_d(u-y-x)\mathcal{P}(u,y,x|\lambda_1+\lambda_2,\beta_1)dxdy\Big] \\ &= q\psi_1(u) + (1-q)\psi_2(u). \end{aligned} \tag{4.38}$$

Setting $s = u - y$, we can rewrite $\psi_1(u), \psi_2(u)$ as follows

$$\begin{aligned} \psi_1(u) =& e^{-\zeta_1 u} + \frac{\lambda_1\xi_1\xi_2}{(\lambda_1+\delta+\beta_1)(\xi_1+\xi_2)}\Big[\int_u^\infty e^{\xi_2(u-s)}(\sigma_{d,1}(s) - \sigma_{d,2}(s))ds \\ &- \int_0^\infty e^{-\xi_1 u - \xi_2 s}(\sigma_{d,1}(s) - \sigma_{d,2}(s))ds\Big] \\ &+ \frac{\lambda_1\zeta_1\zeta_2}{(\lambda_1+\delta)(\zeta_1+\zeta_2)}\Big[\int_u^\infty e^{\zeta_2(u-s)}\sigma_{d,2}(s)ds - \int_0^\infty e^{-\zeta_1 u - \zeta_2 s}\sigma_{d,2}(s)ds\Big], \end{aligned} \tag{4.39}$$

$$\psi_2(u) = e^{-\zeta_1' u} + \frac{(\lambda_1+\lambda_2)\xi_1'\xi_2'}{(\lambda_1+\lambda_2+\delta+\beta_1)(\xi_1'+\xi_2')}\Big[\int_u^\infty e^{\xi_2'(u-s)}(\sigma_{d,1}(s)-\sigma_{d,2}(s))ds - \int_0^\infty e^{-\xi_1' u-\xi_2' s}(\sigma_{d,1}(s)-\sigma_{d,2}(s))ds)\Big] + \frac{(\lambda_1+\lambda_2)\zeta_1'\zeta_2'}{(\lambda_1+\lambda_2+\delta)(\zeta_1'+\zeta_2')}\Big[\int_u^\infty e^{\zeta_2'(u-s)}\sigma_{d,2}(s)ds - \int_0^\infty e^{-\zeta_1' u-\zeta_2' s}\sigma_{d,2}(s)ds\Big], \tag{4.40}$$

where

$$\sigma_{d,1}(u) = \int_0^u \psi_d(u-x)f_1(x)dx;$$
$$\sigma_{d,2}(u) = \int_0^u \psi_d(u-x)f_2(x)dx.$$

As in the proof of Theorem 3, the results (4.35)-(4.37) could be certificated.

5 Concluding remarks

In this paper, we show how to calculate the ruin probabilities with Gerber-Shiu function in a class of delayed and perturbed risk model (Type I and Type II). We derive the integro-differential equations of the Gerber-Shiu function when ruin is caused by claims and by oscillations in Type I and Type II respectively. It's worth pointing out that, if during the time between the last claim before time 0 and the first claim after time 0 is exponentially, the distribution at time 0 has the same exponential density, regardless of when the last claim before time 0 occurred, as follows from the memoryless property of the exponential distribution. Which is very useful in our disscussion. On the central problem of risk in the insurance industry, estimating the probability of ruin,can effectively avoid huge losses of many insurance companies. The proof of Theorem 1-4 are basically from [14] and [15], which play of importance role in our paper. In the course of discussing the case of the roots of the Lundberg-type equation , we mainly use the Rouche's theorem, which can be seen it's specific operation principle in detail in [12].

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