Original Research Article

An A- Stable block integrator scheme for the solution of first order system of IVP of ordinary differential equations

Abstract

In this article, we present an A- stable block integrator scheme for the solution of first order system of IVP of ordinary differential equations. The block scheme at a single integration step produces four approximate solution values of y_{n+1}, y_{n+2}, y_{n+3} and y_{n+4} at point x_{n+1}, x_{n+2}, x_{n+3} and x_{n+4} respectively. The order and stability property of the scheme are checked, the method is zero stable, A–stable and of order 6. Some test problems are solved with the proposed scheme and the result are compared with some existing method. The proposed method found to have advantages in terms of accuracy, minimum errors and less computational time. Hence, the method is recommended for solving first order system of IVP of ordinary differential equations.

Keyword: Zero stable, A-stable, IVPs, Order, Ordinary Differential equation.

1. Introduction

A number of real life issues that we encounter, especially in the field of engineering, sciences both physical, social and life sciences can be modeled in Mathematics as differential equations. Considering the vast application of differential equations, analytical and numerical methods are being developed to find solutions.

This study consider a method for solving system of first order initial value problem of ordinary differential equation of the form

$$y' = f(x,\ \hat{Y}), \qquad \hat{Y}(a) = \eta, \qquad a \le x \le b \tag{1}$$
$$\hat{Y} = (y_1, y_2, y_3, \dots \dots \dots y_n), \qquad \hat{\eta} = (\eta_1, \eta_2, \eta_3, \dots \dots \dots \eta_n)$$

Ordinary differential equations can be solved by analytical and numerical methods. The solutions generated by the analytical method are generally exact values, whereas with the numerical method an approximation is given as a solution approaching the real value (Fatokun *et al* 2005). Implicit numerical schemes proved to be more efficient in solving problems than explicit ones. Most common implicit algorithms are based on Backward Differentiation Formula (BDF). The BDF first appeared in the work of (Curtiss and Hirschfelder 1952). Researchers continued to improve on the BDF methods. Such improvements include the Extended Backward Differential Formula by (Cash, 1980), modified extended backward differential formula by (Cash, 2000), 2 point diagonally implicit super class of backward differentiation formula (Musa *et al.*, 2016), an order five implicit 3-step block method for solving ordinary differential equation (Yahaya and Sagir, 2013), Implicit r-point block backward differentiation formula for solving first- order stiff ODEs (Ibrahim *et al.*, 2007), a new variable step size block backward differentiation formula for solving stiff initial value problems (Suleiman *et al.*, 2013), a new fifth order implicit block method for solving first order stiff ordinary differential equations by (Musa *et al* 2014), an accurate computation of block hybrid method for solving stiff ODEs (Sagir, 2012), One-leg Multistep Method for first Order Differential Equations (Fatunla, 1984), Sagir (2014), Numerical Treatment of Block Method for the Solution of Ordinary Differential equations, Order and Convergence of Enhanced 3 point fully implicit super class of block backward differentiation formula for solving first order stiff initial value problems (Abdullahi & Musa, 2021). All the schemes mentioned above developed by different scholars possesses various sort of accuracy, minimum error and less computation time at one step or the other. However, there is need of developing a numerical algorithm that will solve system of ODEs with minimal computational time and converge faster, hence the motivation for this research.

2. Preliminaries

The following are definition of the basic terms used in this research.

Definition 1 (Ordinary Differential Equation)

A differential equation involving derivatives with respect to a single independent variable is called an ordinary differential equation.

Definition 2 (Order of the Differential Equation)

The order of a differential equation is the order of the highest differential coefficient present in the equation. A differential equation that has the second derivatives as the highest derivatives is said to be of second order.

Definition 3 (Solution of ODEs)

An equation containing dependent variable y and independent variable x and free from the derivative, which satisfies the differential equation is called the solution (primitive) of the differential equation.

Definition 4 (Initial Value problems)

A differential equation along with initial conditions on the unknown function and its derivatives, all given at the same value\ of the independent variable, constitutes an initial-value problem.

Definition 5 (Linear multi-step method)

A general linear multi-step method (LMM) has the following form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j}$$

Where $\alpha_j and\ \beta_j$ are constants and $\alpha_k \neq 0$. $\alpha_0 and\ \beta_0$ Cannot both be zero at the same time, for any linear k-step method, α_k is normalized to 1.

Definition 6 (Explicit and Implicit method)

The general linear multi-step method is said to be Explicit if $\beta_k = 0$, otherwise it is Implicit$(i.e\ \ \beta_k \neq 0)$.

Definition 7 (Linear Difference Operator L)

The linear difference operator L associated with the linear multi-step method is defined by

$$L\{y(x), h\} = \sum_{j=0}^{k}\left[\alpha_j y(x + jh) - h\beta_j y'(x + jh)\right].$$

Where $y(x)$ is an arbitrary test function and it is continuously differential on [a, b].Expanding $y(x + jh)$ and $y'(x + jh)$ as a Taylor's series about$x$, and expanding the common the terms yields:

$$L\{y(x), h\} = c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + \cdots + c_q h^q y^q(x_n) + \cdots$$

Where c_q are common constants given by

$$c_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k$$

$$c_1 = \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

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$$c_q = \frac{1}{q!}\left(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_q\right) - \frac{1}{(q-1)!}\left(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_k\right)$$

$$q = 2,3, \ldots \ldots \ldots$$

Definition 8 (Zero stability)

A linear multi-step method (2) is said to be zero stable if all the roots of first characteristics polynomial have modulus less than or equal to unity and those roots with modulus unity are simple.

Definition 9 (A- stability)

A linear multi-step method (2) is said to be A-stable if the stability region covers the entire negative half plane

Definition 10 (Block method)

A method is called Block if it computes more than one solution values at different points per step concurrently.

Let $Y_m\ and\ F_m$ be vectors defined by

$Y_m = [y_n, y_{n+1}, y_{n+2}, \dots, y_{n+r-1}]^t$.

$Y_m = [f_n, f_{n+1}, f_{n+2}, \dots, f_{n+r-1}]^t$.

Then a general k-block, r-point method is a matrix of finite difference equation of the form

$Y_m = \sum_{i=0}^{k} A_i Y_{m-1} + \sum_{i=0}^{k} B_i F_{m-1}$

$A_i and\ B_i$ are $r \times r$ coefficient matrices.

3. Analysis of the proposed Method

3.1 Formulation of the Method

Consider the general k-step linear multistep method in definition (5)

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_{kf_{n+k}} \qquad (2)$$

This study consider adding a future point in (2), with three step backward, to came-up with the formula of the form

$$\sum_{j=0}^{7} \alpha_j y_{n+j-3} = h\beta_{kf_{n+k-3}} \quad k = 1,2,3,4 \qquad (3)$$

The implicit four point method (3) is constructed using a linear operator L_i. To derive the four point, define the linear operator L_i associated with (3) as

$$L_i[y(x_n), h)]: \alpha_0\, y_{n-3} + \alpha_1\, y_{n-2+}\alpha_2\, y_{n-1} + \alpha_3\, y_n + \alpha_4\, y_{n+1} + \alpha_5\, y_{n+2} + \alpha_6\, y_{n+3} + \alpha_7\, y_{n+4} - h\beta_k f_{n+k-3} = 0 \quad k = i = 1,2,3,4 \qquad (4)$$

To derive the first, second, third, and fourth points as $y_{n+1},\ y_{n+2},\ y_{n+3}\ and\ \ y_{n+4}$ respectively Using Taylor series expansion in (4) and normalizing $\alpha_3 = 1$, $\alpha_4 = 1$, $\alpha_5 = 1$and$\alpha_6 = 1$ as coefficient's of the four points, $k = 1$, $k = 2$, $k = 3$ and $k = 4$respectively. To obtain

$$y_{n+1} = -\frac{1298881}{341643939}\,\mathrm{y}_{\mathrm{n}-3} + \frac{341643939}{569406565}\mathrm{y}_{\mathrm{n}-2} - \frac{72003623}{113881313}\,\mathrm{y}_{\mathrm{n}-1} + \frac{426060731}{341643939}\mathrm{y}_{\mathrm{n}} + \frac{6274637}{16268759}\mathrm{y}_{\mathrm{n}+2} - \frac{143998979}{1708219695}\mathrm{y}_{\mathrm{n}+3} + \frac{1847955}{113881313}\mathrm{y}_{\mathrm{n}+4} - \frac{9603792}{113881313}\mathrm{f}_{\mathrm{n}-2}$$

$$y_{n+2} = -\frac{79696}{845265}\,\mathrm{y}_{\mathrm{n}-3} + \frac{41929759}{9861425}\mathrm{y}_{\mathrm{n}-2} - \frac{68414023}{3944570}\,\mathrm{y}_{\mathrm{n}-1} + \frac{189894686}{5916855}\mathrm{y}_{\mathrm{n}} - \frac{7210474}{394457}\mathrm{y}_{\mathrm{n}+1} + \frac{21582821}{59168550}\mathrm{y}_{\mathrm{n}+3} + \frac{14016}{1972285}\mathrm{y}_{\mathrm{n}+4} + \frac{19789614}{1972285}\mathrm{f}_{\mathrm{n}-1} \qquad (5)$$

$$y_{n+3} = \frac{70450}{1797393}\,\mathrm{y}_{\mathrm{n}-3} - \frac{1295843}{1198262}\mathrm{y}_{\mathrm{n}-2} + \frac{5593225}{599131}\,\mathrm{y}_{\mathrm{n}-1} + \frac{676840}{105729}\mathrm{y}_{\mathrm{n}} - \frac{11495780}{599131}\mathrm{y}_{\mathrm{n}+1} + \frac{6496015}{1198262}\mathrm{y}_{\mathrm{n}+2} + \frac{42690}{599131}\mathrm{y}_{\mathrm{n}+4} + \frac{845710}{46087}\mathrm{f}_{\mathrm{n}}$$

$$y_{n+4} = -\frac{338687}{348237}\,\mathrm{y}_{\mathrm{n}-3} - \frac{353855969}{77076456}\mathrm{y}_{\mathrm{n}-2} + \frac{2326014617}{19269114}\,\mathrm{y}_{\mathrm{n}-1} - \frac{4938738481}{115614684}\mathrm{y}_{\mathrm{n}} + \frac{1117145237}{19269114}\mathrm{y}_{\mathrm{n}+1} - \frac{11296250177}{77076456}\mathrm{y}_{\mathrm{n}+2} + \frac{495749336}{28903671}\mathrm{y}_{\mathrm{n}+3} + \frac{951570371}{3211519}\mathrm{f}_{\mathrm{n}+1}$$

3.2 Order of the Method

In this section, we derive the order of the methods (5). It can be transform to a general matrix form as follows

$$\sum_{j=0}^{1} C_j^* Y_{m+j-1} = h \sum_{j=0}^{1} D_j^* Y_{m+j-1},$$

Let C_0^*, C_1^*, D_0^* and D_1^* be block matrices defined by

$C_0^* = [C_{0,} C_1, C_2, C_3], C_1^* = [C_4, C_5, C_6, C_7],\ \ D_0^* = [D_0, D_1, D_2, D_3], \mathrm{D}_1^* = [D_4, D_5, D_6, , D_7]$

Where C_0^*, C_1^*, D_0^* and D_1^* are square matrices and $\mathrm{Y}_{\mathrm{m}-1}, \mathrm{Y}_{\mathrm{m},} \mathrm{F}_{\mathrm{m}-1}$ and F_{m} are column vectors defined by

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \\ y_{3m+4} \end{bmatrix}, \quad Y_{m-1} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \\ y_{3(m-1)+4} \end{bmatrix}, \quad F_{m-1} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_{3(m-1)+1} \\ f_{3(m-1)+2} \\ f_{3(m-1)+3} \\ f_{3(m-1)+4} \end{bmatrix}$$

$$F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} = \begin{bmatrix} f_{3m+1} \\ f_{3m+2} \\ f_{3m+3} \\ f_{3m+4} \end{bmatrix}$$

Thus, equations (5) can be rewritten as

$$\begin{bmatrix} -\frac{1298881}{341643939} & \frac{341643939}{569406565} & -\frac{72003623}{113881313} & \frac{42606073}{34164393} \\ -\frac{79696}{845265} & \frac{41929759}{9861425} & -\frac{68414023}{3944570} & \frac{18989468}{5916855} \\ \frac{70450}{1797393} & -\frac{1295843}{1198262} & \frac{5593225}{599131} & \frac{676840}{105729} \\ -\frac{338687}{348237} & -\frac{353855969}{77076456} & \frac{2326014617}{19269114} & -\frac{4938848}{1156146} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} +$$

$$\begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{170821969} & -\frac{9603792}{1138813} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145231}{19269114} & \frac{7129625017}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = h$$

$$\begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{84571}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \qquad (6)$$

From the (6) we have

$$C_0^* = \begin{bmatrix} -\frac{1298881}{341643939} & \frac{341643939}{569406565} & -\frac{72003623}{113881313} & \frac{42606073}{34164393} \\ -\frac{79696}{845265} & \frac{41929759}{9861425} & -\frac{68414023}{3944570} & \frac{18989468}{5916855} \\ \frac{70450}{1797393} & -\frac{1295843}{1198262} & \frac{5593225}{599131} & \frac{676840}{105729} \\ -\frac{338687}{348237} & -\frac{353855969}{77076456} & \frac{2326014617}{19269114} & -\frac{4938848}{1156146} \end{bmatrix}$$

$$C_1^* = \begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{170821969} & -\frac{9603792}{1138813} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145231}{19269114} & \frac{7129625017}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix}$$

$$D_0^* = \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{84571}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad D_1^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix}$$

Where

$$C_0 = \begin{bmatrix} -\frac{1298881}{3416439} \\ -\frac{79696}{845265} \\ \frac{70450}{1797393} \\ -\frac{338687}{348237} \end{bmatrix} C_1 = \begin{bmatrix} \frac{34164393}{56940656} \\ -\frac{4192975}{9861425} \\ -\frac{1295843}{1198262} \\ -\frac{3538559}{348237} \end{bmatrix} C_2 = \begin{bmatrix} -\frac{7200362}{1138813} \\ -\frac{6841402}{3944570} \\ \frac{5593225}{599131} \\ \frac{23260146}{19269114} \end{bmatrix} C_3 = \begin{bmatrix} \frac{42606073}{34164393} \\ \frac{18988946}{5916855} \\ \frac{676840}{105729} \\ -\frac{4938848}{1156146} \end{bmatrix}$$

$$C_4 = \begin{bmatrix} 1 \\ \frac{7210474}{394457} \\ \frac{11495780}{599131} \\ -\frac{11171452}{1926911} \end{bmatrix} C_5 = \begin{bmatrix} -\frac{6274637}{1626875} \\ 1 \\ -\frac{6496015}{1198262} \\ \frac{112962507}{77076456} \end{bmatrix} C_6 = \begin{bmatrix} \frac{14399897}{17082196} \\ -\frac{2158282}{5916855} \\ 1 \\ -\frac{4957493}{2890367} \end{bmatrix} C_7 = \begin{bmatrix} -\frac{9603792}{1138813} \\ -\frac{14016}{1972285} \\ -\frac{42690}{599131} \\ 1 \end{bmatrix}$$

$$D_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_1 = \begin{bmatrix} \frac{9603792}{1138813} \\ 0 \\ 0 \\ 0 \end{bmatrix} D_2 = \begin{bmatrix} 0 \\ \frac{1978961}{197228} \\ 0 \\ 0 \end{bmatrix} D_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{84571}{46087} \\ 0 \end{bmatrix} D_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{9515703}{3211519} \end{bmatrix}$$

$$D_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{9515703}{3211519} \end{bmatrix} D_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From definatio()the order of the block method (5) and its associated linear operator are given by

$$L[y(x); h] = \sum_{j=0}^{7} [C_j y(x + jh)] - h \sum_{j=0}^{7} [D_j y'(x + jh)]$$

Where p is unique integer such that

$E_q = 0, \; q = 0,1, \ldots \; p$ and $E_{p+1} \neq 0$, where the E_q are constant Matrix.
With

$$E_0 = \sum_{J=0}^{7} C_J = 0,$$

$$E_1 = \sum_{J=0}^{7} \left[jC_j - 2D_j \right] = 0,$$

$$E_2 = \sum_{J=0}^{7} \left[\frac{1}{2!} j^2 C_j - 2jD_j \right] = 0,$$

$$E_3 = \sum_{J=0}^{7} \left[\frac{1}{3!} j^3 C_j - 2\frac{1}{2!} j^2 D_j \right] = 0,$$

$$E_4 = \sum_{J=0}^{7} \left[\frac{1}{4!} j^4 C_j - 2\frac{1}{3!} j^3 D_j \right] = 0,$$

$$E_5 = \sum_{J=0}^{7} \left[\frac{1}{5!} j^5 C_j - 2\frac{1}{4!} j^4 D_j \right] = 0,$$

$$E_6 = \sum_{J=0}^{7} \left[\frac{1}{6!} j^6 C_j - 2\frac{1}{5!} j^5 D_j \right] = 0$$

$$E_7 = \sum_{J=0}^{7} \left[\frac{1}{7!} j^7 C_j - 2\frac{1}{6!} j^6 D_j \right] \neq 0$$

Therefore, the method is of order 6, with error constant as: $E_7 = \begin{bmatrix} -\frac{210}{829358} \\ \frac{324}{3184255} \\ -\frac{981}{692640} \\ \frac{563}{594758} \end{bmatrix}$ (7)

3.3 Zero Stability of the Method.

The method (5) is converted into matrix form as:

$$\begin{bmatrix} 1 & -\frac{6274637}{162687591} & \frac{143998979}{170821969} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{7129625017}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = h \begin{bmatrix} \frac{1298881}{341643939} & -\frac{341643939}{569406565} & \frac{72003623}{113881313} & -\frac{4260607}{3416439} \\ \frac{79696}{845265} & -\frac{41929759}{9861425} & \frac{68414023}{3944570} & -\frac{1898946}{591685} \\ -\frac{70450}{1797393} & \frac{1295843}{1198262} & -\frac{5593225}{599131} & -\frac{676840}{105729} \\ \frac{338687}{348237} & \frac{353855969}{77076456} & -\frac{2326014617}{19269114} & \frac{4938848}{11561468} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + h \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{84571}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \quad (8)$$

The equation above can be written in matrix form as:

$$A_0 Y_m = A_1 Y_{m-1} + h(B_0 F_{m-1} + B_1 F_m) \quad (9)$$

Where

$$A_0 = \begin{bmatrix} 1 & -\frac{6274637}{162687591} & \frac{143998979}{170821969} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{7129625017}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} \frac{1298881}{341643939} & -\frac{341643939}{569406565} & \frac{72003623}{113881313} & -\frac{4260607\ldots}{3416439\ldots} \\ \frac{79696}{845265} & -\frac{41929759}{9861425} & \frac{68414023}{3944570} & -\frac{18989468\ldots}{5916855\ldots} \\ -\frac{70450}{1797393} & \frac{1295843}{1198262} & -\frac{5593225}{599131} & -\frac{676840\ldots}{105729\ldots} \\ \frac{338687}{348237} & \frac{353855969}{77076456} & -\frac{2326014617}{19269114} & \frac{4938848\ldots}{11561468\ldots} \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{84571\ldots}{46087\ldots} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix}$$

Y_{m-1} , Y_m , F_{m-1} and F_m are column vectors defined by

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \\ y_{3m+4} \end{bmatrix}, \quad Y_{m-1} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \\ y_{3(m-1)+4} \end{bmatrix}, \quad F_{m-1} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_{3(m-1)+1} \\ f_{3(m-1)+2} \\ f_{3(m-1)+3} \\ f_{3(m-1)+4} \end{bmatrix}$$

$$F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} = \begin{bmatrix} f_{3m+1} \\ f_{3m+2} \\ f_{3m+3} \\ f_{3m+4} \end{bmatrix}$$

Substituting scalar test equation $y' = \lambda y$ ($\lambda < 0$, λ complex) into (9) and using $\lambda h = \bar{h}$ gives

$$A_0 Y_m = A_1 Y_{m-1} + \bar{h}(B_0 Y_{m-1} + B_1 Y_m) \qquad (10)$$

The stability polynomial of (5) is obtained by evaluating
$det\left[(A_0 - \bar{h}B_1)t - (A_1 + \bar{h}B_0)\right] = 0$
to get

$$R(\bar{h}, t) = \frac{996090316807547559435103303 3}{13297532593636682035736546 0}h - \frac{446737709680296868675844731429}{1063802607490934562858923 68}t - \frac{267896832298507582085725524 9}{664876629681834101786827 3}t^4h - \frac{193733184956304804387420096}{6648766296818341017868273}t^3h^2 - \frac{2560816946243088126164260 8}{6648766296818341017868273}h^2 + \frac{638171663697310966422440976921}{106380260749093456285892368}t^2 - \frac{121326420211773039356715312 7537}{19946298890455023053604819 0}th - \frac{567630082507188660567238515954 1}{398925977809100461072096380}t^2h - \frac{156704338863926834777833911288 6}{33243831484091705089341365}t^3h + \frac{636761357124782350302383414 4}{511443561293718539836021}t^2h^2 - \frac{28326438751226632796183511 36}{6648766296818341017868273}th^2 + \frac{305957123435183189886671552 47}{531901303745467281429461840} - \frac{921565130889493728528082 09}{511443561293718539836021 0}t^4 - \frac{153560628738985568751170538 3}{8350098959897445548343 20}t^3 \qquad (11)$$

By putting $\bar{h} = 0$ in (11), we obtain the first characteristic polynomial as

$$R(0,t) = -\frac{446737709680296868675844731429}{106380260749093456285892368}t + \frac{638171663697310966422440976921}{106380260749093456285892368}t^2 - \frac{153560628738985568751170538 3}{835009895989744554834320}t^3 - \frac{921565130889493728528082 09}{511443561293718539836021 0}t^4 + \frac{30595712343518318988667155247}{53190130374546728142946184 0} \quad (12)$$

Since, the roots of (12) are $t_1 = 1$ *and* $t_2, t_3, t_4 \leq 0$
Therefore, the method (5) is zero Stable by definition (8).

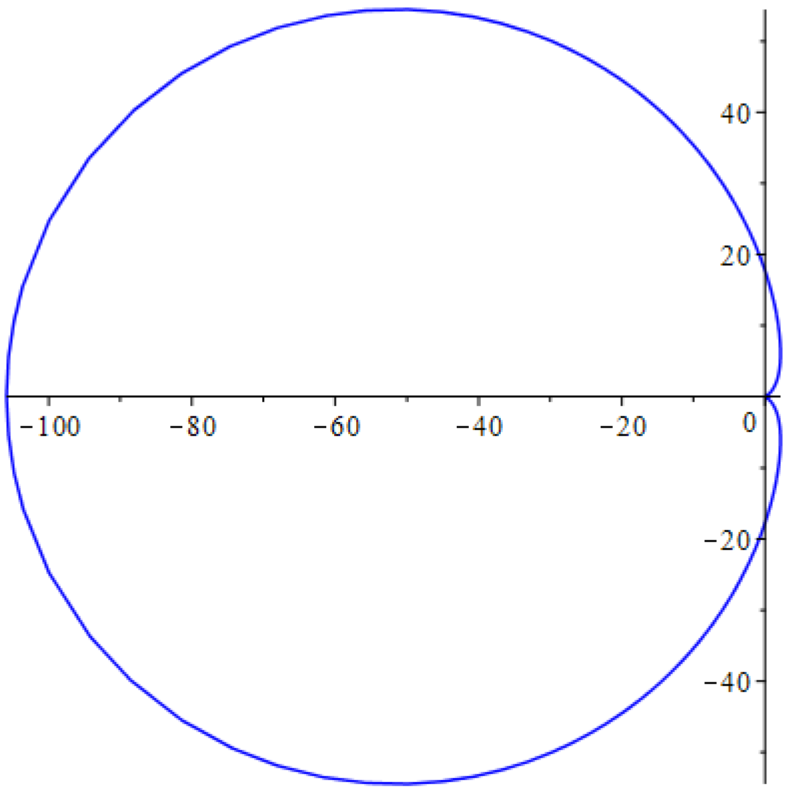

Figure1. The A-stability region of the proposed scheme ABISBDF.
The new method is A-stable by definition (9)

4 Implementation of the Method

4.1 Test Problems

To validate the method developed, the following stiff IVPs are solved.

Problem 1 : $y_1' = y_2 y_1(0) = 1$
$y_2' = y_1 y_2(0) = 1 \quad 0 \leq x \leq 100$
Exact Solution
$y_1(x) = e^x$

$y_2(x) = e^x$
Source: (Bronson, 1973)

$Problem\ 2$: $y_1' = 198y_1 + 199y_1 y_1(0) = 1$ $\quad 0 \le x \le 10$

$y_2' = -398y_1 - 399y_2 \quad y_2(0) = -1$
Exact solution
$y_1(x) = e^{-x}$

$y_2(x) = -e^{-x}$
Eigen values -1 and -200
Source: (Ibrahim *et al.*, 2007);

$Problem\ 3$: $y_1' = y_2\ y_1(0) = 0$ $\quad 0 \le x \le 20$

$y_2' = -y_1\ y_2(0) = 1$
Exact solution
$y_1(x) = sinx$

$y_2(x) = cosx$
Source: (shampine*et al*, 1975)

Problem 4 : $y_1' = y_2 y_1(0) = 0$
$y_2' = -2y_2 y_2(0) = 0 \quad 0 \le x \le 4$л
$y_3' = y_2 + 2y_3 y_3(0) = 1$
Exact Solution
$y_1(x) = 2cosx + 6sinx - 6x - 2$
$y_2(x) = -2sinx + 6cosx - 6$
$y_3(x) = 2sinx - 2cosx + 3$
Source: (Sulaiman,1989)

4.2 Numerical Results

The problems sampled in this research are solved using the developed scheme. The results are tabulated, compared; and the graphs highlighting the performance of these methods are plotted. The acronyms below are used in the tables.
h = step-size;
MAXE = Maximum Error;
T=Time in second;
3ESBBDF = 3 Point enhanced fully implicit Super Class of Block Backward Differentiation
$F_1$3ESBBDF = Family of block 3 Super class of Block Backward Differentiation
ABISBDF = A-stable block integrator scheme of Backward Differentiation Formula for solving Stiff IVPs.

Table 1: Comparison of Errors between Proposed Method and other Methods for Problem 1 & 2

Numerical Result for Problem 1				Numerical Result for Problem 2			
h	Method	MAXE	TIME	h	Method	MAXE	TIME
10^{-2}	$F_1$3SBBDF	3.30736e-002	4.23434e-1	10^{-2}	$F_1$3SBBDF	3.23032e-002	3.77590e-002
	3ESBSBDF	3.51456e-002	3.52416e-4		3ESBSBDF	3.98707e-002	2.63337e-002
	ABISBDF	5.83217e-004	4.23441e-5		ABISBDF	5.83217e-003	5.68676e-002
10^{-3}	$F_1$3SBBDF	5.41853e-003	1.81850e-3	10^{-3}	$F_1$3SBBDF	4.76165e-003	5.66636e-001
	3ESBSBDF	5.20191e-003	2.50367e-3		3ESBSBDF	4.40956e-003	2.60816e-001
	ABISBDF	6.95338e-005	4.65467e-4		ABISBDF	6.05338e-005	5.64515e-001
10^{-4}	$F_1$3SBBDF	5.44701e-005	1.71443e-2	10^{-4}	$F_1$3SBBDF	4.66516e-004	5.64385e-001
	3ESBSBDF	5.20417e-005	2.36918e-2		3ESBSBDF	5.08942e-005	2.60725e-001
	ABISBDF	6.95692e-007	4.48433e-3		4BSBDF	6.26692e-007	5.68143e+000
10^{-5}	$F_1$3SBBDF	5.44971e-007	1.70042e-1	10^{-5}	$F_1$3SBBDF	4.68707e005	5.63788e+000
	3ESBSBDF	5.25030e-007	2.34808e-1		3ESBSBDF	5.21534e-007	2.60597e+000
	ABISBDF	6.959740e-009	4.58687e-2		ABISBDF	6.32740e-009	5.59821e+001
10^{-6}	$F_1$3SBBDF	5.44998e-009	1.70308e0	10^{-6}	$F_1$3SBBDF	4.69123e-006	5.65356e+001
	3ESBSBDF	5.25648e-009	2.35791e0		3ESBSBDF	5.89872e-009	2.60700e+001
	ABISBDF	7.186362e-011	4.23434e-1		ABISBDF	6.33362e-011	5.53567e+002

Table 2: Comparison of Errors between Proposed Method and other Methods for Problem 3 & 4

Numerical Result for Problem 3				Numerical Result for Problem 4			
h	Method	MAXE	TIME	h	Method	MAXE	TIME
10^{-2}	$F_1$3SBBDF	2.07208e-002	1.37500e-2	10^{-2}	$F_1$3SBBDF	2.83032e-002	3.67590e-002
	3ESBSBDF	2.54347e-002	1.20394e-3		3ESBSBDF	2.48705e-002	2.63337e-002
	ABISBDF	2.83117e-004	7.36289e-2		ABISBDF	3.83217e-003	5.58676e-002
10^{-3}	$F_1$3SBBDF	3.20160e-004	2.72200e-2	10^{-3}	$F_1$3SBBDF	3.76163e-003	8.56636e-002
	3ESBSBDF	3.02893e-004	1.25972e-2		3ESBSBDF	3.40956e-003	2.60816e-001
	ABISBDF	4.05338e-006	5.81512e-2		ABISBDF	4.05338e-005	5.54515e-001
10^{-4}	$F_1$3SBBDF	3.20233e-006	2.02700e-1	10^{-4}	$F_1$3SBBDF	3.76514e-005	8.54385e-001
	3ESBSBDF	3.09895e-006	1.25148e-1		3ESBSBDF	3.48942e-005	2.60725e+000
	ABISBDF	4.26592e-008	5.81491e-1		ABISBDF	4.26690e-007	5.58143e-001
10^{-5}	$F_1$3SBBDF	3.20261e-008	1.92600e0	10^{-5}	$F_1$3SBBDF	3.70705e005	8.53788e+000
	3ESBSBDF	3.10157e-008	1.25471e0		3ESBSBDF	3.58532e-005	2.60597e+001
	ABISBDF	4.32640e-010	5.81122e0		ABISBDF	4.32740e-009	5.49821e+000
10^{-6}	$F_1$3SBBDF	3.20263e-010	1.91700e1	10^{-6}	$F_1$3SBBDF	3.71121e-007	8.53356e+001
	3ESBSBDF	3.41129e-010	1.24892e1		3ESBSBDF	3.69872e-007	2.60700e+002
	ABISBDF	4.33262e-012	5.79987e1		ABISBDF	4.3335e-009	5.43567e+001

Similarly, to highlight the performance of the proposed methods, ABISBDF in relation to the other methods, 3ESBSBDF and $F_1$3SBBDF. The graphs of *Log10(MAXE)* against the step size, *h* for the 4 problems are plotted accordingly as shown below:

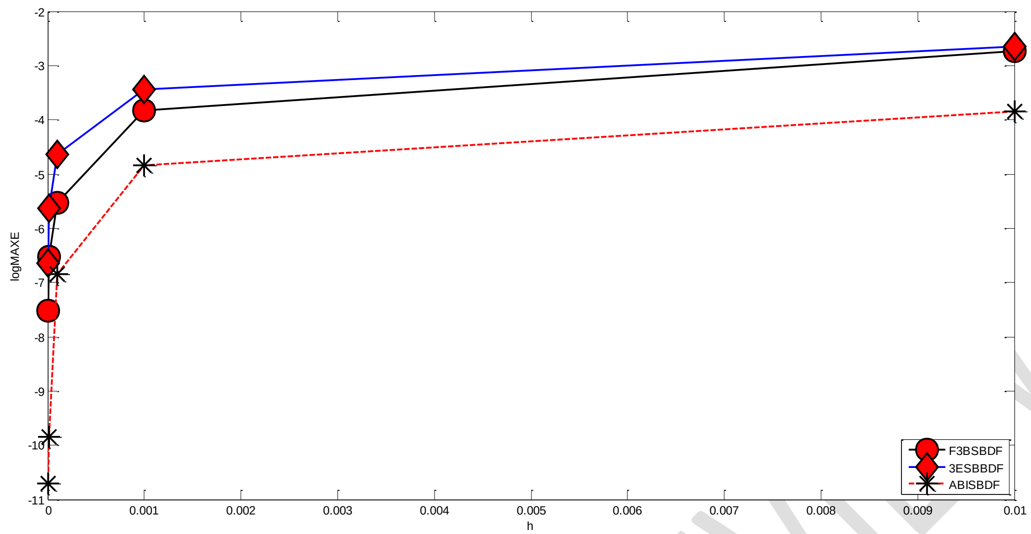

Figure 2: Graph of $\mathrm{Log}_{10}(MAXE)$ against h for problem 1

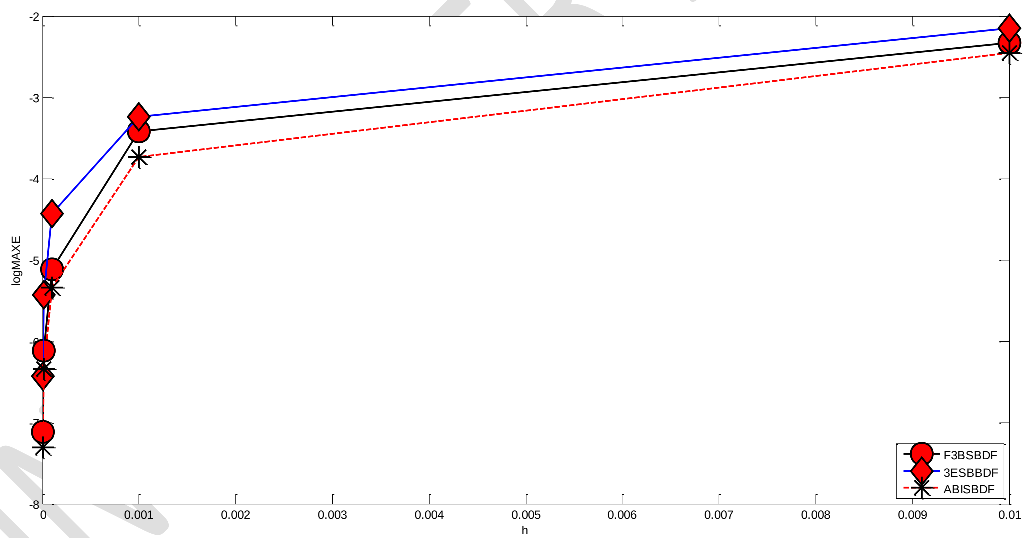

Figure 3: Graph of $\mathrm{Log}_{10}(MAXE)$ against h for problem 2

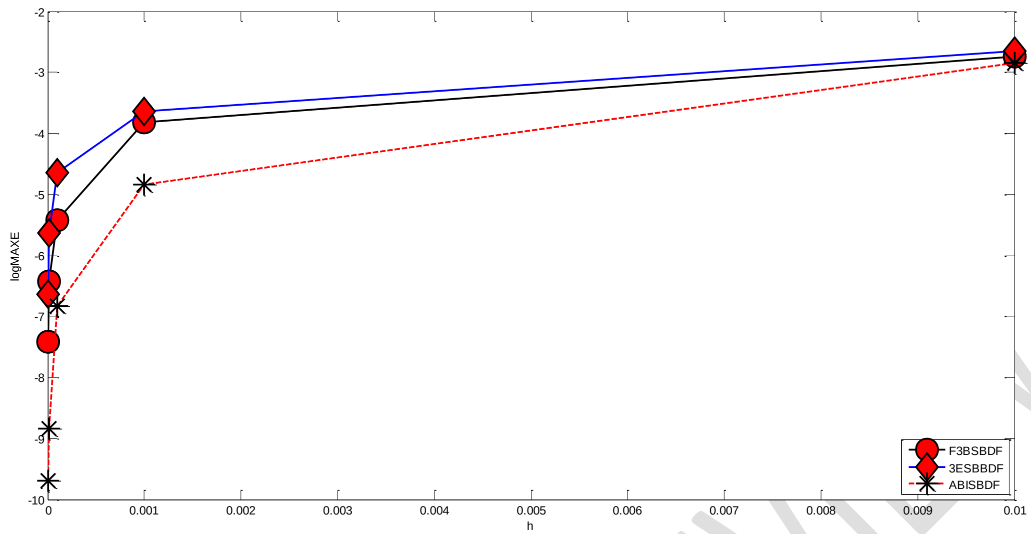

Figure 4: Graph of $\text{Log}_{10}(MAXE)$ against h for problem 3

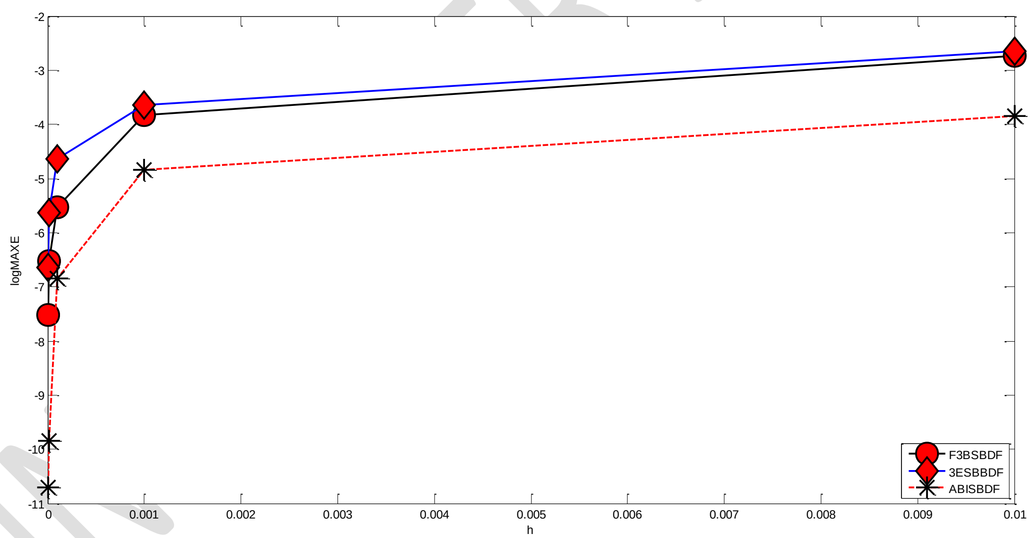

Figure 5: Graph of $\text{Log}_{10}(MAXE)$ against h for problem 4

4.3 Discussion of the results

From the numerical problems solved in the Table 1 (comprising problem 1&2), it has been shown that the proposed scheme, ABISBDF outperformed both the 3ESBSDF and $F_1$3SBBDF in terms of minimum error and less computational time. Also, from table 2 (comprising problem 2&3) the proposed scheme have good advantage in terms of scale error over the two methods compared. But, $F_1$3SBBDF has advantages over the new method ABISBDF in execution time.

To visibly highlight the performance of the proposed method, ABISBDF in relation to the other methods, 3ESBSBDF and $F_1$3SBBDF. The graphs of *Log10 (MAXE)* against the step size, *h* for the 1-4 problems are plotted accordingly in figure (2,3,4,5), the method has minimum scaled error in the entire problems considered. The proposed scheme is recommended for solving first order system of initial value problems of ordinary differential equation.

Conclusion

An A stable block integrator scheme is proposed. The order and stability properties of the method are investigated, the scheme found to be zero stable, A-stable and of order 6. The developed method is implicit methods, can computes four solution values at a time per step, concurrently. The results from the tested problems shows that the new method has advantages in terms of accuracy of the scaled error and computational time when compared with the 3ESBSBDF and also has advantages in terms of accuracy of the scaled error over $F_1$3SBBDF method. The proposed scheme can be used in solving a system of first order initial value problem of ordinary differential equations.

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