Original Research Article

ON A METHOD OF BIAS REDUCTION IN THE PRODUCT METHOD OF ESTIMATION

ABSTRACT

In this paper, we focused our attention on the creation of an almost unbiased predictive product estimator after estimating and correcting bias of the classical product estimator under predictive approach. Considering mean square error as the performance measure, superiority of the proposed estimator has been analyzed compared to the classical product estimator and Robson's (1957) unbiased product estimator under (i) a finite population set-up, (ii) an infinite population set-up assuming bivariate normal distribution between the variables, and (iii) the assumption of a super-population model.

Keywords: Almost unbiased product estimator, predictive approach, super-population model.

1. INTRODUCTION

Let y and x denote the survey variable and an auxiliary variable taking values y_i and x_i respectively on the ith unit of a finite population $U = \{1, 2, \ldots, i, \ldots, N\}$ of N units. Define $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$, $\overline{X} = \frac{1}{N}\sum_{i=1}^{N} x_i$ as the population means and $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \overline{Y})^2$, $S_x^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{X})^2$ as the population variances of y and x, and $S_{yx} = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \overline{Y})(x_i - \overline{X})$ as the population covariance between y and x. Assume that a random sample s of n units is drawn from U according to simple random sampling without replacement (SRSWOR) in order to estimate unknown mean $\overline{Y}$ when $\overline{X}$ is known accurately. Let $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ be the sample means, $s_y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$ and $s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$ be the sample variances, and $s_{yx} = \frac{1}{n-1}\sum_{i=1}^{n}(y_i - \bar{y})(x_i - \bar{x})$ be the sample covariance.

When the correlation coefficient between y and x has a very high negative value, product method of estimation as complementary to the ratio method of estimation for estimating $\bar{Y}$ is recommended as an error reducing technique. But, in the literature, product method of estimation has not yet been given as much emphasis as ratio method of estimation. Because, many survey statisticians are on the opinion that the occurrence of negatively correlated auxiliary variables is a rare phenomenon. It is of course true that positively correlated variables are easily encountered in practice. But, in the context of sample surveys, it is not very uncommon to observe highly negatively correlated variables. For example, in real life situations we observe that the correlations between yield of paddy per plant (y) and percentage of sterility (x); average miles per gallon (y) and engine HP of passenger cars (x); child mortality (y) and female literacy rate (x); the loss of body weight (y) and time spend for practicing exercise (x); selling of chocolate products (y) and atmospheric temperature (x); egg production (y) and age of chicken (x) are highly negative. A negatively correlated auxiliary variable is also generated taking inverse transformation of a positively correlated variable. Some recent papers highlighting useful theoretical results on the product method of estimation are Mandowara and Mehta (2016), Kamba *et al.* (2019), Kumar and Chhaparwal (2020), Brar *et al.* (2020), Kumar and Sharma (2020), Sahoo *et al.* (2021) among others.

The classical product estimator of the population mean $\bar{Y}$ is defined by

$$\ell_P = \frac{\bar{y}\bar{x}}{\bar{X}}$$

[*cf.*, Murthy (1964)], which performs better than the mean per unit estimator $\bar{y}$ when $\rho C_y/C_x < -1/2$, where $C_y = S_y/\bar{Y}$ and $C_x = S_x/\bar{X}$ are coefficients of variation of y and x respectively, and ρ is the coefficient of correlation between them. However, ℓ_P is a biased estimator of $\bar{Y}$ and the exact expression for the bias is

$$B(\ell_P) = E(\ell_P) - \bar{Y} = \theta\frac{S_{yx}}{\bar{X}}, \tag{1}$$

where $\theta = \frac{N-n}{Nn}$. Although the bias may be small for large samples, in small samples its impact may be important enough not to be ignored. Ordinarily, the survey statisticians avoid estimators that are considerably biased, because valid confidence intervals cannot be obtained if bias is substantial. The most important estimators in survey sampling are therefore either unbiased or approximately unbiased.

Estimation of S_{yx} by its unbiased estimator s_{yx} and then correction of bias given in (1) lead to define an unbiased estimator

$$\ell_{RP} = \ell_P(1 - \theta c_{yx}),$$

where $c_{yx} = s_{yx}/\bar{y}\bar{x}$. This unbiased estimator was framed by Robson (1957). Srivastava *et al.* (1981) compared variance expressions up to terms of order n^{-2} and concluded that ℓ_{RP} is more efficient than ℓ_P. Considering exact variance expressions under finite and infinite population's set-up, Chaubey *et al.* (1990) established that ℓ_{RP} is better than ℓ_P when $\rho^2 > (n-2)^{-1}$.

Referring to Sahoo's (1983) work, Singh (1989) constructed an almost unbiased product estimator (unbiased up to terms of $O(n^{-1})$) of the form

$$\ell_{SP} = \ell_P/(1 + \theta c_{yx}).$$

The followed technique is to consider expected value of ℓ_P *i.e.*,

$$E(\ell_P) = \bar{Y}(1 + \theta C_{yx}), \tag{2}$$

where $C_{yx} = S_{yx}/\bar{Y}\bar{X}$, and then dividing ℓ_P by the estimate of the terms in the parentheses to get the proposed estimator. But the two estimators ℓ_{RP} and ℓ_{SP} are virtually equivalent in the sense that they use the same statistics for their computation and moreover their variances are equal to $O(n^{-2})$.

From (1), we note that the bias of ℓ_P is a function of the parameters $\overline{X}$ and S_{yx}. Since $\overline{X}$ is known, estimation of bias needs an estimation of S_{yx} from the sample data. But in this work, instead of estimating the covariance S_{yx} under classical approach, we estimate the parameter under prediction approach in order to have an almost unbiased product estimator.

2. A PREDICTIVE ESTIMATION OF S_{yx}

Let us decompose U into two mutually exclusive sets s and r of n and N units respectively, where $r = U - s$ denotes the collection of un-sampled units of U. Then, under the usual prediction criterion [*c.f.*, Bolfarine and Zacks (1992), p.12], it is possible to express the population covariance S_{yx} in the following form:

$$(N-1)S_{yx} = (n-1)s_{yx} + (N-n-1)S_{yx(r)} + (1-f)n(\bar{y} - \overline{Y}_r)(\bar{x} - \overline{X}_r), \qquad (3)$$

where $f = \frac{n}{N}$, $\overline{Y}_r = \frac{1}{N-n}\sum_{i\in r} y_i$, $\overline{X}_r = \frac{1}{N-n}\sum_{i\in r} x_i$ and $S_{yx(r)} = \frac{1}{N-n-1}\sum_{i\in r}(y_i - \overline{Y}_r)\,(x_i - Xr$.

According to the equation (3), note that s_{yx} and $\overline{X}_r = \frac{N\overline{X} - n\bar{x}}{N-n}$ are known quantities whereas $\overline{Y}_r$ and $S_{yx(r)}$ are unknown. Hence, prediction of $(N-1)S_{yx}$ needs simultaneous prediction of $\overline{Y}_r$ and $S_{yx(r)}$ by some means from the sample data. Letting M_r and C_r as their respective predictors, a predictor of S_{yx} can be formed from the following equation:

$$(N-1)\hat{S}_{yx} = (n-1)s_{yx} + (N-n-1)C_r + (1-f)n(\bar{y} - M_r)(\bar{x} - \overline{X}_r). \qquad (4)$$

Most of the predictions are based either on the distributional forms or an assumed model [*cf.*, Royall (1988), Bolfarine and Zacks (1992)]. However, Sampford (1978) argued that a model free prediction can generate a new estimator possessing some desirable properties. Basu (1971) also encouraged the use of tools of the classical estimation theory to find out suitable predictors for $\overline{Y}$. Inspired by Basu (1971), Biradar and Singh (1998) and Nayak and Sahoo (2012) formulated some predictive estimators of the population variance S_y^2 in terms of the auxiliary variable x under the classical estimation tool.

Under classical approach, the predictive equation (4) provides a family of estimators of S_{yx} for various selections of C_r and M_r. But to avoid complexity, we concentrate on the simple selections. Towards this motivation, let us now consider $C_r = s_{yx}$ and $M_r = \bar{y} + d(\overline{X}_r - \bar{x})$, a difference estimator where d is a suitably chosen constant which in particular may a random variable converging in probability to a constant D. But here we need to determine d such that the resulting estimator of S_{yx} is unbiased. The predictive equation after simple algebra yields the following estimator:

$$\hat{S}^*_{yx} = \frac{N-2}{N-1} s_{yx} + \frac{1}{(N-1)\theta} d\left(\bar{x} - \overline{X}\right)^2.$$

To determine d we have to satisfy $E\left(\hat{S}^*_{yx}\right) = S_{yx}$ that gives

$$\frac{N-2}{N-1} S_{yx} + \frac{d}{(N-1)} S_x^2 = S_{yx}$$

$$\Longrightarrow \quad d = \frac{S_{yx}}{S_x^2} = \beta_{yx},$$

the regression coefficient of y on x. Hence, for $d = \beta_{yx}$, $M_r = \bar{y} + \beta_{yx}\left(\overline{X}_r - \bar{x}\right)$ and $\hat{S}^*_{yx}$ defines an unbiased estimator for S_{yx}.

In actual practice the composite parameter β_{yx} may not be known in advance and ordinarily estimated by its plausible consistent estimate $b_{yx} = s_{yx}/s_x^2$, the sample regression coefficient of y on x. Hence, on the consideration of $C_r = s_{yx}$ and $M_r = \bar{y} + b_{yx}\left(\overline{X}_r - \bar{x}\right)$, (4) provides us the following new predictive estimator for S_{yx}:

$$T_{yx} = \frac{N-2}{N-1} s_{yx} + \frac{1}{(N-1)\theta} b_{yx}\left(\bar{x} - \overline{X}\right)^2.$$

It may be remarked that T_{yx} is no more a completely unbiased estimator but an almost unbiased estimator for S_{yx} *i.e.*, unbiased to $O(n^{-1})$. To justify this let us write

$$T_{yx} = \frac{N-2}{N-1} s_{yx} + \frac{\overline{X}^2 \beta_{yx}}{(N-1)\theta} (\delta\bar{x})^2 \left(1 + \delta s_{yx}\right)(1 + \delta s_x^2)^{-1} \qquad (5)$$

where $\delta\bar{x} = \frac{\bar{x}-\overline{X}}{\overline{X}}$, $\delta s_{yx} = \frac{s_{yx}-S_{yx}}{S_{yx}}$ and $\delta s_x^2 = \frac{s_x^2 - S_x^2}{S_x^2}$. Further, let us assume that $|\delta s_x^2| \leq 1$ [vide Sukhatme *et al*. (1984), p.238], so that $(1 + \delta s_x^2)^{-1}$ can be validly expanded as a power series in δs_x^2. After considerable simplification and then retaining terms up to degree 2 for δ's, we obtain

$$T_{yx} = \frac{N-2}{N-1} s_{yx} + \frac{\overline{X}^2 \beta_{yx}}{(N-1)\theta} (\delta\bar{x})^2$$

$$\Longrightarrow \quad E\left(T_{yx}\right) = \frac{N-2}{N-1} E\left(s_{yx}\right) + \frac{\overline{X}^2 \beta_{yx}}{(N-1)\theta} E(\delta\bar{x})^2$$

$$\Longrightarrow \quad E\left(T_{yx}\right) \cong \frac{N-2}{N-1} S_{yx} + \frac{\overline{X}^2 \beta_{yx}}{(N-1)\theta} \theta \frac{S_x^2}{\overline{X}^2} = S_{yx}. \qquad (6)$$

This means that T_{yx} is almost unbiased.

3. THE PROPOSED ALMOST UNBIASED PRODUCT ESTIMATOR

Estimating S_{yx} by T_{yx} in equation (1), the bias of ℓ_P is estimated by

$$\text{Est } B(\ell_P) = \theta \frac{T_{yx}}{\overline{X}}. \qquad (7)$$

Then adjusting ℓ_P for its bias, we compose the following almost unbiased estimator for $\overline{Y}$:

$$\ell_{AP} = \ell_P - \theta\frac{T_{yx}}{\overline{X}}$$

i.e., $$\ell_{AP} = \frac{1}{\overline{X}}\left[\bar{y}\bar{x} - \theta\frac{N-2}{N-1}s_{yx} - \frac{1}{N-1}b_{yx}\left(\bar{x}-\overline{X}\right)^2\right].$$

To check almost unbiasedness property of ℓ_{AP} see from (1) that $E(\ell_P) = \overline{Y} + \theta\frac{S_{yx}}{\overline{X}}$, and from (6) up to $O(n^{-1})$ that $E\left(T_{yx}\right) = S_{yx}$. Hence, finally we have

$$E(\ell_{AP}) = E(\ell_P) - \theta E\left(\frac{T_{yx}}{\overline{X}}\right) \cong \overline{Y}. \qquad (8)$$

4. EFFICIENCY OF THE PROPOSED ESTIMATOR

In order to study the efficiency aspect of ℓ_{AP} compared to ℓ_P, ℓ_{RP} and ℓ_{SP}, we need expression for its mean square error (MSE). But, our main concern here is that ℓ_{AP} is a nonlinear function of four statistics *viz.*, $\bar{y}$, $\bar{x}$, s_{yx} and s_x^2. Therefore, it is often impossible to derive exact results on its MSE under a finite population set-up. For this reason, we have to rely on the asymptotic results *i.e.*, expressions up to a desired order of approximation. To achieve this we may take the help of Taylor linearization technique [*cf.*, Sarndal, Swensson and Wretman (2003, p.172)]. But to circumvent much difficulty, we consider the power series expansion method using $\delta -$ notations as in section 2.

Denoting $\delta\bar{y} = \frac{\bar{x}-\overline{Y}}{\overline{Y}}$ and then using (5), we write

$$\ell_{AP} - \overline{Y} = \overline{Y}[\delta\bar{y} + \delta\bar{x} + \delta\bar{y}\delta\bar{x}] - \theta\frac{N-2}{(N-1)\overline{X}}S_{yx}\left(1+\delta s_{yx}\right)$$
$$-\frac{\overline{X}}{(N-1)}\cdot\beta_{yx}(\delta\bar{x})^2\left[1+\delta s_{yx} - \delta s_x^2 + (\delta s_x^2)^2 - \left(\delta s_{yx}\right)(\delta s_x^2) + \cdots\right] \qquad (9)$$

Squaring both sides of (9), simplifying and keeping terms up to degree 2 for δ's, and taking expectation term-by- term, we obtain MSE of ℓ_{AP} to order n^{-1} as

$$M(\ell_{AP}) = E\left(\ell_{AP} - \overline{Y}\right)^2 = \theta\left[C_y^2 + 2C_{yx} + C_x^2\right]. \qquad (10)$$

This is also MSE expression of ℓ_P, ℓ_{RP} and ℓ_{SP} up to $O(n^{-1})$. It means that the four estimators ℓ_{AP}, ℓ_P, ℓ_{RP} and ℓ_{SP} are asymptotically equally well under MSE criterion. In view of this, we further need a comparison of MSEs considering terms up to $O(n^{-2})$.

Evaluating the expression $E\left(\ell_{AP} - \overline{Y}\right)^2$ and considering term-by-term expectations, after some algebraic manipulation we derive MSE of ℓ_{AP} to terms of $O(n^{-2})$ as given below. Here we follow the same notations and approximations used by Tin (1965) [also see Kendall, Stuart and Ord (1983), Vol. III, p.2421].

$$M(\ell_{AP}) = \overline{Y}^2\theta\left[\Delta_1 - 2\frac{1}{N-1}\Delta_2 + \theta\left\{C_{20}C_{02} - \left(\frac{N-2}{N-1}\right)^2 C_{11}^2\right\}\right] - 2\theta\frac{D}{N-1}\overline{Y}^2\Delta_3, \qquad (11)$$

where $\Delta_1 = C_{20} + 2C_{11} + C_{02}$, $\Delta_2 = C_{21} + C_{12}$, $\Delta_3 = \frac{C_{11}(C_{12}+C_{03})}{C_{02}}$, $D = \frac{N-2n}{Nn}$ and $C_{rs} = K_{rs}/\overline{Y}^r\overline{X}^s$, K_{rs} being the $(r,s)^{th}$ cumulant in y and x.

Precisely, in a similar way Srivastava *et al.* (1981) obtained MSEs of ℓ_P and ℓ_{RP} to order n^{-2}. But from Singh's (1989) derived results, up to this order of approximation,

$M(\ell_{RP}) \cong M(\ell_{SP})$. So, we ignore ℓ_{SP} from the comparison and rewrite below the results for ℓ_P and ℓ_P:

$$M(\ell_P) = \overline{Y}^2 \theta[\Delta_1 + 2D\Delta_2 + \theta(C_{20}C_{02} + 2C_{11}^2)] \tag{12}$$

$$M(\ell_{RP}) = \overline{Y}^2 \theta\left[\Delta_1 - 2\frac{1}{N}\Delta_2 + \theta(C_{20}C_{02} + C_{11}^2)\right] \tag{13}$$

From the equations (11), (12) and (13), we have the following results:

(i) $$\frac{M(\ell_P) - M(\ell_{RP})}{2\theta^2\overline{Y}^2} = \Delta_2 + C_{11}^2 \tag{14}$$

It means that ℓ_{RP} is more efficient that ℓ_P when

$$\Delta_2 + C_{11}^2 > 0. \tag{15}$$

This result is due to Srivastava *et al*. (1981). Since $C_{11}^2 > 0$, a sufficient condition for ℓ_{RP} to be more efficient than ℓ_P is that $\Delta_2 > 0$.

(ii) $$\frac{M(\ell_P) - M(\ell_{AP})}{\theta\overline{Y}^2} = 2\left(D + \frac{1}{N-1}\right)\Delta_2 + \theta\left[2 + \left(\frac{N-2}{N-1}\right)^2\right]C_{11}^2 + 2\frac{D}{N-1}\Delta_3 \tag{16}$$

$$\frac{M(\ell_{RP}) - M(\ell_{AP})}{\theta\overline{Y}^2} = 2\left(\frac{1}{N-1} - \frac{1}{N}\right)\Delta_2 + \theta\left[1 + \left(\frac{N-2}{N-1}\right)^2\right]C_{11}^2 + 2\frac{D}{N-1}\Delta_3 \tag{17}$$

In survey sampling literature, $n < N/2$ is one of the most commonly accepted theoretical conditions so that D is most likely to be positive. This implies that the coefficients of Δ_2, C_{11}^2 and Δ_3 in (16) and (17) are positive. Hence, when $\Delta_2 > 0$, ℓ_{AP} would be more efficient than both ℓ_{RP} and ℓ_P if $\Delta_3 > 0$. As $C_{11} < 0$, fulfillment of the sufficient condition $\Delta_3 > 0$ is possible only when $C_{12} + C_{03} < 0$. But, depending on the distribution of the variables under consideration, the parametric function $(C_{12} + C_{03})$ may assume either positive or negative values. In view of this, we tentatively draw the following conclusion:

Under the situation $\Delta_2 > 0$ i.e., when ℓ_{RP} is more precise than ℓ_P, ℓ_{AP} would be preferable to both ℓ_{RP} and ℓ_P if either $\Delta_3 > 0$ or the contribution of the third term $\left(= 2\frac{D}{N-1}\Delta_3\right)$ in the right hand sides of (16) and (17) is negligible in comparison to the preceding terms.

In some practical situations, it is not so easy to check the feasibility of the derived sufficient conditions to draw any meaningful conclusion as they depend on the survey situations, unknown population parameters, composition of population units, joint distribution of the variables and many other constraints. This may mislead our efficiency comparison. However, this comparison clearly indicates that there is enough scope for using our proposed estimator in place of its competitors.

To make our efficiency comparison more viable, we further continue our analytical comparison under two noteworthy assumptions – bivariate normal distribution of the variables and a super-population model.

4.1 Efficiency Comparison under the Assumption of Bi-Variate Normality

Let us assume that the random sample of n units is drawn from an infinite population in which the joint distribution of x and y is bivariate normal. Then $C_{12} = C_{21} = C_{03} = 0 \Rightarrow \Delta_2 = \Delta_3 = 0$ and as $\frac{N-2}{N-1} \leq 1$ and $N \to \infty$, we may easily assume that $\frac{N-2}{N-1} \approx 1$. Hence, after some algebra, from (11), (12) and (13) we obtain the following MSE expressions for ℓ_{AP}, ℓ_P and ℓ_{RP} to $O(n^{-2})$:

$$M(\ell_{AP}) = \frac{\overline{Y}^2}{n}\left[\left(C_y^2 + 2\rho C_y C_x + C_x^2\right) + \frac{C_y^2 C_x^2}{n}\{1 - \rho^2\}\right] \quad (18)$$

$$M(\ell_P) = \frac{\overline{Y}^2}{n}\left[\left(C_y^2 + 2\rho C_y C_x + C_x^2\right) + \frac{C_y^2 C_x^2}{n}\{1 + 2\rho^2\}\right] \quad (19)$$

$$M(\ell_{RP}) = \frac{\overline{Y}^2}{n}\left[\left(C_y^2 + 2\rho C_y C_x + C_x^2\right) + \frac{C_y^2 C_x^2}{n}\{1 + \rho^2\}\right] \quad (20)$$

Thus, it follows that under bivariate normality assumption, both ℓ_P and ℓ_{RP} are inferior to ℓ_{AP} under MSE criterion.

To make an idea on the efficiency gain in estimation quantitatively by the different estimators compared to $\bar{y}$ whose variance under normality assumption is

$$V(\bar{y}) = \frac{\overline{Y}^2}{n} C_y^2, \quad (21)$$

we computed numerical values of their percentage relative efficiencies (RE) for some selected values of n, C_y, C_x and ρ as shown in table 1. For a given value of C_y, values of C_x and ρ are chosen so as to satisfy the condition $\rho C_y / C_x < -0.5$ with a view to make the product method of estimation more effective.

Table 1: Relative Efficiencies of the Estimators w.r.t. $\bar{y}$ (in %)

n	C_y	C_x	ρ	ℓ_{AP}	ℓ_P	ℓ_{RP}
10	1.0	0.3	−0.4	116.62	116.03	116.22
		0.6	−0.6	150.83	142.47	145.15
		0.9	−0.8	250.56	180.31	198.88
		1.2	−1.0	2500.00	211.86	304.87
20	1.5	0.8	−0.4	113.04	111.11	111.74
		0.9	−0.6	150.17	140.92	143.88
		1.0	−0.8	252.66	203.34	217.49
		1.1	−1.0	1406.25	395.91	520.59
30	2.0	0.7	−0.4	116.79	115.73	116.08
		0.9	−0.6	147.10	141.05	143.02
		1.1	−0.8	228.83	194.39	204.65
		1.3	−1.0	816.32	343.08	425.26
40	2.5	1.0	−0.4	116.14	114.55	115.07
		1.5	−0.6	147.93	135.73	139.57
		2.5	−0.8	219.18	132.23	152.38
		3.5	−1.0	625.00	92.70	129.45

Results of table 1 indicates clearly that for given values of n and C_y, RE of all estimators increases with increase in the value of ρ except that of ℓ_P and ℓ_{RP} for $n = 40$

and $C_y = 2.5$. But the gain in efficiency of ℓ_{AP} over ℓ_P and ℓ_{RP} is usually very high except for $\rho = -0.4$ where it is only marginal. This means that higher negative values of the correlation coefficient between y and x favors ℓ_{AP} for its efficient use.

4.2 Efficiency Comparison under a Super Population Model

To study performance of ℓ_{AP} compared to others, we consider a super-population model for which

$$y_i = \beta x_i + e_i, i = 1, 2, \ldots\ldots, N, \tag{22}$$

where β is an unknown real constant, e_i's are uncorrelated random errors such that $E(e_i|x_i) = 0$ and $E(e_i^2|x_i) = \vartheta x_i^t$ for all i with $0 < \vartheta < \infty$ and $0 \le t \le 2$. Further, x_i's are assumed to be $i.i.d.$ gamma variates with a common parameter $h\ (> 0)$ taken equal to the mean $\overline{X}$.

By the direct substitution under the model and then after some algebraic manipulations, we see that ℓ_{AP} is completely unbiased *i.e.*, model-unbiased. At the same time we also see that ℓ_{RP} is model-unbiased whereas ℓ_P and ℓ_{SP} are not model-unbiased.

Expressing population parameters *i.e.*, C_{rs}'s in terms of the model parameters, from equations (11), (12) and (13) we directly obtain the following model-based MSE results up to $O(n^{-2})$:

$$M(\ell_{AP}) = \beta^2\theta\left[4h - 8\frac{D+1}{N-1} - \left(\frac{N-2}{N-1}\right)^2 + \theta\right] + H\theta\left(1 - \frac{2t}{(N-1)h} + \frac{\theta}{h}\right) \tag{23}$$

$$M(\ell_P) = \beta^2\theta[4h + 8D + 2\theta] + H\theta\left(1 + \frac{2Dt}{h} + \frac{\theta}{h}\right) \tag{24}$$

$$M(\ell_{RP}) = \beta^2\theta\left[4h - \frac{8}{N} + 2\theta\right] + H\theta\left(1 - \frac{2t}{Nh} + \frac{\theta}{h}\right) \tag{25}$$

where $H = \vartheta\frac{\Gamma(t+h)}{\Gamma(h)}$.

Comparing (23), (24) and (25), we straight-forwardly conclude that

$$M(\ell_{AP}) \le M(\ell_{RP}) \le M(\ell_P),$$

which implies that ℓ_{AP} performs better than ℓ_P and ℓ_{RP} on the ground of MSE. But, when $t = 0$, that is under the assumption of homoscedasticity, $M(\ell_{AP}) = M(\ell_{RP}) = M(\ell_P)$ and the estimators appear to be equally efficient.

In table 2, we display numerical values of the deciding factors – the coefficients of $\theta\beta^2$ and θH in the MSE expressions of the comparable estimators for a few combinations of the parametric values. Values are given for $N = 50$, $h = 2.0$, $n = 5, 10, 15, 20$ and $t = 0.0, 0.5, 1.0, 1.5, 2.0$.

From the tabulated numerical results, it is crucial to note that the values of the said coefficients in each case decrease as n increases. On the other hand, it is also noted that (i) for increased value of t, coefficient values of θH for $M(\ell_{AP})$ and $M(\ell_{RP})$ decrease whereas for $M(\ell_P)$ these values increase, and (ii) differences between coefficients values of θH for $M(\ell_{AP})$ and $M(\ell_{RP})$ are just marginal. On the whole from the calculated coefficient values, we see that $M(\ell_{AP})$ is the least for all cases indicating ℓ_{AP} as the most efficient estimator.

Table 2: Coefficients of $\theta\beta^2$ and θH

n	Coefficient of $\theta\beta^2$			t	Coefficient of θH		
	$M(\ell_{AP})$	$M(\ell_P)$	$M(\ell_{RP})$		$M(\ell_{AP})$	$M(\ell_P)$	$M(\ell_{RP})$
5	7.0316	9.6400	8.2000	0.0	1.0900	1.0900	1.0900
				0.5	1.0798	1.1700	1.0800
				1.0	1.0696	1.2500	1.0700
				1.5	1.0594	1.3300	1.0600
				2.0	1.0492	1.4100	1.0500
10	6.9476	8.6400	8.0000	0.0	1.0400	1.0400	1.0400
				0.5	1.0298	1.0700	1.0300
				1.0	1.0196	1.1000	1.0200
				1.5	1.0094	1.1300	1.0100
				2.0	0.9992	1.1600	1.0000
15	6.9198	8.3060	7.9332	0.0	1.0233	1.0233	1.0233
				0.5	1.0131	1.0366	1.0133
				1.0	1.0029	1.0499	1.0033
				1.5	0.9927	1.0632	0.9933
				2.0	0.9825	1.0765	0.9833
20	6.9056	8.1400	7.9000	0.0	1.0150	1.0150	1.0150
				0.5	1.0048	1.0200	1.0050
				1.0	0.9946	1.0250	0.9950
				1.5	0.9844	1.0300	0.9850
				2.0	0.9742	1.0350	0.9750

5. CONCLUDING REMARKS

Our preceding discussions show that the proposed estimator is no way inferior to the classical, Robson's (1957) unbiased and Singh's (1989) almost unbiased product estimators. Because it is not only approximately unbiased but also more efficient under a variety of easily acceptable conditions and assumptions relating to the population. As our estimator is structurally complex, from the computational point it may not be preferred to others. But, this drawback is not a matter of great concern for our purpose. However, the new estimation mechanism formulated here has a greater scope for further development of a wide variety of estimators.

Acknowledgement: The authors are grateful to the reviewers whose constructive comments led to an improvement in the paper.

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