BIVARIATE COMPOUND EXPONENTIATED SURVIVAL FUNCTION OF THE LOMAX DISTRIBUTION: ESTIMATION AND PREDICTION

Abstract:

In this paper, bivariate compound exponentiated survival function of the Lomax distribution is constructed based on the technique considered by AL-Hussaini (2011). Some properties of the distribution are derived. Maximum likelihood estimation and prediction of the future observations are considered. Also, Bayesian estimation and prediction are studied under squared error loss function. The performance of the proposed bivariate distribution is examined using a simulation study. Finally, a real data set is analyzed under the proposed distribution to illustrate its flexibility for real-life application.

Keywords: *Lomax distribution; Bivariate distributions; Compound exponentiated survival functions; Maximum likelihood estimators; Prediction; Bayes estimators; Monte Carlo simulations.*

1. Introduction

Although bivariate extensions of univariate distributions are useful, it has not been applied in practice due to shortage of inferential procedures caused by numerical complexity. Moreover, generalization of univariate models is not straightforward in the sense that certain desirable properties may hold for more than one multivariate model.

One of the objectives of this paper is to construct a *bivariate compound exponentiated survival function of the Lomax* (BCESFLO) distribution; based on the technique considered by AL-Hussaini (2011) who constructed a class of multivariate distributions. It could be useful in studying reliability maintainability of complicated systems.

This paper consists of six sections. In Section 2, construction of BCESFLO distribution based on the technique proposed by AL-Hussaini (2011), also some properties of the distribution are obtained. Maximum likelihood estimation and prediction are considered in Section 3. In Section 4, simulation study and a data analysis are presented to illustrate the theoretical results derived for ML estimation and prediction. In Section 5, Bayesian estimation; for the unknown parameters, rf and hrf of BCESFLO distribution, are derived, also Bayesian prediction is considered. Finally, a simulation study and a data analysis for the results of Bayesian estimation and prediction are given in Section 6.

2. Construction of a Compound Exponentiated Survival of the Lomax Distribution

Recently, in the statistical literature several methodologies of constructing bivariate and multivariate distributions based on marginal and conditional distributions have been proposed see Arnold *et al*. (1999, 2001), Kotz *et al*. (2000), Sarabia and Gomez-Deniz (2008), Balakrishnan and Lai (2009) among others.

Bivariate survival data arise when each study subject experiences two events. Examples include failure times of paired human organs, kidneys, eyes, lungs, breasts and others, as well as first and second occurrences of given disease. Moreover, bivariate survival data may consist of time to diagnosis or hospitalization and the time to eventual death from a fatal disease. Moreover, it is appropriate to emphasize that in the medical literature the paired organs of an individual are considered as a two-component system, which work under interdependency circumstances. Specifically, in industrial applications these data types may come from system whose survival depends on the survival of two similar components. For an example, the breakdown times of dual generators in a power plant or failure times of twin engines in a 2-engine airplane are illustrations of bivariate survival data. In fact, there are many bivariate distributions that can be employed for the analysis of paired data, see Kotz *et al*. (2000).
Regarding the bivariate Pareto distribution, we highlight that two bivariate Pareto distributions were suggested by Mardia (1962), which are called bivariate Pareto of the first kind and bivariate Pareto of the second kind. Arnold (1983) suggested a distribution of the fourth kind and presented three methods to derive this model. Moreover, Muliere and Scarini (1987) proposed a bivariate Pareto survival function which was characterized by Padamadan and Nair (1994) using the survival function of the marginal distributions.

In this section, two cases of the construction of the *compound exponentiated survival function* of the *Lomax* (CESFLO) distribution, univariate and bivariate, are introduced.

2.1 Construction of the univariate compound exponentiated survival of the Lomax distribution

AL-Hussaini (2011) introduced the construction of a class of distributions by compounding the exponentiated *survival function* (sf) with the gamma *probability density function* (pdf). The obtained class includes all distributions with positive domain. Such domain could be the whole positive half of the real line or subset of it. A particular class of such distributions is the univariate CESFLO distribution.

Next, we will obtain the univariate CESFLO distribution. Suppose that the random variable T has *Lomax* (LO) distribution whose pdf and *cumulative distribution function* (cdf), are given, respectively, by

$$g(t,\alpha) = \alpha(1+t)^{-(\alpha+1)}, \quad t > 0, \alpha > 0, \tag{1}$$

and

$$G(t,\alpha) = 1-(1+t)^{-\alpha} \;, \quad t > 0, \alpha > 0, \tag{2}$$

Using (1) and (2), we can define the following pdf and cdf, respectively, as follows:

$$q(t|\beta,\alpha) = \alpha\beta[(1+t)^{-\alpha}]^{\beta-1}(1+t)^{-(\alpha+1)},$$

and

$$Q(t|\beta,\alpha) = 1-[\overline{G}(t)]^{\beta} = 1-(1+t)^{-\alpha\beta} \;, \tag{3}$$

where $t, \beta, \alpha > 0$. From (3), it follows that the corresponding sf is

$$s(t|\beta,\alpha) = (1+t)^{-\alpha\beta}.$$

Now, using the idea proposed by AL-Hussaini (2011), we will define the pdf of CESFLO distribution, f, as the compounding of q with the gamma pdf. That is,

$$f(t|\alpha,a,b) = \int_0^\infty f(t|\beta)\eta(\beta)d\beta,$$

where

$$\eta(\beta) = \frac{b^a}{\Gamma(a)}\beta^{a-1}\,\exp(-b\beta)\,, \qquad \beta, a, b > 0, \tag{4}$$

Notice that using the equality $(1+t)^{-\alpha\beta} = e^{\beta ln(1+t)^{-\alpha}}$
we can construct the following gamma density $\frac{r^q}{\Gamma(q)} x^{q-1} e^{-rx}$ with $r = b + ln(1+t)^{-\alpha}$,

and $q = a + 1$,
to ensure that the pdf of the CESFLO distribution is given by

$$f(t) = \frac{a}{b}\left[\frac{\alpha}{1+t}\right]\left[1 + \frac{\alpha}{b}\ln(1+t)\right]^{-(a+1)}, t > 0, \alpha, \beta > 0. \quad (5)$$

From (5) the cdf of CESFLO distribution can be written as

$$F(t) = 1 - \left[1 + \frac{\alpha}{b}\ln(1+t)\right]^{-a}.$$

The *hazard rate function* (hrf) corresponding to $H(t)$ is

$$\lambda_H(t) = \frac{f(t)}{1 - F(t)}$$

$$= \frac{a}{b}\frac{\alpha}{(1+t)}\left[1 + \frac{\alpha}{b}\ln(1+t)\right]^{-1}.$$

The *reversed hazard rate function* (rhrf) is given by

$$\lambda_H^*(t) = \frac{\frac{a}{b}\frac{\alpha}{(1+t)}\left[1 + \frac{\alpha}{b}\ln(1+t)\right]^{-(a+1)}}{1 - \left[1 + \frac{\alpha}{b}\ln(1+t)\right]^{-a}}.$$

2.2 Construction of the bivariate compound exponentiated survival of the Lomax distribution

Suppose that T_i has LO distribution for $i = 1, 2$ with T_1 and T_2 are independent random variables.

Again, using (1) and (2), we can define the following cdf and pdf

$$Q(t_i|\beta, \alpha_i, \theta_i) = 1 - [(1+t_i)^{-\alpha_i}]^{\theta_i\beta}$$

and

$$q(t_i|\beta, \alpha_i, \theta_i) = \alpha_i\theta_i\beta[(1+t_i)^{-\alpha_i}]^{\theta_i\beta-1}(1+t_i)^{-(\alpha_i+1)},$$

for $i = 1, 2$, respectively, where $t_i, \alpha_i, \theta_i, \beta > 0$.
Let φ be defined by

$$\varphi\left(\underline{t};\underline{p}\right)=\prod_{i=1}^{2}q(t_i\,|\beta,\alpha_i,\theta_i),$$

where $\underline{t}=(t_1,t_2)$ with $t_i>0$ for i=1,2 and $\underline{p}=(\alpha_1,\alpha_2,\theta_1,\theta_2,\beta)$ with $\alpha_i,\theta_i,\beta>0$, for $i=1,2$.

Again, using the idea proposed by AL-Hussaini (2011) we will define pdf of BCESFLO distribution, f, as the compounding of φ with η where η is given by (4). That is,

$$f\left(\underline{t}\,|\beta,\alpha_i,\theta_i\right)=\int_0^\infty\prod_{i=1}^{2}q(t_i\,|\beta,\alpha_i,\theta_i)\,\eta(\beta)d\beta\,, \tag{6}$$

Next, we will derive the function, $f\left(\underline{t}\,|\beta,\alpha_i,\theta_i\right)$. Observe that

$$\prod_{i=1}^{2}f(t_i\,|\alpha_i,\theta_i)=\prod_{i=1}^{2}\theta_i\,[(1+t_i)^{-\alpha_i}]^{\theta_i\beta}\,\frac{\alpha_i(1+t)^{-(\alpha_i+1)}}{(1+t_i)^{-\alpha_i}}$$
$$=\beta^2\prod_{i=1}^{2}\theta_i\,\frac{\alpha_i(1+t)^{-(\alpha_i+1)}}{(1+t_i)^{-\alpha_i}}\exp\left[\sum\theta_i\,\beta ln((1+t_i)^{-\alpha_i})\right]. \tag{7}$$

Substituting (7) and (4) in (6), one obtains

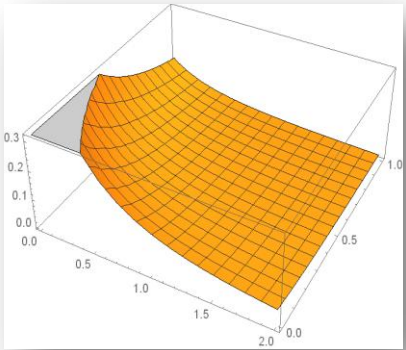
$$f\left(\underline{t}|\alpha_1,\alpha_2,\theta_1,\theta_2\right)=\prod_{i=1}^{2}\theta_i\,\frac{\alpha_i(1+t)^{-(\alpha_i+1)}}{(1+t_i)^{-\alpha_i}}\frac{b^a}{\Gamma(a)}\int_0^\infty\beta^{a+1}e^{-\beta[b-\sum_{i=1}^{2}\theta_i\,ln((1+t_i)^{-\alpha_i})]}d\,\beta$$
$$=\frac{\Gamma(a+2)}{\Gamma(a)}\left[\prod_{i=1}^{2}\left(\frac{\theta_i}{b}\right)\frac{\alpha_i(1+t)^{-(\alpha_i+1)}}{(1+t_i)^{-\alpha_i}}\right]\left[1-\sum_{i=1}^{2}\left(\frac{\theta_i}{b}\right)\ln(1+t_i)^{-\alpha_i}\right]^{-(a+2)}.$$

Then the pdf in the bivariate case is

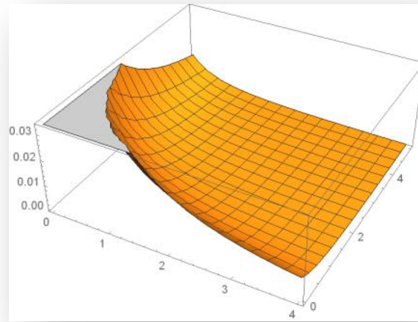
$$f(t_1,t_2|\alpha_1,\alpha_2,\theta_1,\theta_2)=a(a+1)\left[\left(\frac{\theta_1}{b}\left(\frac{\alpha_1}{1+t_1}\right)\right)\left(\frac{\theta_2}{b}\left(\frac{\alpha_2}{1+t_2}\right)\right)\right]$$
$$\times\left[1+\left\{\frac{\theta_1\;\alpha_1}{b}\;ln(1+t_1)+\frac{\theta_2\;\alpha_2}{b}\;ln(1+t_2)\right\}\right]^{-(a+2)}, \tag{8}$$

where $t_i,\alpha_i,\theta_i>0$, for $i=1,2$.

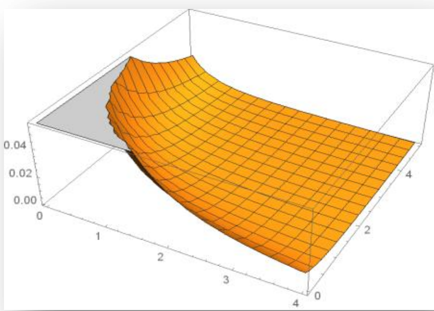
The contour plots of the joint pdf of BCESFLO distribution for different parameter values are presented in Figure 1.



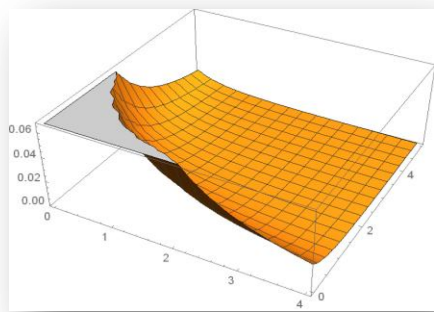
(1.a)



(1.b)



(1.c)



(1.d)

Figure 1. The contour plots of the joint pdf of BCESFLO distribution for different parameter values

(1.a) $(a = 0.8, \mathrm{b} = 0.8, \alpha_1 = 0.8, \alpha_2 = 0.8, \theta_1 = 0.8, \theta_2 = 0.8)$,
(1.b) $(a = 1.5, \mathrm{b} = 0.5, \alpha_1 = 2, \alpha_2 = 1, \theta_1 = 1, \theta_2 = 0.5)$,
(1.c) $(a = 1, \mathrm{b} = 1, \alpha_1 = 1, \alpha_2 = 1, \theta_1 = 1, \theta_2 = 0.5)$,
and (1.d) $(a = 1, \mathrm{b} = 1, \alpha_1 = 1, \alpha_2 = 1, \theta_1 = 1, \theta_2 = 1)$.

On the other hand the cdf of BCESFLO distribution is given by

$$\mathrm{F}(\mathrm{t}_1, \mathrm{t}_2 | \alpha_1, \alpha_2, \theta_1, \theta_2) = \int_0^{\mathrm{t}_1} \int_0^{\mathrm{t}_2} f(t_1, t_2)\, dt_2 dt_1$$

$$= \int_0^{t_1}\int_0^{t_2} a(a+1)\left[\left(\frac{\theta_1}{b}\left(\frac{\alpha_1}{1+t_1}\right)\right)\left(\frac{\theta_2}{b}\left(\frac{\alpha_2}{1+t_2}\right)\right)\right]$$
$$\times\left[1+\left\{\frac{\theta_1\ \alpha_1}{b}\ ln(1+t_1)+\frac{\theta_2\ \alpha_2}{b}\ ln(1+t_2)\right\}\right]^{-(a+2)} dt_2 dt_1$$
$$= 1+\left[1+\left\{\frac{\theta_1\ \alpha_1}{b}\ \ln(1+t_1)+\frac{\theta_2\ \alpha_2}{b}\ \ln(1+t_2)\right\}\right]^{-a}-\left[1+\left\{\frac{\theta_1\ \alpha_1}{b}\ ln(1+t_1)\right\}\right]^{-a}$$
$$-\left[1+\left\{\frac{\theta_2\ \alpha_2}{b}\ ln(1+t_2)\right\}\right]^{-a}, \qquad (9)$$

where $t_i, \alpha_i, \theta_i > 0$, for $i = 1,2$. Moreover, the marginal's pdf and cdf of BCESFLO distribution can be written, respectively, as

$$f(t_i|\alpha_i,\theta_i) = \frac{a\theta_i}{b}\left(\frac{\alpha_i}{1+t_i}\right)\left[1+\frac{\theta_i\ \alpha_i}{b}\ln(1+t_i)\right]^{-(a+1)}, \qquad i = 1,2,$$

and

$$F(t_i|\alpha_i,\theta_i) = 1-\left[1+\frac{\theta_i\ \alpha_i}{b}\ ln(1+t_i)\right]^{-a}, \qquad i = 1,2,$$

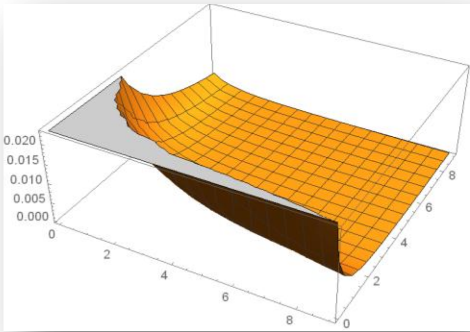
where $t_i, \alpha_i, \theta_i > 0$, for $i = 1,2$, the joint *reliability function* (rf) of BCESFLO distribution is given by:

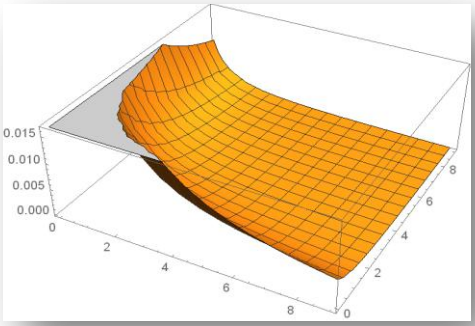
$$R(t_1,t_2) = p(T_1 > t_1, T_2 > t_2) = 1 - F(t_1) - F(t_2) + F(t_1,t_2)$$
$$= \left[1+\left\{\frac{\theta_1\ \alpha_1}{b}\ ln(1+t_1)+\frac{\theta_2\ \alpha_2}{b}\ ln(1+t_2)\right\}\right]^{-a}, \quad i = 1,2. \qquad (10)$$

where $t_i, \alpha_i, \theta_i > 0$, for $i = 1,2$. Also, the joint hrf of BCESFLO distribution can be defined as

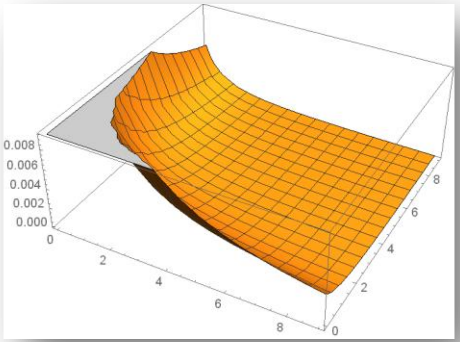
$$h(t_1,t_2) = \frac{f(t_1,t_2)}{R(t_1,t_2)} = a(a+1)\left[\left(\frac{\theta_1}{b}\left(\frac{\alpha_1}{1+t_1}\right)\right)\left(\frac{\theta_2}{b}\left(\frac{\alpha_2}{1+t_2}\right)\right)\right]$$
$$\times\left[1+\left\{\frac{\theta_1\ \alpha_1}{b}\ ln(1+t_1)+\frac{\theta_2\ \alpha_2}{b}\ ln(1+t_2)\right\}\right]^{-2}, \quad i = 1,2. \qquad (11)$$

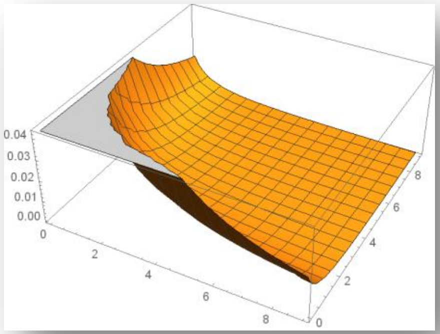
where $t_i, \alpha_i, \theta_i > 0$, for $i = 1,2$.
Just express that h is a decreasing function in t_1 and t_2 the probabilistic was clarified at the beginning of the section. The contour plots of the joint hrf of BCESFLO distribution for different parameter values are presented in Figure 2.





(2.a) (2.b)





(2.c) (2.d)

Figure 2: The contour plots of the joint hazard of BCESFLO distribution for different parameter values

(2.a) ($a = 1.1, \mathrm{b} = 2, \alpha_1 = 1.5, \alpha_2 = 2.2, \theta_1 = 1.9, \theta_2 = 3.1$),

(2.b) ($a = 1, \mathrm{b} = 1, \alpha_1 = 1, \alpha_2 = 0.5, \theta_1 = 0.5, \theta_2 = 0.5$),

(2.c) ($a = 0.8, \mathrm{b} = 0.8, \alpha_1 = 0.8, \alpha_2 = 0.4, \theta_1 = 0.4, \theta_2 = 0.4$),

and (2.d) ($a = 1, \mathrm{b} = 1, \alpha_1 = 1, \alpha_2 = 1, \theta_1 = 1, \theta_2 = 1$).

3. Maximum Likelihood Estimation

In this section, the ML estimation and prediction for the vector of the parameters

$\underline{\omega} = (a, b, \alpha_1, \alpha_2, \theta_1, \theta_2)$ of BCESFLO distribution will be considered.

3.1 Maximum likelihood estimation of the parameters

The likelihood function of BCESFLO distribution can be derived using the pdf in (8) directly, but compounding of $\prod_{i=1}^{k} f\left(t_i \middle| \underline{\omega}\right)$ and $\eta(\beta)$ can be applied to make the ML estimation easier, hence

$$L(\alpha_1, \alpha_2, \theta_1, \theta_2, a, b; t_1, t_2, \beta) = \prod_{j=1}^{n} f(t_{1j}, t_{2j}; \alpha_1, \alpha_2, \theta_1, \theta_2, , a, b, \beta_j)$$

$$= \prod_{j=1}^{n} \left[\prod_{i=1}^{2} f(\, t_{ij} \middle| \alpha_1, \alpha_2, \theta_1, \theta_2, , a, b, \beta_j) \eta\left(\beta_j\right) \right]$$

$$= \frac{b^{an}}{[\Gamma(a)]^n} \left(\prod_{j=1}^{n} \beta_j \right)^{a+1} \theta_1^n \theta_2^n \alpha_1^n \alpha_2^n \prod_{j=1}^{n} (1 + t_{1j})^{-1} \prod\nolimits_{j=1}^{n} \left(1 + t_{2j}\right)^{-1}$$

$$\times\, exp \left\{ -\sum_{j=1}^{n} \beta_j \left[\theta_1 \alpha_1 \ln\left(1 + t_{1j}\right) + \theta_2 \alpha_2 ln\left(1 + t_{2j}\right) + b \right] \right\},$$

$$t_{ij} = (t_{i1}, \dots, t_{in}), for\ i = 1{,}2\ and\ \beta_j = (\beta_1, \dots, \beta_n), \qquad (12)$$

where $\underline{\omega} = (a, b, \alpha_1, \alpha_2, \theta_1, \theta_2)$. The log likelihood function is given by

$$\ell\left(\underline{\omega}; t_1, t_2, \beta\right) = n\, a \ln(b) - n\, ln\left(\Gamma(a)\right) + (a + 1) \left(\sum_{j=1}^{n} \ln\left(\beta_j\right) \right) + n \ln(\theta_1) + n \ln(\theta_2)$$

$$+ n\, ln(\alpha_1) + n \ln(\alpha_2) - \sum_{j=1}^{n} \ln(1 + t_{1j}) - \sum_{j=1}^{n} \ln(1 + t_{2j})$$

$$- \left\{ \sum_{j=1}^{n} \beta_j \left[\theta_1 \alpha_1 \ln\left(1 + t_{1j}\right) + \theta_2 \alpha_2 ln\left(1 + t_{2j}\right) + b \right] \right\}, \qquad (13)$$

To obtain the ML estimators for the parameters, Equation (13) is differentiated with respect to the parameters. Hence, the resulting non-linear system of likelihood equations are given below

$$\partial \ell / \partial a = n \ln b - n\psi(a) + \sum_{j=1}^{n} \ln(\beta_j),$$

$$\partial \ell / \partial b = na/b - \sum_{j=1}^{n} \beta_j\,,$$

$$\partial\ell/\partial\alpha_1 = n/\alpha_1 - \sum_{j=1}^{n} \beta_j \theta_1 ln\big(1 + t_{1j}\big),$$

$$\partial\ell/\partial\theta_1 = n/\theta_1 - \sum_{j=1}^{n} \beta_j \alpha_1 ln\big(1 + t_{1j}\big),$$

$$\partial\ell/\partial\alpha_2 = n/\alpha_2 - \sum_{j=1}^{n} \beta_j \theta_2 ln\big(1 + t_{2j}\big)\,,$$

and

$$\partial\ell/\partial\theta_2 = n/\theta_2 - \sum_{j=1}^{n} \beta_j \alpha_2 ln\big(1 + t_{2j}\big),$$

where $\psi(\hat{a}) = \grave{\Gamma}(a)/\Gamma(a)$.

Setting the previous non-linear system of likelihood equations to zero and then solving numerically, the ML estimates can be obtained. The invariance property of the ML estimators can be applied to obtain the ML estimators for $R(t_1, t_2)$ and $h(t_1, t_2)$ by replacing the parameters in (10) and (11) by their ML estimators as given below

$$\hat{R}(t_{01}, t_{02}) = \left[1 + \left\{\frac{\hat{\theta}_1\ \hat{\alpha}_1}{\hat{b}}\ ln(1 + t_{01}) + \frac{\hat{\theta}_2\ \hat{\alpha}_2}{\hat{b}}\ ln(1 + t_{02})\right\}\right]^{-\hat{a}},$$

and

$$\hat{h}(t_{01}, t_{02}) = \hat{a}(\hat{a} + 1)\left[\left(\frac{\hat{\theta}_1}{\hat{b}}\left(\frac{\hat{\alpha}_1}{1 + t_{01}}\right)\right)\left(\frac{\hat{\theta}_2}{\hat{b}}\left(\frac{\hat{\alpha}_2}{1 + t_{02}}\right)\right)\right]$$
$$\times\left[1 + \left\{\frac{\hat{\theta}_1\ \hat{\alpha}_1}{\hat{b}}\ ln(1 + t_{01}) + \frac{\hat{\theta}_2\ \hat{\alpha}_2}{\hat{b}}\ ln(1 + t_{02})\right\}\right]^{-2}.$$

Hence the $\hat{R}(t_{01}, t_{02})$ and $\hat{h}(t_{01}, t_{02})$ can be calculated numerically.

3.2 Two-sample maximum likelihood prediction

Considering two- sample prediction, the two samples are assumed to be independent and drawn from the same distribution. In univariate case, the density of the *s-th* order statistic in the future sample is used to obtain the predictive pdf of the *s-th* ordered statistic. The first variable in the vector of bivariate distribution is the ordered observation and the second variable is its concomitants, therefore the joint pdf of the ordered observations and the concomitants is needed to obtain the joint predictive density function of future ordered observations and their concomitants.

For a future bivariate sample of size m, the joint pdf of future *s-th* ordered observation and its *s-th* concomitant denoted by $\left(y_{1(s:m)}, y_{2(s:m)}\right)$, $s = 1,2,\dots,m$, has the joint pdf which is given by (8) after replacing t_1 by $y_{1(s:m)}$ and t_2 by $y_{2(s:m)}$. For simplicity, it can be written as $\left(y_{1(s)}, y_{2(s)}\right)$ instead of $\left(y_{1(s:m)}, y_{2(s:m)}\right)$. Then the joint pdf of $\left(y_{1(s)}, y_{2(s)}\right)$ can be derived as follows:

$$f_{s:m}\left(y_{1(s)}, y_{2(s)}; \underline{\omega}\right) = \frac{m!}{(s-1)!\,(m-s)!} f\left(y_{1(s)}, y_{2(s)}; \underline{\omega}\right)\left[F\left(y_{1(s)}, y_{2(s)}\right)\right]^{s-1}\left[1 - F\left(y_{1(s)}, y_{2(s)}\right)\right]^{m-s},$$

using binomial expansion to simplify the last term in the previous equation, one gets

$$\left[1 - F\left(y_{1(s)}, y_{2(s)}\right)\right]^{m-s} = \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j \left[F\left(y_{1(s)}, y_{2(s)}\right)\right]^j.$$

Thus, the joint probability density of $\left(y_{1(s)}, y_{2(s)}\right)$ is

$$f_{s:m}\left(y_{1(s)}, y_{2(s)}; \underline{\omega}\right) = f\left(y_{1(s)}, y_{2(s)}; \underline{\omega}\right) \sum_{j=0}^{m-s} C_{m,s,j} \left[F\left(y_{1(s)}, y_{2(s)}\right)\right]^{s+j-1},$$

where $\underline{\omega} = (a, b, \alpha_1, \alpha_2, \theta_1, \theta_2)$, and $y_{1(s)}, y_{2(s)}, a, b, \alpha_1, \alpha_2, \theta_1, \theta_2 > 0$, with

$$C_{m,s,j} = \frac{m!}{(s-1)!\,(m-s-j)!\,(j)!} (-1)^j\ .$$

Substituting $f(t_1, t_2)$ given in (8) and $F(t_1, t_2)$ in (9) after replacing t_1 by $y_{1(s)}$ and t_2 by $y_{2(s)}$ then, the joint ML predictive pdf of the ordered observations and their concomitants is given by

$$f_{s:m}\left(y_{1(s)}, y_{2(s)}; \underline{\widehat{\omega}}_{ML}\right) =$$

$$\hat{a}(\hat{a}+1)\left[\left(\frac{\hat{\theta}_1}{\hat{b}}\left(\frac{\hat{\alpha}_1}{1+y_{1(s)}}\right)\right)\left(\frac{\hat{\theta}_2}{\hat{b}}\left(\frac{\hat{\alpha}_2}{1+y_{2(s)}}\right)\right)\right]$$
$$\times\left[1+\left\{\frac{\hat{\theta}_1\ \hat{\alpha}_1}{\hat{b}}\ ln\left(1+y_{1(s)}\right)+\frac{\hat{\theta}_2\ \hat{\alpha}_2}{\hat{b}}\ ln\left(1+y_{2(s)}\right)\right\}\right]^{-(\hat{a}+2)}$$
$$\times\sum_{j=0}^{m-s} C_{m,s,j}\left\{1+\left[1+\left\{\frac{\hat{\theta}_1\ \hat{\alpha}_1}{\hat{b}}\ \ln\left(1+y_{1(s)}\right)+\frac{\hat{\theta}_2\ \hat{\alpha}_2}{\hat{b}}\ \ln\left(1+y_{2(s)}\right)\right\}\right]^{-\hat{a}}\right.$$
$$\left.-\left[1+\left\{\frac{\hat{\theta}_1\ \hat{\alpha}_1}{\hat{b}}\ \ln\left(1+y_{1(s)}\right)\right\}\right]^{-\hat{a}}-\left[1+\left\{\frac{\hat{\theta}_2\ \hat{\alpha}_2}{\hat{b}}\ ln\left(1+y_{2(s)}\right)\right\}\right]^{-a}\right\}^{s+j-1}, \quad (14)$$

where $y_{1(s)}, y_{2(s)}, a, b, \alpha_1, \alpha_2, \theta_1, \theta_2 > 0$.

The point predictors of the future ordered observations and their concomitants $\left(Y_{1(s)}, Y_{2(s)}\right)$, $s = 1,2,\dots,m$, can be obtained as follows:

$$Y_1 = E(y_{1(s)}; \underline{\widehat{\omega}}_{ML}) = \int_{y_{1(s)}=0}^{\infty} y_{1(s)} \int_{y_{2(s)}}^{\infty} f\big(y_{1(s)}, y_{2(s)}; \underline{\widehat{\omega}}_{ML}\big) dy_{2(s)} dy_{1(s)}, \qquad (15)$$

and

$$Y_2 = E(y_{2(s)}; \underline{\widehat{\omega}}_{ML}) = \int_{y_{2(s)}=0}^{\infty} y_{2(s)} \int_{y_{1(s)}}^{\infty} f\big(y_{1(s)}, y_{2(s)}; \underline{\widehat{\omega}}_{ML}\big) dy_{1(s)} dy_{2(s)}, \qquad (16)$$

where $\underline{\widehat{\omega}}_{ML} = \big(\hat{a}, \hat{b}, \hat{\alpha}_1, \hat{\alpha}_2, \hat{\theta}_1, \hat{\theta}_2\big)$.

From (15) and (16), the ML point predictors Y_1 and Y_2 cannot be obtained in closed form. Then, the joint point predictors of the future ordered observations is

$$Y_1, Y_2 = E\big(y_{1(s)}, y_{2(s)}; \underline{\widehat{\omega}}_{ML}\big) = \int_0^{\infty}\int_0^{\infty} y_{1(s)} y_{2(s)} f\big(y_{1(s)}, y_{2(s)}; \underline{\widehat{\omega}}_{ML}\big) dy_{2(s)} dy_{1(s)},$$

which can be evaluated numerically.

4. Numerical Illustration

This section aims to clarify the theoretical results for both estimation and prediction on the basis of simulated and real data set.

4.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented ML estimates based on the generated data from BCESFLO distribution. The ML averages of the estimates of the parameters, rf and hrf are computed. Moreover, *confidence intervals* (CIs) of the parameters, rf and hrf are calculated. Simulation studies are performed using Mathematica 11 for illustrating the obtained results.

The steps of the simulation procedure are as follows:

a) For given values of $\underline{\omega}$ (where $\underline{\omega} = (a, b, \alpha_1, \alpha_2, \theta_1, \theta_2)$), random samples of size *n* are generated from BCESFLO distribution.
b) For each sample size sort $t_{ij}^{\prime}$s, such that $(t_{11}, t_{21}), (t_{12}, t_{22}), \ldots, (t_{1n}, t_{2n})$.
c) Repeat the previous two steps N times, where N represents a fixed number of simulated samples.

- For the number of the population parameter values the Newton-Raphson method can be used, the ML averages and the CIs of the parameters are obtained. Also, the rf, hrf and their CIs are calculated using the ML averages of the parameters.
- Evaluating the performance of the estimates is considered through some measurements of accuracy. To study the precision and variation of the estimates, it is convenient to use the *estimated risk* (ER)
- $ER = \frac{\sum_{i=1}^{N}(\text{estimator}-\text{true value})^2}{\text{N}}$.
- Simulation results of the ML estimates are displayed in Tables 1, 2, where N = 10000 is the number of repetitions, samples of size (*n*=30, 50, 100), and the population parameter values are
  $(a = 1.1, b = 2, \alpha_1 = 1.5, \alpha_2 = 2.2, \theta_1 = 1.9, \theta_2 = 3.1)$,
  and $(a = 0.6, b = 1.2, \alpha_1 = 0.79, \alpha_2 = 1.1, \theta_1 = 0.95, \theta_2 = 1.55)$.
- Tables 1 and 2 present the ML averages, ERs, and CIs of the unknown parameters based. While Tables 3 and 4 display the ML averages, Ers and CIs of the rf and hrf for different values of time t_{01}, t_{02}. The ML two-sample predictors are presented in Table 8.

4.2 Example data set

In this example, a data set is analyzed from a Sankaran-Nair bivariate Pareto distribution [see Sankaran-Nair (1994) and Sankaran and Kundu (2014)]. The generated data set for *n*=30 is:
(0.252, 8.400), (1.105, 0.458), (0.427, 1.602), (12.491, 2.383), (0.260, 0.106), (0.240, 1.769), (4.888, 0.758), (0.870, 0.572), (0.036, 0.254), (1.537, 0.023), (1.508,0.535), (0.239, 1.4120), (0.173, 0.011), (1.090, 1.278), (6.002, 0.017), (0.897, 2.032), (0.690, 0.138), (1.883, 0.398), (0.960, 0.257), (0.561, 0.573), (5.370, 0.325), (0.167, 0.260), (13.602, 0.364), (3.922, 0.938), (0.132, 0.547), (0.603,0.102), (0.226, 0.481), (0.143, 0.779), (0.643, 0.071), (0.349, 1.586).

The Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model. The p values are given, respectively 0.808 and 0.393. The p values showed that the model fits the data very well. Table 5 displays the ML estimates and *standard errors* (Se) of the unknown parameters for the real data set. While Tables 6 and 7 present the ML estimates, Se and CIs of the rf and hrf for different values of time t_{01}, t_{02}. Table 8 gives the ML two-sample predictors for the future observation.

Table 1

ML averages, variance, estimated risks and 95% confidence intervals of the parameters

$(N = 10000, a = 1.1, b = 2, \alpha_1 = 1.5, \alpha_2 = 2.2, \theta_1 = 1.9, \theta_2 = 3.1)$

	Parameters	Averages	Var	ER	UL	LL	Length
30		0.9764	0.0010	0.0163	1.0393	0.9134	0.1259
		1.7883	0.0030	0.0478	1.8963	1.6804	0.2159
		1.5112	9.6547e-06	0.0001	1.5173	1.5051	0.0122
		2.2207	0.00003	0.0005	2.2319	2.2095	0.0224
		1.9142	0.00002	0.0002	1.9219	1.9065	0.0154
		3.1292	0.00007	0.0009	3.1450	3.1134	0.0315
50		0.9951	0.00002	0.0110	1.0040	0.9861	0.0179
		1.8199	0.00007	0.0325	1.8359	1.8038	0.0321
		1.5086	1.4879e-08	0.0001	1.5089	1.5084	0.0005
		2.2168	9.5819e-08	0.0003	2.2174	2.2162	0.0012
		1.9109	2.3872e-08	0.0001	1.9113	1.9106	0.0006
		3.1237	1.9025e-07	0.0006	3.1245	3.1228	0.0017
100		0.9969	1.7630e-06	0.0106	0.9996	0.9944	0.0052
		1.8235	5.2811e-06	0.0312	1.8279	1.8189	0.0090
		1.5089	1.4657e-9	0.0001	1.5089	1.5088	0.0002
		2.2167	1.0896e-08	0.0003	2.2169	2.2166	0.0004
		1.9112	2.3517e-09	0.0001	1.9113	1.9111	0.0002
		3.1236	2.1634e-08	0.0005	3.1239	3.1233	0.0006

Table 2

ML averages, variance, estimated risks and 95% confidence intervals of the parameters

$(N = 10000, a = 0.6, b = 1.2, \alpha_1 = 0.79, \alpha_2 = 1.1, \theta_1 = 0.95, \theta_2 = 1.55)$

n	Parameters	Averages	Var	ER	UL	LL	Length
30		0.9567	0.0005	0.1277	1.0020	0.9113	0.0908
		1.7614	0.0018	0.3169	1.5254	1.6791	0.1647
		1.5235	9.5511e-07	0.5983	1.5254	1.5216	0.0038
		2.2370	7.6043e-06	1.29277	2.2424	2.2316	0.0108
		1.9298	1.5324e-06	0.9599	1.9322	1.9273	0.0049
		3.1521	0.00002	2.5668	3.1598	3.1445	0.0152
50		0.9425	1.2119e-06	0.1173	0.9447	0.9404	0.0043
		1.7359	3.3451e-06	0.2872	1.7395	1.7323	0.0072
		1.5232	9.6456e-07	0.5978	1.5251	1.5212	0.0038
		2.2376	3.2349e-08	1.2941	2.2379	2.2372	0.0007
		1.9293	1.5476e-06	0.9591	1.9318	1.9269	0.0049
		3.1529	6.4230e-08	2.5694	3.1534	3.1525	0.0009
100		0.9411	5.4548e-07	0.1164	0.9426	0.9397	0.0029
		1.7335	1.5400e-06	0.2846	1.7359	1.7311	0.0049
		1.5238	4.0737e-07	0.5988	1.5251	1.5226	0.0025
		22382	1.0435e-09	1.2955	2.2383	2.2381	0.0001
		1.9302	6.5359e-07	0.9608	1.9317	1.9286	0.0032
		3.1538	2.0718e-09	2.5723	3.1539	3.1537	0.0002

Table 3

ML averages, relative absolute biases, variance, estimated risks and 95% confidence intervals of the reliability and hazard rate function

$(\boldsymbol{N=10000, a=1.1, b=2, \alpha_1=1.5, \alpha_2=2.2, \theta_1=1.9, \theta_2=3.1, t_{01}=2, t_{02}=3})$

	rf and hrf	Averages	RAB	Var	ER	UL	LL	Length
30		0.0756	0.2194	0.00002	0.0001	0.0831	0.0681	0.0150
		0.0051	0.1669	5.7242e-08	0.0016	0.0055	0.0046	0.0009
50		0.0761	0.2178	0.00001	0.0001	0.0831	0.0691	0.0140
		0.0051	0.1666	5.2811e-08	0.0016	0.0055	0.0046	0.0009
100		0.0759	0.2160	0.0000	0.0001	0.0826	0.0688	0.0138
		0.0050	0.1654	5.0551e-08	0.0015	0.0055	0.0046	0.0009

Table 4

ML averages, relative absolute biases, variance, estimated risks and 95% confidence intervals of the reliability and hazard rate functions

$(\boldsymbol{N=10000, a=0.6, b=1.2, \alpha_1=0.75, \alpha_2=1.1, \theta_1=0.95, \theta_2=1.55, t_{01}=2, t_{02}=4})$

	rf and hrf	Averages	RAB	Var	ER	UL	LL	Length
30		0.0524	0.8323	1.2496e-08	0.0226	0.0527	0.0522	0.0004
		0.0014	0.1493	5.0258e-10	0.0009	0.0016	0.0013	0.0003
50		0.0522	0.8321	1.8122e-09	0.0224	0.0525	0.0523	0.0002
		0.0013	0.1386	1.8525e-10	0.0006	0.0015	0.0012	0.0003
100		0.0520	0.8320	1.7635e-09	0.0221	0.0521	0.0519	0.0002
		0.0011	0.1382	2.5761e-12	0.0002	0.0012	0.0010	0.0002

Table 5

ML estimates and standard errors of the parameters for the real data set

Parameters	Estimates	Se
	0.9952	0.0109
	1.8246	0.0307
	1.5123	0.0002
	2.2275	0.0008
	1.9156	0.0002
	3.1387	0.0015

Table 6

ML estimates and standard errors of the reliability and hazard rate functions for the real data set

rf and hrf	Estimates	Se
	0.0727	0.0001
	0.0052	0.0044

Table 7

ML estimates and standard errors of the reliability and hazard rate functions for the real data set

rf and hrf	Estimates	Se
	0.0654	0.0001
	0.0034	0.0038

Table 8
ML predictors and bounds of the future observation under two-sample prediction
$(\mathbf{n = 30}, \boldsymbol{a = 0.3}, \boldsymbol{b = 0.5}, \boldsymbol{\alpha_1 = 0.7}, \boldsymbol{\alpha_2 = 2}, \boldsymbol{\theta_1 = 1.9}, \boldsymbol{\theta_2 = 3.1})$

		Estimates	UL	LL	Length
7		0.0068	0.0448	0.0000	0.0448
		0.1141	0.3374	0.0147	0.3227
15		0.0295	0.0452	0.0001	0.0451
		0.2062	0.3563	0.0113	0.3450
18		0.4845	2.1960	0.0222	2.1738
		0.6831	1.3169	0.2638	1.0532

4.3 Concluding remarks

1. It is noticed, from Tables 1 and 2 that the ML averages are very close to the population parameter values as the sample size increases. Also, ER is decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the true parameter values as the sample size increases.
2. The lengths of the CIs of the parameters become narrower as the sample size increases.
3. The ML averages for the rf and hrf perform better as the sample size increases. Also, ER is decreasing when the sample size is increasing.
4. The length of the CI for the first future order statistic is smaller than the length of the CI for the last future order statistic [Tables 8 and 9].
5. The ML interval includes the estimates (between the LL and UL).

5. Bayesian Method

In this section Bayesian estimation and prediction for the vector of parameters $\underline{\omega} = (a, b, \alpha_1, \alpha_2, \theta_1, \theta_2)$ of BCESFLO distribution will be studied.

5.1 Bayesian estimation

AL-Hussaini and Ateya (2005) estimated the parameters under a *squared error loss* (SEL) function using Tierney- Kadane's (1986) approximation form. Iliopoulos *et al*. (2005) considered bivariate gamma distribution for estimating the unknown parameters based on SEL function. Chadi *et al*. (2013) estimated the parameters and the mean time between failures of a bivariate exponential model under various loss functions, namely SEL, absolute error, DeGroot, LINEX and Entropy loss functions. Lin *et al*. (2013) obtained the estimators for the parameters of Moran-Downton bivariate exponential distribution based on complete and Type-II censoring. Independent gamma priors were assumed for scale parameters and beta distribution for correlation parameter. Pradhan and Kundu (2015) derived the estimators for the parameters of the Block and Basu bivariate Weibull distribution.

Considering $(a, b), (\alpha_1, \theta_1)$ and (α_2, θ_2) are independent, a prior density function of

$\underline{\omega} = (a, b, \alpha_1, \theta_1, \alpha_2, \theta_2)$ is given by

$$\pi(\underline{\omega}) = \pi_1(a, b)\, \pi_2(\alpha_1, \theta_1)\, \pi_3(\alpha_2, \theta_2), \tag{17}$$

where the first prior is

$\pi_1(a, b) = \pi_{11}(a|b)\, \pi_{12}(b),$ Assuming that $\pi_{11} \sim Gamma(c_1, b)$ and $\pi_{12} \sim Gamma(c_2, c_3)$, which is more suitable and easier to do the calculations.

the second prior is

$\pi_2(\alpha_1, \theta_1) = \pi_{21}(\alpha_1|\theta_1)\, \pi_{22}(\theta_1),$ Assuming that $\pi_{21} \sim Gamma(c_4, \theta_1)$ and $\pi_{22} \sim Gamma(c_5, c_6)$, which is more suitable and easier to do the calculations.

and the third prior

$\pi_3(\alpha_2, \theta_2) = \pi_{31}(\alpha_2|\theta_2)\, \pi_{32}(\theta_2),$ Assuming that $\pi_{31} \sim Gamma(c_7, \theta_2)$ and $\pi_{32} \sim Gamma(c_8, c_9)$, which is more suitable and easier to do the calculations.

The three priors can be written as

$$\pi_1(a, b) \propto b^{c_1+c_2-1} a^{c_1-1} e^{-b(c_3+a)}, \tag{18}$$
$$\pi_2(\alpha_1, \theta_1) \propto \theta_1^{c_4+c_5-1} \alpha_1^{c_4-1} e^{-\theta_1(c_6+\alpha_1)}, \tag{19}$$
and
$$\pi_3(\alpha_2, \theta_2) \propto \theta_2^{c_7+c_8-1} \alpha_2^{c_7-1} e^{-\theta_2(c_9+\alpha_2)}. \tag{20}$$

Bayes' Theorem for probability distributions is often stated as:

Posterior $\propto$ Likelihood $\times$ Prior.

Now, substituting from (18)-(20) in (17) and using the likelihood function in (12), then the posterior density function will separate into three posteriors, which are

$$\pi_1^*(a,b|t_1,t_2,\beta) \propto \frac{b^{na+c_1+c_2-1}}{\left(\Gamma(a)\right)^n}\left(\prod_{j=1}^{n}\beta_j\right)^{a+1} a^{c_1-1} exp\left[-b\left(\sum_{j=1}^{n}\beta_j + c_3 + a\right)\right], \quad (21)$$

$$\pi_2^*(\alpha_1,\theta_1|t_1,t_2,\beta) \propto$$
$$\theta_1^{c_4+c_5+n-1}\alpha_1^{c_4+n-1}\prod_{j=1}^{n}(1+t_{1j})^{-1}\, exp\left[-\theta_1\sum_{j=1}^{n}\alpha_1\beta_j \ln\left(1+t_{1j}\right) + c_6 + \alpha_1\right], \quad (22)$$

and

$$\pi_3^*(\alpha_2,\theta_2|t_1,t_2,\beta) \propto$$
$$\theta_2^{c_7+c_8+n-1}\alpha_2^{c_7+n-1}\prod_{j=1}^{n}(1+t_{2j})^{-1}\, exp\left[-\theta_2\sum_{j=1}^{n}\alpha_2\beta_j \ln\left(1+t_{2j}\right) + c_9 + \alpha_2\right], \quad (23)$$

where $t_i = (t_{i1}, \dots, t_{in})$, for $i = 1,2$, and $\beta_j = (\beta_1, \dots, \beta_n)$.

By using (21)-(23), hence the posterior density function is given by

$$\pi^*\left(\underline{\omega}|t_1,t_2,\beta\right) \propto \pi_1^*(a,b|t_1,t_2,\beta)\, \pi_2^*(\alpha_1,\theta_1|t_1,t_2,\beta)\, \pi_3^*(\alpha_2,\theta_2|t_1,t_2,\beta) \quad (24)$$

where $\underline{\omega} = (a,b,\alpha_1,\theta_1,\alpha_2,\theta_2)$, $t_j = (t_{i1}, \dots, t_{in})$and $\beta_j = (\beta_1, \dots, \beta_n)$.

The Bayes estimators, $\omega^*_{(SE)}$ are the posterior means under SEL function

$$\omega^*_{(SE)} = E(\omega|t_1,t_2) = \int_{\underline{\omega}} \omega\, \pi^*\left(\underline{\omega}|t_1,t_2,\beta\right) d\underline{\omega},$$

$\omega^*_{(SE)}$ can be evaluated numerically to obtain the Bayes estimates for the parameters.

The Bayes estimators of the $R(t_1,t_2)$ and $h(t_1,t_2)$ can be obtained using (10), (11) and (24), respectively, as given below

$$R^*_{SE}(t_{01},t_{02}) = E\left(R(t_{01},t_{02})|\underline{\omega}\right) = \int_{\underline{\omega}} R(t_{01},t_{02})\, \pi^*\left(\underline{\omega}|t_{01},t_{02}\right) d\underline{\omega}, \quad (25)$$

and

$$h^*_{SE}(t_{01},t_{02}) = E\left(h(t_{01},t_{02})|\underline{\omega}\right) = \int_{\underline{\omega}} h(t_{01},t_{02})\, \pi^*\left(\underline{\omega}|t_{01},t_{02}\right) d\underline{\omega}\,. \quad (26)$$

Equations (25) and (26) can be calculated numerically to obtain the Bayes estimates of the parameters, rf and hrf based on SEL function.

5.2 Bayesian prediction

The joint pdf of $\left(y_{1(s)}, y_{2(s)}\right)$ has the form as given in (14), and hence the joint Bayes predictive density of the ordered observations and their concomitants is given by

$$h\left(y_{1(s)}, y_{2(s)}|\underline{\omega}\right) = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty f\left(y_{1(s)}, y_{2(s)}|\underline{\omega}\right)\pi^*\left(\underline{\omega}|y_{1s}, y_{2s}\right) dadb\, d\alpha_1 d\alpha_2 d\theta_1 d\theta_2. \quad (27)$$

Substituting (14) and (24) in (27), yields the joint Bayes predictive density of $\left(y_{1(s)}, y_{2(s)}\right)$ as

$$h\left(y_{1(s)}, y_{2(s)}|\underline{\omega}\right) = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty I_1 I_2 I_3 I_4 I_5\, dadb\, d\alpha_1 d\alpha_2 d\theta_1 d\theta_2$$

where

$$I_1 = \frac{(a+1)a^{c_1}}{\left(\Gamma(a)\right)^n}\, b^{na+c_1+c_2-3}\theta_1^{c_4+c_5+n}\alpha_1^{c_4+n}\theta_2^{c_7+c_8+n}\alpha_2^{c_7+n},$$

$$I_2 = e^{-\left[b\left(\sum_{j=1}^n \beta_j + c_3 + a\right) + \theta_1(c_6+\alpha_1) + \theta_2(c_9+\alpha_2)\right]}\left(1+y_{1(s)}\right)^{-1}\left(1+y_{2(s)}\right)^{-1},$$

$$I_3 = \left[1 + \left\{\frac{\theta_1\ \alpha_1}{b}\ ln\left(1+y_{1(s)}\right) + \frac{\theta_2\ \alpha_2}{b}\ ln\left(1+y_{2(s)}\right)\right\}\right]^{-(a+2)},$$

$$I_4 = \left(\prod_{j=1}^n\left(1+y_{1j}\right)\right)^{-1}\left(\prod_{j=1}^n\left(1+y_{2j}\right)\right)^{-1}\left(\prod_{j=1}^n \beta_j\right)^{a+1} e^{-\left[\theta_1\alpha_1\sum_{j=1}^n \beta_j y_{1j}^{-\alpha_1} + \theta_2\alpha_2\sum_{j=1}^n \beta_j y_{2j}^{\alpha_2}\right]}$$

and

$$I_5 = \sum_{j=0}^{m-s} C_{m,s,j}\left\{1 + \left[1 + \left\{\frac{\theta_1\ \alpha_1}{b}\ \ln\left(1+y_{1(s)}\right) + \frac{\theta_2\ \alpha_2}{b}\ \ln\left(1+y_{2(s)}\right)\right\}\right]^{-a} - \left[1 + \left\{\frac{\theta_1\ \alpha_1}{b}\ \ln\left(1+y_{1(s)}\right)\right\}\right]^{-a} - \left[1 + \left\{\frac{\theta_2\ \alpha_2}{b}\ ln\left(1+y_{2(s)}\right)\right\}\right]^{-a}\right\}^{s+j-1}. \quad (28)$$

where $y_{1(s)}, y_{2(s)}, a, b, \alpha_1, \alpha_2, \theta_1, \theta_2 > 0$.

The Bayes point predictors of the future ordered observation and their concomitants $\left(Y_{1(s)}, Y_{2(s)}\right)$,
$s = 1,2, \dots, m$, under SEL function can be obtained as follows:

$$Y_{1B} = E\left(y_{1(s)}|\underline{\omega}\right) = \int_0^\infty y_{1(s)} \int_0^\infty h\left(y_{1(s)}, y_{2(s)}|\underline{\omega}\right) dy_{2(s)} dy_{1(s)}, \quad (29)$$

and

$$Y_{2B} = E\left(y_{2(s)}\middle|\underline{\omega}\right) = \int_0^\infty y_{2(s)} \int_0^\infty h\left(y_{1(s)}, y_{2(s)}|\underline{\omega}\right) dy_{1(s)} dy_{2(s)}, \tag{30}$$

where $y_{1(s)}, y_{2(s)}, a, b, \alpha_1, \alpha_2, \theta_1, \theta_2 > 0$.

From (29) and (30), the Bayes point predictors Y_{1B} and Y_{2B} cannot be obtained in closed form. The joint Bayes points predictors of future ordered observation is

$$Y_{1B}, Y_{2B} = E\left(y_{1(s)}, y_{2(s)}|\underline{\omega}\right) = \int_0^\infty \int_0^\infty y_{1(s)} y_{2(s)} f\left(y_{1(s)}, y_{2(s)}|\underline{\omega}\right) dy_{1(s)} dy_{2(s)}. \tag{31}$$

6. Numerical Illustration

This section aims to investigate the precision of the theoretical results of Bayesian estimation and prediction based on the simulated and real data set.

6.1 Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the presented Bayes estimates based on generated data from BCESFLO distribution. Bayes averages of the estimates for the parameters, rf and hrf are computed. Moreover, credible intervals of the parameters, rf and hrf are calculated, Bayes point predictors for a future observation from BCESFLO distribution are computed for the two-sample case. All simulation studies are performed using R programming language.

Simulation algorithm

A. In similar manner to the steps used in Subsection 4.1, different samples can be generated.

B. The Bayes estimates of $a, b, \alpha_1, \alpha_2, \theta_1$ and θ_2 are obtained by following the steps:

1. Assuming the population parameters and the sample size *n*.
2. Generate random samples with different sizes (30, 50, and 100) from the population distribution under study.
3. Repeat Step 2, N times, where N =10000.
4. ω^* is an estimate of ω, and is given by $\overline{\omega^*} = \frac{1}{N}\sum_{j=1}^{N} \omega_j^*$.
5. The ER of ω^*, over the N samples is given by

$$ER(\omega^*) = \frac{1}{N}\sum_{j=1}^{N} \left(\omega_j^* - \omega\right)^2.$$

Using Steps 4 and 5, compute $\overline{a^*}, \overline{b^*}, \overline{\alpha_1^*}, \overline{\alpha_2^*}, \overline{\theta_1^*}, \overline{\theta_2^*},$

$ER(a^*), ER(b^*), ER(\alpha_1^*), ER(\alpha_2^*), ER(\theta_1^*)$ and $ER(\theta_2^*)$.

In the case of two-sample Bayesian prediction

1. Assuming the population parameters and the sample size *n*.
2. Generate a bivariate random sample of size n, say $(T_1, Y_1), (T_2, Y_2)$ as shown in the beginning of this algorithm.
3. Follow steps in Subsection 5.2.

- The underlying population in Tables 10 and 11 displays the averages estimates, ERs and variances of the Bayes case based on sample of different sizes n, and N=10000 repetitions with informative prior. The generated population parameters are $(a = 0.6, \mathrm{b} = 0.8, \alpha_1 = 1.1, \alpha_2 = 1.5, \theta_1 = 1.2, \theta_2 = 1.7)$ and $(a = 2.5, \mathrm{b} = 0.67, \alpha_1 = 3, \alpha_2 = 2.5, \theta_1 = 7.4, \theta_2 = 5.1)$, the given vector of hyper parameters is $(c_1 = 0.1, c_2 = 0.2, c_3 = 0.3, c_4 = 0.4, c_5 = 0.5, c_6 = 0.6, c_7 = 0.7, c_8 = 0.8, c_9 = 0.9)$. Tables 12 and 13 present the Bayes averages, ERs and credible intervals of rf and hrf for different values of the time t_{01}, t_{02} based on informative priors.
- The Bayes two-sample predictors under informative priors are presented in Tables 18 and 19.
- Considering the two-sample prediction and using informative prior, in Tables 18 and 19 the hyper parameters are $(c_1 = 0.1, c_2 = 0.2, c_3 = 0.3, c_4 = 0.4, c_5 = 0.5, c_6 = 0.6, c_7 = 0.7, c_8 = 0.8, c_9 = 0.9)$, the population parameters are $(a = 0.6, b = 0.8, \alpha_1 = 1.1, \alpha_2 = 1.5, \theta_1 = 1.2, \theta_2 = 1.7)$ and $(a = 1.5, b = 0.55, \alpha_1 = 5.8, \alpha_2 = 3.5, \theta_1 = 3.5, \theta_2 = 2.5)$.

6.2 Example data set

The data set is given in Subsection 4.2 and analyzed to illustrate the theoretical results of Bayesian estimation and prediction. Tables 14- 17 present the Bayes averages and ERs, of the estimates of the parameters, rf and hrf, for the real data set under informative prior. Bayes predictors and Se of the future observation are given in Table 19.

6.3 Concluding remarks

In our study we observe the following

1. The variance of the estimates is inversely proportional to the sample size and that the variance of an estimate tends to zero as the sample size tends to infinity.
2. The lengths of the CIs of the parameters become narrower as the sample size increases.
3. The Bayes averages for the rf and hrf performs better as the sample size increases. Also, ER is decreasing when the sample size is increasing.
4. It is interesting to notice that if the variables of the prior density are independent and if the likelihood function factors out with respect to these variables, then the variables of the posterior given data are also independent.

- That if $\pi(\omega_1, \dots, \omega_k) = \prod_{i=1}^{k} \pi(\omega_i)$ and if $L\left(\omega_1, \dots, \omega_k \middle| \underline{t}\right) = \prod_{i=1}^{k} L\left(\omega_i \middle| \underline{t}\right)$, then

$$\pi^*\left(\omega_1, \dots, \omega_k \middle| \underline{t}\right) \propto \pi(\omega_1, \dots, \omega_k) L\left(\omega_1, \dots, \omega_k \middle| \underline{t}\right) = \prod_{i=1}^{k} \pi(\omega_1)\, L\left(\omega_i \middle| \underline{t}\right)$$
$$= \prod_{i=1}^{k} \pi^*\left(\omega_1 \middle| \underline{t}\right) \Rightarrow \left(\omega_1 \middle| \underline{t}, \dots, \omega_k \middle| \underline{t}\right),$$

are independent, the analysis will be easier.

5. The likelihood function of BCESFLO distribution can be derived using the pdf in (8) directly but compounding of $\prod_{i=1}^{k} f\left(t_i \middle| \underline{\omega}\right)$ and $\eta(\beta)$ can be applied to make the ML estimation easier. The results become better as the informative sample size gets larger. In all cases, the simulated percentage coverage is at least 95%.

Table 9

Bayes averages, relative absolute biases, estimated risks and 95% credible intervals for the parameters of BCESFLO

	Parameters	Averages	RAB	ER	UL	LL	Length
30	a	0.60059	9.9478e-04	4.6360e-07	0.6012	0.5986	0.0026
	b	0.8007	0.0008	8.9479e-07	0.8019	0.7992	0.0028
		1.0985	1.3765e-03	2.8541e-06	1.1002	1.0973	0.0029
		1.4993	0.0015	3.9345e-06	1.2030	1.1998	0.0032

$$(\mathbf{N = 10000}, \boldsymbol{a = 0.6}, \boldsymbol{b = 0.8}, \boldsymbol{\alpha_1 = 1.1}, \boldsymbol{\alpha_2 = 1.5}, \boldsymbol{\theta_1 = 1.2}, \boldsymbol{\theta_2 = 1.7})$$

		1.2018	0.0005	8.7362e-07	1.5004	1.4982	0.0021
		1.7015	0.0009	3.7897e-06	1.7037	1.6998	0.0039
50	*a*	0.5998	0.0004	4.4439e-07	0.6006	0.5982	0.0024
	b	0.7996	4.5745e-4	6.5481e-07	0.8008	0.7982	0.0026
		1.0997	0.0003	3.6181e-07	1.1001	1.0978	0.0023
		1.5002	9.6223e-03	1.7531e-06	1.2023	1.1996	0.0027
		1.2011	1.1497e-04	3.7496e-07	1.5012	1.4989	0.0024
		1.7006	3.6611e-04	8.0349e-07	1.7016	1.6994	0.0021
100	*a*	0.5997	5.3088e-04	2.3494e-07	0.6004	0.5988	0.0017
	b	0.7999	7.8312e-05	3.9512e-07	0.8009	0.7986	0.0024
		1.0999	4.1095e-05	2.9580e-07	1.1007	1.0986	0.0022
		1.5001	6.3095e-04	8.3359e-07	1.2002	1.1982	0.0019
		1.1992	3.4961e-05	3.2374e-07	1.5007	1.4988	0.0019
		1.6996	2.1334e-04	4.4509e-07	1.7006	1.6985	0.0021

Table 10

Bayes averages, relative absolute biases, estimated risks and 95% credible intervals for the parameters of BCESFLO

($\mathbf{N = 10000}, \boldsymbol{a = 2.5}, \boldsymbol{b = 0.67}, \boldsymbol{\alpha_1 = 3}, \boldsymbol{\alpha_2 = 2.5}, \boldsymbol{\theta_1 = 7.4}, \boldsymbol{\theta_2 = 5.1}$)

	Parameters	Averages	RAB	ER	UL	LL	Length
30		2.5014	0.0005	2.6580e-06	2.5024	2.4994	0.0029
		0.6710	0.0015	2.0137e-06	0.6724	0.6694	0.0029
		3.0017	0.0005	4.5668e-06	3.0044	2.9998	0.0045
		2.5017	0.0007	3.9767e-06	2.5029	2.4998	0.0030
		7.3985	0.0006	3.2725e-06	7.4001	7.3970	0.0020
		5.1023	0.0004	7.0838e-06	5.1039	5.0998	0.0041
50		2.4991	0.0003	1.0481e-06	2.4999	2.4978	0.0020
		0.6709	0.0013	1.8134e-06	0.6720	0.6695	0.0025
		3.0007	0.0002	1.1669e-06	3.0025	2.9982	0.0043
		2.4984	0.0001	3.0778e-06	2.4999	2.4971	0.0028
		7.4008	0.0001	9.0744e-07	7.4016	7.3995	0.0020
		5.1013	2.6363e-04	2.5136e-06	5.1029	5.0996	0.0033
100		2.4994	2.1057e-04	6.5731e-07	2.5002	2.4984	0.0018
		0.6698	2.5083e-04	4.2225e-07	0.6710	0.6686	0.0023
		3.0001	6.5163e-05	9.1350e-07	3.0015	2.9974	0.0041
		2.5003	5.7028e-05	1.3608e-06	2.5010	2.4993	0.0016
		7.4004	1.4222e-04	3.3561e-07	7.4023	7.4005	0.0018
		5.0988	0.0002	1.5726e-06	5.0997	5.0975	0.0021

Table 11
Bayes averages, relative absolute biases, estimated risks and 95% credible intervals for the reliability and hazard rate functions of BCESFLO
$(\mathbf{N = 10000, a = 0.6, b = 0.8, \alpha_1 = 1.1, \alpha_2 = 1.5, \theta_1 = 1.2, \theta_2 = 1.7, t_{01} = 2, t_{02} = 4})$

	rf and hrf	Averages	RAB	ER	UL	LL	Length
30		0.2886	0.0006	5.4372e-07	0.2897	0.2874	0.0024
		0.0032	0.4062	6.6236e-06	0.0055	0.0012	0.0042
50		0.2881	0.0011	3.3358e07	0.2888	0.2868	0.0019
		0.0055	0.0387	2.8049e-07	0.0063	0.0042	0.0022
100		0.2884	6.3252e-05	1.2286e-07	0.2890	0.2877	0.0014
		0.0055	2.6847e-02	2.7532e-07	0.0062	0.0043	0.0019

Table 12
Bayes averages, relative absolute biases, estimated risks and 95% credible intervals for the reliability and hazard rate functions of BCESFLO
$(\mathbf{N = 10000, a = 1.5, b = 0.55, \alpha_1 = 5.8, \alpha_2 = 3.5, \theta_1 = 3.5, \theta_2 = 2.5, t_{01} = 2, t_{02} = 3})$

n	rf and hrf	Averages	RAB	ER	UL	LL	Length
30		0.0035	0.7896	5.5620e-06	0.0061	0.0009	0.0052
		0.0472	0.0402	4.2526e-06	0.04841	0.0449	0.0034
50		0.0025	0.2684	4.1521e-07	0.0031	0.0014	0.0017
		0.0441	0.0268	2.5180e-06	0.0453	0.0429	0.0024
100		0.0019	0.0051	1.4099e-07	0.0026	0.0009	0.0016
		0.0456	0.0059	2.8211e-07	0.0464	00447	0.0017

Table 13
Bayes estimates and standard errors for the parameter of BCESFLO

Parameters	Estimate	Se
	0.6013	0.0009
	0.8001	0.0008
	1.1025	0.0009
	1.5003	0.0008
	1.2003	0.0007
	1.6999	0.0004

Table 14

Bayes estimates and standard errors for the parameters of BCESFLO

Parameters	Estimates	Se
	1.5018	0.0014
	0.5510	0.0005
	5.8005	0.0007
	3.5027	0.0008
	3.5027	0.0014
	2.4978	0.0013

Table 15

Bayes estimates and standard errors for the reliability and hazard rate functions of BCESFLO

rf and hrf	Estimates	Se
	0.2897	0.0004
	0.0035	0.0008

Table 16

Bayes estimate and standard errors for the reliability and hazard rate functions

	rf and hrf	Estimate	Se
30		0.0029	0.0009
		0.0461	0.0005

Table 17

Bayes predictors, relative absolute biases, estimated risks and 95% credible interval of the future observation

$(\mathbf{N = 10000}, \boldsymbol{a = 0.6}, \boldsymbol{b = 0.8}, \boldsymbol{\alpha_1 = 1.1}, \boldsymbol{\alpha_2 = 1.5}, \boldsymbol{\theta_1 = 1.2}, \boldsymbol{\theta_2 = 1.7})$

	s		Averages	RAB	ER	UL	LL	Length
30	1		3.9999	2.7731e-05	5.0017e-07	4.0009	3.9982	0.0026
			7.0003	4.0064e-05	3.5349e-07	7.0011	6.9989	0.0021
	12		4.0003	6.6825e-05	8.0271e-07	4.0015	3.9986	0.0029
			6.9982	2.5996e-04	3.7367e06	6.9994	6.9968	0.0026
	18		4.0009	0.0002	1.6289e-06	4.0022	3.9992	0.0031
			6.9981	0.0003	5.1302e-06	7.0001	6.9959	0.0042
50	1		4.0005	1.1660e-04	4.6721e-07	4.0013	3.9993	0.0019
			7.0000	2.0907e-06	1.6751e-07	7.0006	6.9989	0.0017
	12		4.0007	1.6923e-04	1.0526e-06	4.0016	3.9989	0.0027
			6.9994	8.2851e-05	5.9514e-07	7.0004	6.9983	0.0020
	18		4.0014	0.0004	3.2526e-06	4.0028	3.9998	0.0030
			7.0008	0.0001	1.5965e-06	7.0024	6.0024	0.0033
100	1		3.9997	6.9537e-05	2.9892e-07	4.0006	3.9989	0.0017
			7.0004	51594e-05	2.9367e-07	7.0009	6.9994	0.0016
	12		3.9997	8.0968e-05	3.7275e-07	4.0005	3.9983	0.0022
			7.0014	2.0482e-04	2.4549e-06	7.0023	6.9996	0.0027
	18		4.0002	5.7244e-05	7.7567e-07	4.0016	3.9983	0.0032

			6.9985	2.1164e-04	3.8607e-06	7.0005	6.9964	0.0041

Table 18
Bayes predictors, relative absolute biases, estimated risks and
95% credible intervals of the future observation
$(\mathbf{N = 10000}, \boldsymbol{a = 1.5}, \boldsymbol{b = 0.55}, \boldsymbol{\alpha_1 = 5.8}, \boldsymbol{\alpha_2 = 3.5}, \boldsymbol{\theta_1 = 3.5}, \boldsymbol{\theta_2 = 2.5})$

	s		Averages	RAB	ER	UL	LL	Length
30	1		3.9986	0.0003	3.1005e-06	4.0007	3.9982	0.0025
			7.0008	1.1791e-04	1.1019e-06	7.0016	6.9991	0.0026
	12		3.9983	4.3381e-04	3.2831e-06	3.9993	3.9957	0.0036
			6.9994	7.2306e-05	1.6241e-06	7.0013	6.9969	0.0043
	18		3.9978	5.4197e-04	6.6847e-06	3.9999	3.9956	0.0043
			7.0022	0.0003	7.1597e-06	7.0041	6.9994	0.0048
50	1		3.9997	7.5194e-05	3.0675e-07	4.0005	3.9989	0.0017
			7.0002	2.6742e-05	2.5213e-07	7.0011	6.9990	0.0021
	12		4.0001	2.7094e-05	4.2356e-07	4.0010	3.9986	0.0024
			6.9983	0.0002	3.5432e-06	6.9999	6.9971	0.0029
	18		4.0008	0.0002	1.0349e-06	4.0018	3.9993	0.0025
			7.0023	3.3174e-04	7.4926e-06	7.0049	6.9995	0.0054
100	1		3.9997	8.631e-05	3.6679e-07	4.0004	3.9989	0.0015
			6.9999	1.5412e-05	2.2928e-07	7.0006	6.9988	0.0019
	12		4.0007	0.0001	1.0153e-06	4.0019	3.9995	0.0024
			6.9989	0.0001	1.8428e-06	7.0003	6.9975	0.0028
	18		3.9996	9.3514e-05	1.1006e-06	4.0010	3.9979	0.0031
			7.0027	3.9138e-04	9.4936e-06	7.0048	6.9999	0.0049

Table 19
Bayes predictors and standard errors
of the future observation

s		Estimate	Se
1		3.9999	0.0005
		7.0009	0.0006
12		3.9993	0.0007
		7.0021	0.0007
18		3.9987	0.0008
		7.00248	0.0019

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