The Riemann Zeta Function and its Zeroes

Abstract

This dissertation includes a detailed of the Riemann zeta functions, with a particular focus its analytic continuation, functional equation and application. We will start with the historical background. Following this we cover certain important preliminaries which are needed to use the functional equation. We then define the Riemann zeta function and prove the functional equation. In addition to this that, we show the Riemann zeta function has generalization in form of the Dirichlet L-function. Then, the zeroes of the Riemann zeta function will be studied. Finally, we establish the zero free region of Riemann zeta function.

1 Introduction

Number theory is among the many branches of mathematics. In order to make progress in number theory, numerous mathematicians must sometimes use techniques from several areas of mathematics, such as complex analysis. The connection between number theory and complex analysis is referred to as Analytic Number Theory.

In 1737, the Swiss mathematician Leonhard Euler put forth the zeta function. Following this, in 1859, the German mathematician Bernhard Riemann introduced the Riemann zeta function. He published an eight page-paper 'On the Number of Prime Number Less than a Given Magnitude'. He presented that this is the link between the zeta function and distribution of prime numbers and he showed the zeta function to be a homomorphic function in a complex plane. However, he did not prove that some the zeros of £ lie on the line $Re(s) = \frac{1}{2}$|, this is called the Riemann hypothesis. This function is useful function in mathematics and is particularly it is important in number theory.

The aim of this dissertation is to focus on studying of the Riemann zeta function, its analytic continuation, functional equation and applications.

This dissertation is divided in to seven sections, In the first section, a brief historical background of the Riemann zeta function will be provided to illustrate the history of function. In the second section, the theory of analytic continuation will be proved.

Following this, third section, provides some useful preliminaries for analytic which will help to us prove function equation of the Riemann zeta function and Dirichlet L function; indeed, these preliminaries are vital when it comes to reaching our aim. In the fourth section, the definition of the Riemann zeta function will be presented in two ways, namely the Dirichlet series and the Euler product. In addition, the analytic continuation of the Riemann zeta function will be proved. In the fifth section, the Riemann zeta function has been generalised and one of these generalizations namely the Dirichlet L-function will be presented, alongside its analytic continuation. Moreover, the functional equation of the Dirichlet L −function will be set out. In the sixth section, the zero of the Riemann zeta functions will be studied. Finally, in the seventh section, the applications of the Riemann zeta function will be provided.

2 Historical background

The series $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$
is the harmonic series. This is one of the most familiar example of an infinite series. The number theory is fundamentally about the positive integers *n*= 1, 2, 3

This series is very interesting in the context of number theory. Unfortunately, it diverges, but only: the sum of the first *n* terms is about Inn and as $n \to \infty$ the sum tends to $+\infty$.

When we replace its general term $\frac{1}{n}$ with the smaller $\frac{1}{n^s}$ *where s* > 1, this makes $\sum\frac{1}{n}$ converge without loss, which is an important property of number theory.
Clearly, this gives rise to £(s) defined by $£(s)=\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$
this series is named after Riemann, who published the fundamental paper on the £(s) properties in 1859. Euler was the first mathematician to introduce the £(s) approximately 120 years earlier he stated that the Riemann zeta function can be expanded as product

$$£(s)=\prod_{\mathrm{p}}\left(\frac{1}{1-\mathrm{p}^{-\mathrm{s}}}\right) \quad \textit{where p all primes'''}$$

this is a very important and powerful result because it means that method analysis can be applied to study the prime number. Euler defined £(s) as a function of the real variable s, whereas Riemann improved the zeta function depending on whether it was allowed to be complex number. Riemann's tried to extend the definition £(s) by Euler from R *to* C. Moreover, he studied the analytic continuation of £(s) and obtained the functional equation. When he studied the zeta function then he found the £(s) to be complex function, he also laid out the key to more thoroughly investigating the distribution of the primes lies. Before proving the Riemann conjecture or the result about prime, he proved

the two main results.
(a) The Riemann zeta function can achieve analytic continuity over the whole s-plan with a simple poles at s equal to 1 with residue 1, such that $£(s) - (s — l)^{-1}$ is integral (entire) function.
(b) The Riemann zeta function satisfies the functional equation

$$\pi^{\frac{-s}{2}}\Gamma\left(\frac{s}{2}\right)£(s) = \pi^{\frac{-1}{2}(1-s)}\Gamma\left(\frac{1-s}{2}\right)£(1-s)$$

we can say that the function on the left side is even a function of $(s-\frac{1}{2})$. It can be concluded that this functional has properties of $£(s)$ for σ less than zero to be deduced from properties for σ greater than 1. In particular, the Riemann zeta function had zero for $\sigma < 0$ at the poles of $\Gamma\left(\frac{1}{2}s\right)$ $at\ s = -2, -4, -6$ ..... These zeroes called the trivial zero, the reminder of the plane where $0 \le \sigma \le 1$ there are non trivial zeros is called the critical strip.

Moreover, Riemann made some remarkable conjectures
(1) In the critical strip the £(s) has several zeros with respect to a =
(2) the integral (entire) function £(s) defined as

$$£(s) = \frac{1}{2}s(s-1)\pi^{\frac{-1}{2}s}\Gamma\left(\frac{1}{2}s\right)£(s)$$

the £(s) has not pole for a greater than or equal $\frac{1}{2}$ and the integral function is an even function of $(s — 1)$. In addition, it has product representation

$$£(s) = e^{A+Bs}\prod_{p}^{\infty}\left(1-\frac{s}{p}\right)e^{\frac{s}{p}}$$

where p runs over the zeros of the Riemann zeta function in the critical strip and A and B are constants. In 1893 Hadamard proved it. Note that we take the information from Jones [5] (p. g. 163) and Davenport [1] (p. g.59).

3 Preliminaries of complex number

In this section we present the important part of complex analysis that will be help to know the basic of analytic number theory.

3. 1 Convergence

Definition 3. 1. [6]
We say a sequence $\{Z_1, Z_{2,....}\}$ *of C* converge to ω in complex number if $\lim_{n\to\infty}|z_n - \omega| = 0$ *and we write* $\omega = \lim_{n\to\infty} z_n$

Definition 3. 2. [20]
Let S is non empty set in C. (F_n) be sequence converges point wise F on S, for every
s in S and $F_n(s)$ tend to $F(s)$. the (F_n) is uniformly convergent to F on S if $\forall\, \varepsilon > 0$ there exist $N = N(\varepsilon)$ such that $\forall n > N$

$$|F(s) - F_n(s)| < \varepsilon \qquad , \forall a \in S$$

the uniformly in this definition that means a number N not depending on s.

3.2 Identity theorem

Let f and h be two analytic functions in region R. Assume that there is $\{s_n\}$ sequence of the different point of R converging to a point $s_0 \in R$ such that the function $f(s_n)$ and $h(s_n)$ have the same values for all $= 1,2,3 \dots$. Then $f = h$ on all of R.

these were taken from ([2] theorem 1).

3.3 Analytic function

We can express that is ANALYTIC at the point z if it is a complex differentiable on open set x which contain the point z. Clearly, f is analytic on D if f is a complex differentiable on a domain D in C. The information in this section was taken from Stein [6] and Mutry [19] and Neubrander [22].

Definition 3.3. Let f be a complex function define on an open D in $\mathbb{C}$, $f : D \to \mathbb{C}$ we can say if there is some number $f'(a)$ then f is complex differentiable at a if

$$\lim_{z\to a} \frac{f(z)-f(a)}{z-a} = f'(a)$$

Definition 3.4. If f is a complex differentiable in $a \in D \subset \mathbb{C}$ then

$$f'(a) = f(a) = \frac{df}{dz}(a) = \lim_{z\to a} \frac{f(z)-f(a)}{z-a}$$

Definition 3.5. Let $S \subseteq \mathbb{C}$ be an open subset and f is a function from S tend to $\mathbb{C}$ is called complex differentiable or homomorphic on S if

$$\lim_{n\to 0} \frac{f(z+h)-f(z)}{h}$$

exist and is finite $\forall z \in S$.

If $\forall z \in S$, f is analytic on S if the f equals its own Taylor series in the neighborhood of z

$$f(z+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z)}{n!} h^n \qquad \textit{for small } h$$

Proposition 3.6. If f and g are analytic (homomorphic) in Ω then

1) $f + g$ *analytic in* Ω *and* $(f + g)' = f' + g'$
2) fg *analytic in* Ω *and* $(fg)' = f'g + fg'$
3) *If* $g(z_0)$ *does not equal* 0 *then* f/g *is holomorhic at* z_0 *and* $(f/g)' = \frac{f'g - fg'}{g^2}$

3.4 Analytic continuation

If the function f is analytic in a region D we can say that the function f will be continued analytically. If f_1 is analytic on a domain D_1 and f_2 is analytic on a domain D_2, where the intersection of the domain does not equal φ and f_1 (s) = f_2 (s) for all s ∈ $D_1 \cap D_2$ then we can state that f_2 is the direct analytic continuation of f_1 to the domain D_2 such that f_2 must be unique for if g is analytic on D_2 and if $g(s) = f_1(s)$ for all s in $D_1 \cap D_2$ then $f_2(s) = g(s)$ $\forall\, s$ is open set $D_1 \cap D_2$ and $f_2(s) = g(s)$ $\forall s \in D_2$.

Example 3.7. let $f_1(x) = \sum_{n=0}^{\infty} s^n$ for (|s| < 1) and $f_2(x) = \frac{1}{(1-s)}$ for (s ∈ $\mathbb{C}\backslash\{1\}$)

then f_2 is direct analytic of f_1 and f_1 is the interior of unit disc and f_2 is on the whole of ($\mathbb{C}\backslash\{1\}$)

Example 3.8. Let $f(x) = 1 + x + x_2 + x_3 + \ldots$.. the series is converges for $|x| < 1$. In fact $f(x) = \frac{1}{(1-x)}$ for such z then $F(x) = \frac{1}{(1-x)}$ $\forall\, z \neq 1$ and is in fact differentiable and analytic ∀ such z and $F(x) = f(x)$ $\forall\, |z| < 1$ then we can express that F is an analytic continuation of f.

Note that all information are taken from Stein [6] and [18].

4 Preliminaries on complex function

We mentioned before, Euler defined the Riemann zeta function for the real variables. However, Riemann extend this function to $\mathbb{C}$ and he studied the analytic continuation of the Riemann zeta function. The functional equation are very significant in analytic continuation. We need to Know some preliminaries on complex function will help to prove the functional equation. This section divides to sex parts we will start to the gamma function and summation formula. Then we present the theta function and entire function. Finally, we provided the Dirichlet series and Euler product.

4.1 The Gamma Function

During the period spanning (1707-1783) Leonhard Euler first introduced the Gamma function Γ(s). Following this, it was studied by other famous mathematicians such as Carl Gauss, Adrien-Marine Legendre. The gamma function is significant function for analytic number theory.

It appears is several area as deferential integration, such as the zeta function and number theory. In this section we shall require some basic properties of the gamma function Γ(s). Note that these properties are taken from [11] Sebah, [6] Stein, [1] Davenport and [13] Forster.

Definition 4.1. For s be positive integer, the gamma function is defined by
$\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}\,\mathrm{dt}$ the integral converges for $s > 0$ the function defined for $\alpha > 0$ can be continued beyond the line $\alpha = 0$.

Proposition 4.2.
$The\ \Gamma(s) extend\ to\ an\ analytic\ function\ in\ the\ half-plane\ Re(s)\ greater\ than\ 0.$

Definition 4.3. Where C is Euler's constant then the Euler gamma function is defined by

$$\frac{1}{\Gamma(s)}=se^{C_s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}} \qquad for\ all\ s$$

Theorem 4.4. $The\ Euler's\ formula\ of\ the\ Gamma\ function\ is$

$$\Gamma(s)=\frac{1}{s}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1}$$

$Proof$. From the definition infinite product

$$\prod_{n=1}^{\infty}(1+u_n)=(1+u_1)(1+u_2)\ldots\ldots.(1+u_n)\ldots.$$

and from the definition the Euler Gamma function , we obtain

$$\frac{1}{\Gamma(s)}=s\lim_{m\to\infty}e^{s(1+\frac{1}{2}+\cdots+\frac{1}{m}-logm)}\lim_{m\to\infty}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$
$$=s\lim_{m\to\infty}m^{-s}\prod_{n=1}^{m}\left(1+\frac{s}{n}\right)$$

$$=s\lim_{m\to\infty}\prod_{n=1}^{m-1}\left(1+\frac{1}{n}\right)^{-s}\prod_{n=1}^{m}\left(1+\frac{s}{n}\right)$$
$$=s\lim_{m\to\infty}\prod_{n=1}^{m}\left(1+\frac{1}{n}\right)^{-s}\left(1+\frac{s}{n}\right)\left(1+\frac{1}{m}\right)^{s}$$

Letting $m\to\infty$
Then we have

$$=s\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-s}\left(1+\frac{s}{n}\right)$$

Theorem 4.5. $The\ function\ \Gamma(s) satisfies\ the\ functional\ equation$

$$\Gamma(s+1)=s\Gamma(s)$$

$Proof$. From the Euler's formula we have

$$\frac{\Gamma(s+1)}{\Gamma(s)}=\frac{\frac{1}{s+1}}{\frac{1}{s}}\lim_{n\to\infty}\prod_{n=1}^{m}\frac{\left(1+\frac{1}{n}\right)^{s+1}\left(1+\frac{s+1}{n}\right)^{-1}}{\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1}}$$

$$= \frac{s}{s+1} \lim_{n\to\infty} \prod_{n=1}^{m} \frac{\left(\frac{n+1}{n}\right)^{s+1}\left(\frac{n+s+1}{n}\right)^{-1}}{\left(\frac{n+1}{n}\right)^{s}\left(\frac{n+s}{n}\right)^{-1}}$$

then we have

$$= \frac{s}{s+1} \lim_{n\to\infty} \prod_{n=1}^{m} \left(\frac{n+1}{n}\right)\left(\frac{n+s}{n+s+1}\right)$$

passing to the limit as $s \rightarrow 0$ thus

$$= \frac{s}{s+1} \lim_{m\to\infty} \left(\frac{m+1}{m}\right)\left(\frac{s+1}{1+s}\right) = s$$

Theorem 4.6. *The addition formula of the gamma function is*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

for any number s is not an integer.

Corollary 4.7. *The duplication formula of the gamma function is*

$$\Gamma(2n)\Gamma\left(\frac{1}{2}\right) = 2^{2n-1}\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)$$

for n is natural number.

Corollary 4.8. $\Gamma(n+1) = n!$
for every non negative integer n.

Legendre obtained in 1809 the duplication formula.

Theorem 4.9. *The Legendre duplication formula is*

$$\Gamma(x)\Gamma\left(x+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2x+1}}\Gamma(2x)$$

Corollary 4.10. $\int_{-\infty}^{+\infty} e^{-u^2}\, du = \sqrt{\pi}$

Corollary 4.11. $\Gamma(s) = \lim_{n\to\infty} \frac{1.2\ldots.(n-1)n^2}{s(s+1)\ldots.(s+n-1)}$

Corollary 4.12. *The sine product is*

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

for all z is complex number.

4.2 Summation formula

4.2.1 The Euler-Maclaurin formula

Definition 4.13. [16] (p.g. 66)

The sum of $f(n)$ can written as the stieltjes integral

$$\sum_{a<n\leq b} f(n) = \int_a^b f(n)\, d[x],$$

if $a, b \in \mathbb{Z}$ and f is continuous in $[a, b]$ where $[x]$ is integral part of x,

$$[x] = \max\{l \in \mathbb{Z}: l \leq x\}$$

putting $\psi(x) = x - [x] - \frac{1}{2}$ now we can derive the Euler-Maclaurin formula by the partial integration.

Lemma 4.14. [16] *Let* $a, b \in Z$ *with* a *smaller than* b *, we have*

$$\sum_{a<n\leq b} f(n) = \int_a^b (f(x) + \psi(x)f'(x))\, dx + \frac{1}{2}(f(b) - f(a))$$

Theorem 4.15.

Let $f(x)$ *has twice continuously differentiable function on* $[a, b]$ *and define* $\rho(x)$ *and* $\sigma(x)$ *by*

$$\rho(x) = \frac{1}{2} - \{x\} \quad and \quad \sigma(x) = \int_0^x \rho(u)du$$

There for

$$\sum_{a<n\leq b} f(n) = \int_a^b f(x)dx + \rho(b)f(b) - \rho(a)f(a) - \int_a^b \sigma(x)\, f'(x)dx$$

where $f(x)$ be continuous differentiable on [a, b].

4.2.2 The poisson summation formula

Theorem 4.17. [7]

Suppose $f, \hat{f}$ *are Fourier transform, the poisson summation formula is*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

with Fourier transform of

$$f = \hat{f}(t) = \int_{\mathbb{R}} f(x)e^{-2\pi ixt}$$

4.2.3 The partial summation

Lemma 4.18. [14]
let $f(x)$ a continuation differentiable function on the interval $[a,b]$ and let $C(x) = \sum_{a<n<x} c_n$ where c_n be arbitrary complex numbers. then we have

$$\sum_{a<n\leq b} c_n\, f(n) = -\int_a^b C(x)\, f'(x)\mathrm{dx} + \int_a^b C(b)\, f'(b)$$

4.3 Theta Function

Note all the information are taken from [2] Hassen, [8] Segarra, [9] Steiger and [21] Pitman.

Definition 4.19. Jacobi's theta function $\Theta(t)$ is defined by

$$\Theta(t) := \sum_{n=1}^{\infty} exp\,(-n^2\pi t) \qquad (t > 0)$$

Proposition 4.20. $\Theta(t)$ *can be written*

$$\Theta(t) = 1 + 2\sum_{n=-\infty}^{\infty} exp\,(-\pi n^2 t)$$

where Θ is the Jacobi theta function.

Proposition 4.21. *If $t > 0$ then*

$$\Theta(t) = \frac{1}{\sqrt{t}}\,\Theta\left(\frac{1}{t}\right)$$

Proof. we start from definition (4.19)

$$\Theta(t) := \sum_{n=\infty}^{\infty} exp\,(-n^2\pi t)$$

there is very important sum formula is called passion formula (4.17)

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} f(y) e^{-2\pi iky} dy$$

then we have

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} e^{-\pi y^2 x} e^{-2\pi iky} dy$$
$$= \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} e^{-\pi y^2 x - 2\pi iky} dy$$

$$= \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} e^{-\pi x(y^2 + 2i\frac{k}{x}y + i^2\frac{k^2}{x^2} - i^2\frac{k^2}{x^2})} dy$$

From $\left(y^2 + 2i\frac{k}{x}y + i^2\frac{k^2}{x^2}\right) = \left(y + i\frac{k}{x}\right)^2$ then we have

$$= \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} e^{-\pi x\left(\left(y+i\frac{k}{x}\right)^2 - i^2\frac{k^2}{x^2}\right)} dy$$
$$\Rightarrow \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \sum_{n\in\mathbb{Z}} \int_{-\infty}^{+\infty} e^{-\pi k^2\frac{1}{x}} e^{\pi x\left[y+i\frac{k}{x}\right]^2} dy$$
$$\Rightarrow \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \sum_{n\in\mathbb{Z}} e^{-\pi k^2\frac{1}{x}} \int_{-\infty}^{+\infty} e^{\pi x\left[y+i\frac{k}{x}\right]^2} dy$$

the change of variable

$$y + i\frac{k}{x} = \mathrm{z} \quad , \quad \mathrm{dy} = \mathrm{dz} \qquad |_{-\infty}^{+\infty} \to |_{-\infty+i\frac{k}{x}}^{+\infty+i\frac{k}{x}}$$

Then we obtain

$$\Longrightarrow \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \sum_{n\in\mathbb{Z}} e^{-\pi k^2 \frac{1}{x}} \int_{-\infty+i\frac{k}{x}}^{+\infty+i\frac{k}{x}} e^{-\pi x z^2} dz$$

It can be shown that

$$\int_{-\infty+i\frac{k}{x}}^{+\infty+i\frac{k}{x}} e^{-\pi x z^2} dz = \int_{-\mathbb{R}}^{\mathbb{R}} e^{-\pi x z^2} dz$$

we will divide this integral to three parts

$$\int_{-\mathbb{R}}^{\mathbb{R}} e^{-\pi x z^2} dz = \int_{-\mathbb{R}}^{-\mathbb{R}+i\frac{k}{x}} e^{-\pi x z^2} dz + \int_{-\mathbb{R}+i\frac{k}{x}}^{\mathbb{R}+i\frac{k}{x}} e^{-\pi x z^2} dz + \int_{\mathbb{R}+i\frac{k}{x}}^{\mathbb{R}} e^{-\pi x z^2} dz$$

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \sum_{n\in\mathbb{Z}} e^{-\pi k^2 \frac{1}{x}} \int_{-\infty}^{+\infty} e^{-\pi x z^2} dz$$

From Gauss integral

$$\sum_{n\in\mathbb{Z}} e^{-\pi k^2 \frac{1}{x}} \sqrt{\frac{\pi}{\pi x}}$$

then

$$\sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \sum_{n\in\mathbb{Z}} e^{-\pi k^2 \frac{1}{x}} \sqrt{\frac{1}{x}}$$

from

$$\Theta(x) := \sum_{n\epsilon\mathbb{Z}} e^{-\pi n^2 x}$$

then we have

$$\Theta(x) = \frac{1}{\sqrt{x}} \Theta\left(\frac{1}{x}\right)$$

Proposition 4.22. *As* $t \to 0$ *from Proposition* (2.15) *then for some* $c > 0$ *let*

$$\left|\theta(t) - \frac{1}{\sqrt{t}}\right| < e^{\frac{-C}{t}}$$

Proposition 4.23.

$$\Theta\left(\frac{i}{x}\right) = x^{\frac{1}{2}} \Theta(ix)$$

4.4 Entire functions

An entire function (integral function) is an analytic in the whole complex plane.

4.4.1 Entire functions of the finite order

Note we take the information in this section from [1] Davenport, [6] Stein and [10] Rubin. let f(z) be an entire function is said f(z) to be finite order if there exists α greater than 0 such that

$$f(z) = O\left(e^{|z|^{\alpha}}\right) \qquad as \ |z| \to \infty \qquad (4.1)$$

we must $\alpha > 0$ with the property (4.1) is called the order of $f(z)$.

Lemma 4.24. *Let* $f(z)$ *be an entire function of finite order and* $f(z)$ *have no zeros*;

$this\ is\ necessarily\ to\ form\ e\ g(z) where\ g(z) is\ polynomial\ andits\ order\ is\ the\ degree\ of\ polynomial\ g(z) and\ so\ is\ an\ integer.$

Lemma4.25.
$Let\ f\ be\ an\ entire\ function. If\ there\ exist\ a\ positive\ number\ \alpha\ and\ a\ constant\ B > 0\ and\ 0 < R1 < R2 < R3 \ldots..with$

$$\lim_{m\to\infty} R_m = \infty \qquad such\ that\ |f(z)| \le Be^{|z|^{\alpha}}$$

$when\ ever\ |z| \in \{R_1, R_2 \ldots.\} then\ there\ is\ f(z) = e^{g(z)}\ if\ g(z) is\ polynomial.$
$Hence\ f(z)\ is\ of\ finite\ order\ and\ this\ order\ is\ equal\ of\ degree\ g(z).$

4.4.2 Infinite product

Definition 4.26. Let sequence $\{u_n\}$ of complex numbers $P_n = (1+u_1)(1+u_2)\ldots\ldots(1+u_n) = \prod_{k=1}^{\infty}(1+u_k)$ and $p = \lim_{n\to\infty} p_n$ exists. Then we can write the product $p = \prod_{k=1}^{\infty}(1+u_k)$ the p_n are partial product of infinite product. then we express the product $\prod_{k=1}^{\infty}(1+u_k)$ converges if $\{u_n\}$ is converges if there exist the $\lim_{n\to\infty} p_n$.

Proposition 4.27. $Let \sum a_n < \infty$ then $\prod_{n=1}^{\infty}(1+a_n)$ this product is converges to zero if and only if one of its factors is zer*o*.

Proposition 4.28. Assume the sequence $\{F_n\}$ is analytic functions on the open set Ω.
If there exist constant c_n greater than zero such that

$$\sum c_n < \infty \quad \text{and} \quad |F_n(z) - 1| \le c_n \qquad \forall\, z \in \Omega$$

then:

(i) $\prod_{n=1}^{\infty} F_n(z)$ this product converges uniformly in Ω to $F(z)$ analytic function.

(ii) let $\frac{F'(z)}{F(Z)} = \sum_{n=1}^{\infty}\frac{F_n'(z)}{F_n(z)}$ if $F_n(z)$ dose not vanish for any n.

Lemma 4.29. If $u_1, \ldots. u_N \in \mathbb{C}$ and if

$$p_N = \prod_{n=1}^{N}(1+u_n) \qquad p_N^* = \prod_{n=1}^{N}(1+|u_n|)$$

then $p_N^* \le e^{(|u_1|+\cdots+|u_N|)}$ and $|p_N - 1| \le p_N^* - 1$

4.5 Dirichlet series

Definition 4.30. [13]
A Dirichlet series is a series of the form $f(s) = \sum_{n=1}^{\infty}\frac{a_n}{n^s}$ such that s ∈ ℂ and $(a_n)_{n\ge1}$ is a sequence of complex numbers.

Remark 4.31. [4] and [5]
The Dirichlet series $F(x) = \sum_{n=1}^{\infty}\frac{f(x)}{n^s}$ where $s = \alpha + it$, $\alpha > a$ and f is an arithmetic function then the Dirichlet series $\sum_{n=1}^{\infty}\frac{f(x)}{n^s}$ is convergent in a half-plane $\alpha > \alpha_c$ thus the series is an analytic function of s for $\alpha > \alpha_c$.

Theorem 4.32. [4]

$$For\ any \sum f(x)n^{-s}\ with\ \alpha_a\ finite\ we\ have\ \ 0 \leq \alpha_c - \alpha_a \leq 1$$

Theorem 4.33. [13]

suppose $f(s) = \sum_{n=1}^{\infty} \frac{f(x)}{n^s}$ be a Dirichlet series for some s_0 is complex number, it has bounded partial sums then the $\sum_{n=1}^{\infty} \frac{f(x)}{n^s}$ converges with $R(s) > \alpha_0 := R(s_0)$ for all s is complex number and the assume that the converge uniformly on every compact subset of

$$k \subset H(\alpha_0) = \{s \in \mathbb{C} : Re(s) > \alpha_0\}$$

Hence f is analytic function in H(α_0)

Theorem 4.34. [5] (p.g. 180)

Assume that F(s) = X∞ n=1 f(n) ns G(s) = X∞ n=1 g(n) ns H(s) = X∞ n=1 h(n) ns if h = f ∗ g then H(s) = F(s)G(s) such that F(s) and G(s)is absolutely converge for all s Proof. we can multiplay F(s) and G(s) because F(s) and G(s) both converge absolutely

$$F(s)\ G(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{f(k)g(n)}{(kn)^s}$$

We replace $m = kn$ then we have

$$= \sum_{m=1}^{\infty} \sum_{m=kn} \frac{f(k)g(n)}{m^s}$$

$$= \sum_{m=1}^{\infty} \frac{(f * g)(m)}{m^s}$$

$$= \sum_{m=1}^{\infty} \frac{h(m)}{m^s}$$

$$= \mathrm{H(s)}$$

4.6 Euler product

Definition 4.35. [4]

we called f is multiplicative if an arithmetic function f is not identically zero and $f(mn) = f(m)f(n)$ when $(m, n) = 1$ and f multiplicative function is called completely multiplicative if $f(mn) = f(m)f(n)$ for all positive integer m and n.

* The next theorem was discovered in 1737 by Euler.

Theorem 4.36. [5] (p. g. 183)

a) Let $\sum_{n=1}^{\infty} f(n) = \prod_p (1 + f(p) + f(p^2) + \cdots)$ if f is multiplicative arithmetic function such that $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent.
b) Let f is completely multiplicative $\sum_{n=1}^{\infty} f(n)$ is convergent and then
$\sum_{n=1}^{\infty} f(n) = \prod_p \left(\frac{1}{1-f(p)}\right)$ this product is called the Euler product of the series.

5 - The Riemann Zeta Function

The important definition in this dissertation is the definition of the Riemann zeta function. The Riemann zeta function $\zeta(x)$ can be defined in two ways: as a Dirichlet series or as a Euler product. Note that this information was taken from [14] Karatsuba, [23] Batemann, [20] Everest and [3] Karatsuba. We will now introduce the first definition.

Definition 5.1. For Re(s) > 1 the Riemann zeta function ζ(s) is defined by Dirichlet series: $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \cdots$ The series converges $Re(s) > 1$ and it is analytic function in the half plane $Re(s) > 1$.

There is an analogue of the Euler product for $\zeta(s)$.

Lemma 5.2. For $Re(s) > 1$ the Riemann zeta function defined by the Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

Proof. Let $Re(s)$ greater than 1 and $X \geq 2$ be an integer. we can use the series

$$1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots$$

which is absolute converges and from the unique decomposition of a positive integer into prime factors we obtain

$$\prod_{p \leq x} \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_{p \leq x} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right)$$
$$= \sum_{p \leq x} \frac{1}{n^s} + R(s; X)$$

where

$$|R(s;X)| \leq \sum_{n>X} \left|\frac{1}{n^s}\right| = \sum_{n>X} \frac{1}{n^\sigma} \leq \frac{1}{\sigma - 1} X^{1-\sigma}$$

where $\sigma = Re(s) > 1$ passing to the limit $X \to +\infty$
hence,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

Corollary 5.3. *For* $Re(s) = \sigma > 1$ *then,* $\zeta(s) \neq 0$

Proof. This follows from Euler's product formula

$$\frac{1}{|\zeta(s)|} = \left|\prod_p \left(1 - \frac{1}{p^s}\right)\right| \leq \prod_p \left(1 - \frac{1}{p^\sigma}\right)$$

$$< \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \leq 1 + \int_{1}^{\infty} \frac{1}{u^{\sigma}} du$$

$$= 1 + \frac{1}{1-\sigma} u^{-\sigma+1} |_{1}^{\infty}$$

then, we have

$$1 - \frac{1}{1-\sigma} = \frac{\sigma}{\sigma - 1}$$

where $\sigma > 1$ then,

$$|\zeta(s)| > \frac{\sigma}{\sigma - 1} > 0$$

5.1 The zeta function is analytic

The next theorems of convergent uniformly is that it preserves the complex function is analyticity.

Theorem 5.4. [6] $(p.g.169)$
The series of the function $\zeta(s)$ converges for $Re(s) > 1$ and the Riemann zeta function is analytic (homomorphic) in this half plane.

$Proof.$ If $s = \sigma + it$ where σ and t are real then

$$|n^{-s}| = \left|e^{-s\log n}\right| = \left|e^{-\sigma\log n} e^{it\log n}\right| = e^{-\sigma\log n} = n^{-\sigma}$$

As a consequence.
The series of the function zeta(s) is uniformly bounded if $\alpha > 1 + \delta > 1$ by $\sum \frac{1}{n^{1+\delta}}$
then,
$\sum \frac{1}{n^{1+\delta}}$ is converges.
Therefore, the series of Riemann zeta function $\sum \frac{1}{n^{s}}$ converges uniformly on every half-plane $Re(s) > 1 + \delta > 1$
Then,
$\zeta(s)$ is analytic (homomorphic) function in $Re(s) > 1$.

Theorem 5.5. [1]
The function $\zeta(s)$ is homomorphic (analytic) every where except a simple pole at $s = 1$ with residue 1.
Theorem 5.6. [20] ($p.g.173$)
assumes that $S \subseteq \mathbb{C}$ is open and we have F is function from s goes to complex number and sequence of functions F_N from S tend to $\mathbb{C}$ converge to F uniformly on S if the sequence of functions F_N is analytic then F is analytic.

5.2 Continuation of $\zeta(x)$

Analytic continuation is a very important idea from complex analysis. We shall extend $\zeta(s)$ to the half-plane $Re(s) > 0$. Given that the function g is a convergent power series on D is the disk of positive radius, and the analytic function is defined as any domain containing D work with g on D is called continuation of g.

Corollary 5.7. [6] (*corollary* 2.6)
for $Re(s) > 0$ we have

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \delta_n(s)$$

where in the half-plane $Re(s) > 0$ the series $\sum_{n=1}^{\infty} \delta_n(s)$ is homomorphic (analytic).

We will now present the useful proposition before prove the corollary.

Proposition 5.8. [6] (*proposition* 2.5)
The sequence of entire function $\{\delta n(s)\}_{n=1}^{\infty}$ that satisfy the estimate

$$|\delta_n(s)| \leq |s| \,/\, n^{\sigma+1}$$

where $s = \sigma + it$, such that

$$\sum_{1 \leq n < N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1 \leq n < N} \delta_n(s)$$

whenever N is an integer greater than 1
to prove the proposition, we compare

$$\sum_{1 \leq n < N} n^{-s} \quad with \quad \sum_{1 \leq n < N} \int_n^{n+1} x^{-s}\, dx$$

And set

$$\delta_n(s) = \int_n^{n+1} \left[\frac{1}{n^s} - \frac{1}{x^s}\right] dx$$

from the mean-value theorem we apply to $f(x) = x^{-s}$ we get

$$\left|\frac{1}{n^s} - \frac{1}{x^s}\right| \leq \frac{|s|}{n^{\sigma+1}} \quad whenever \quad x \in [n, n+1]$$

then

$$|\delta_n(s)| \leq \frac{|s|}{n^{\sigma+1}}$$

Then,

$$\int_1^N \frac{dx}{x^s} = \sum_{1 \leq n < N} \int_n^{n+1} \frac{dx}{x^s}$$

we will now prove the corollary, we assume that $Re(s) > 1$. From the proposition we let N tend to infinity and we see by the estimate $|\delta_n(s)| \leq |s|/n^{\sigma+1}$.
we have $\sum \delta_n(s)$ is uniform converges in any half plane $Re(s) > 0$.
since the $\sum \frac{1}{n^s}$ converges to $\zeta(s)$ if $Re(s) > 1$.
then, $\sum \delta_n(s)$ is analytic when Re(s) > 0.

then $\zeta(s)$ is analytic continuation.

Lemma 5.9. [14] ($lemma$ 2)
For Re(s) greater than 0 and N ≥ 1 we have

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + s\int_{N}^{\infty} \frac{\rho(u)}{u^{s+1}}du$$

Where $\rho(u) = \frac{1}{2} - \{u\}$

$Proof.$ let $0 < N < M$ be integer numbers. Then we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{N} \frac{1}{n^s} + \sum_{n=N+1}^{\infty} \frac{1}{n^s}$$

$$= \sum_{n=1}^{N} \frac{1}{n^s} + \lim_{M\to\infty} \sum_{N+\frac{1}{2}<n\leq M+\frac{1}{2}}^{\infty} \frac{1}{n^s} \tag{5.1}$$

we will apply the the Euler-Maclaurin sum formula (remark 4.16) for second sum in (5.1) where $\rho(u) = \frac{1}{2} - \{u\}$ and $\{u\} = u - [u]$ thus $\rho(u) = \frac{1}{2} - u - [u]$
we obtain

$$\sum_{N+\frac{1}{2}<n\leq M+\frac{1}{2}}^{\infty} \frac{1}{n^s} = \int_{N+\frac{1}{2}}^{M+\frac{1}{2}} \frac{1}{u^s}du + s\int_{N+\frac{1}{2}}^{M+\frac{1}{2}} \frac{\rho(u)}{u^{s+1}}du \tag{5.2}$$

where $\rho(M + \frac{1}{2}) = \frac{1}{2} - (M + \frac{1}{2}) + [M + \frac{1}{2}] = \frac{1}{2} - M - \frac{1}{2} + M = 0$
also, $\rho(N + \frac{1}{2}) = \frac{1}{2} - (N + \frac{1}{2}) + [N + \frac{1}{2}] = 0$ Now, we solve this integral

$$\int_{N+\frac{1}{2}}^{M+\frac{1}{2}} \frac{1}{u^s}du = \left[\frac{1}{1-s}u^{1-s}\right]_{N+\frac{1}{2}}^{M+\frac{1}{2}} = \frac{1}{1-s}\left(M+\frac{1}{2}\right)^{1-s} - \frac{1}{1-s}\left(N+\frac{1}{2}\right)^{1-s} \tag{3.5}$$

then we solve the second integral

$$s\int_{N+\frac{1}{2}}^{M+\frac{1}{2}} \frac{\rho(u)}{u^{s+1}}du = s\int_{N+\frac{1}{2}}^{N} \frac{\rho(u)}{u^{s+1}}du + s\int_{N}^{M+\frac{1}{2}} \frac{\rho(u)}{u^{s+1}}du \tag{5.4}$$

then

$$s\int_{N+\frac{1}{2}}^{M+\frac{1}{2}} \frac{\rho(u)}{u^{s+1}}du = s\int_{N+\frac{1}{2}}^{N} \frac{\frac{1}{2} - u + [u]}{u^{s+1}}du$$

$$= s\int_{N+\frac{1}{2}}^{N} \frac{\frac{1}{2} + N - u}{u^{s+1}}du$$

$$= s\int_{N+\frac{1}{2}}^{N} \frac{\frac{1}{2} + N}{u^{s+1}}du - s\int_{N+\frac{1}{2}}^{N} \frac{u}{u^{s+1}}du$$

$$= s\left(\frac{1}{2}+N\right)\int_{N+\frac{1}{2}}^{N}\frac{1}{u^{s+1}}\,du + s\int_{N}^{N+\frac{1}{2}}\frac{1}{u^{s}}\,du$$

$$= s\left(\frac{1}{2}+N\right)\left[\frac{u^{-s}}{-s}\right]_{N+\frac{1}{2}}^{N} + s\left[\frac{u^{1-s}}{1-s}\right]_{N}^{N+\frac{1}{2}}$$

$$= -\left(\frac{1}{2}+N\right)N^{-s} + \left(\frac{1}{2}+N\right)\left(\frac{1}{2}+N\right)^{-s} + \frac{s}{1-s}\left(\frac{1}{2}+N\right)^{1-s} - \frac{s}{1-s}N^{1-s}$$

$$= -\frac{1}{2}N^{-s} - N^{1-s} + \left(\frac{1}{2}+N\right)^{1-s} + \frac{s}{1-s}\left(\frac{1}{2}+N\right)^{1-s} - \frac{s}{1-s}N^{1-s}$$

where $\frac{s}{1-s} = \frac{s-1+1}{1-s} = \frac{s-1}{1-s} + \frac{1}{1-s} = -1 + \frac{1}{1-s}$

then

$$= -\frac{1}{2}N^{-s} - N^{1-s} + \left(\frac{1}{2}+N\right)^{1-s} + \left(-1+\frac{1}{1-s}\right)\left(N+\frac{1}{2}\right)^{1-s} - \left(-1+\frac{1}{1-s}\right)N^{1-s}$$

$$= -\frac{1}{2}N^{-s} - \frac{1}{1-s}N^{1-s} - N^{1-s} + \left(\frac{1}{2}+N\right)^{1-s} - \left(N+\frac{1}{2}\right)^{1-s} + \frac{1}{1-s}\left(N+\frac{1}{2}\right)^{1-s} + N^{1-s}$$

$$= -\frac{1}{2}N^{-s} - \frac{1}{1-s}N^{1-s} + \frac{1}{1-s}\left(N+\frac{1}{2}\right)^{1-s} \qquad (5.5)$$

from equations (5.2) , (5.3), (5.4), (5.5) we get

$\sum_{N+\frac{1}{2}<n\leq M+\frac{1}{2}}^{\infty}\frac{1}{n^{s}} =$

$\frac{1}{1-s}\left(M+\frac{1}{2}\right)^{1-s} - \frac{1}{1-s}\left(N+\frac{1}{2}\right)^{1-s} + s\int_{N}^{M+\frac{1}{2}}\frac{\rho(u)}{u^{s+1}}du - \frac{1}{2}N^{-s} - \frac{1}{1-s}N^{1-s} + \frac{1}{1-s}\left(N+\frac{1}{2}\right)^{1-s}$

$$= -\frac{1}{2}N^{-s} - \frac{1}{s-1}N^{1-s} + \frac{1}{1-s}\left(M+\frac{1}{2}\right)^{1-s} + s\int_{N}^{M+\frac{1}{2}}\frac{\rho(u)}{u^{s+1}}\,du$$

thus

$$\sum_{n=N+1}^{\infty}\frac{1}{n^{s}} = \lim_{M\to\infty}\sum_{n=N+1}^{M+1}\frac{1}{n^{s}} = -\frac{1}{2}N^{-s} - \frac{1}{s-1}N^{1-s} + s\int_{N}^{M+\frac{1}{2}}\frac{\rho(u)}{u^{s+1}}\,du$$

thus , from (5.1) we obtain

$$\zeta(s) = \sum_{n=1}^{N}\frac{1}{n^{s}} - \frac{1}{2}N^{-s} - \frac{1}{s-1}N^{1-s} + s\int_{N}^{M+\frac{1}{2}}\frac{\rho(u)}{u^{s+1}}\,du$$

the last integral is analytic function in the half plane $Re(s)$ greater than 0. This lemma now follows by analytic continuation.

Theorem 5.10. [13]

The $\zeta(s)$ has analytic continuation to $\{s \in \mathbb{C} \mid Re(s) > 0\}$ with a simple pole at s equal 1 with residue 1.

5.3 Functional equation of the $\zeta(s)$

Functional equation of the Riemann zeta function is one of the main goals of this dissertation. We will now present two proof of the functional equation by using some of the preliminaries.

Theorem 5.11. [1] ($p.g.$ 61) and [8]
$The\ \zeta(s)\ satisfies\ the\ functional\ equation$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{\frac{-1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

$Proof$. Riemann started from the definition of gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt \qquad (5.6)$$

we will replace s $\rightarrow \frac{s}{2}$ in (5.6)

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty t^{\frac{s}{2}-1}e^{-t}dt$$

valid for $\alpha > 0$. We sub: $t = \pi n2x$ and $dt = \pi n2dx$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty (\pi n^2 x)^{\frac{s}{2}-1}e^{-\pi n^2 x}\pi n^2 dx$$

$$= \int_0^\infty \pi^{\frac{s}{2}-1}n^{2(\frac{s}{2}-1)}x^{\frac{s}{2}-1}e^{-\pi n^2 x}\pi n^2 dx$$

$$= \int_0^\infty \pi^{\frac{s}{2}}\pi^{-1}n^{-2}n^s x^{\frac{s}{2}}x^{-1}e^{-\pi n^2 x}\pi n^2 dx$$

$$= \int_0^\infty \pi^{\frac{s}{2}}n^s x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

then we get

$$\pi^{-\frac{s}{2}}\frac{1}{n^s}\Gamma\left(\frac{s}{2}\right) = \int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

then we add summation

$$\sum_{n=1}^{\infty}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\frac{1}{n^s} = \sum_{n=1}^{\infty}\int_0^\infty x^{\frac{s}{2}-1}e^{-\pi n^2 x}dx$$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^s} = \int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}e^{-\pi n^2 x}\,dx \qquad (5.7)$$

we will use the definition Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\sum_{n=1}^{\infty}e^{-\pi n^2 x}\,dx \qquad (5.8)$$

for $\alpha > 1$, the inversion of order being justified by the convergence of

$$\sum_{n=1}^{\infty}\int_0^\infty x^{\frac{\alpha}{2}-1}\ e^{-n^2\pi x}dx$$

from the definition of the gamma function.
writing

$$\theta(x) = \sum_{n\in\mathbb{Z}}e^{-\pi n^2 x} = 1 + 2\sum_{n=1}^{\infty}e^{-\pi n^2 x} = 1 + 2\psi(x)$$

then

$$\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} \tag{5.9}$$

we putting (5.9) in (5.8) then

$$\int_0^\infty x^{\frac{s}{2}-1} \sum_{n=1}^{\infty} e^{-\pi n^2 x}\, dx = \int_0^\infty x^{\frac{s}{2}-1} \psi(x) dx$$

Then from (5.8)

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \int_0^\infty x^{\frac{s}{2}-1} \psi(x) dx = \int_1^\infty x^{\frac{s}{2}-1} \psi(x) dx + \int_0^1 x^{\frac{s}{2}-1} \psi(x) dx$$

plainly $\theta(x) = 1 + 2\psi(x)$

we will use expression of the θ theta function

$$\theta(x) = \frac{1}{\sqrt{x}} \theta\left(\frac{1}{x}\right) \qquad \text{for } x > 1$$

or

$$2\psi(x) + 1 = \frac{1}{\sqrt{x}} \left[2\psi\left(\frac{1}{x}\right) + 1\right]$$

$$\Rightarrow \psi(x) = \frac{1}{\sqrt{x}} \left[\frac{2}{2} \psi\left(\frac{1}{x}\right) + \frac{1}{2}\right] - \frac{1}{2}$$

we will prove this equation is special case of these satisfied by the Jacobi theta function

$$\psi(x) = \frac{1}{\sqrt{x}} \psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2} \tag{5.11}$$

we apply (5.11) in (5.10)

$$\int_0^1 x^{\frac{s}{2}-1} \psi(x) dx \Rightarrow \int_0^1 x^{\frac{s}{2}-1} \left[\frac{1}{\sqrt{x}} \psi\left(\frac{1}{x}\right) + \frac{1}{2\sqrt{x}} - \frac{1}{2}\right] dx$$

$$\Rightarrow \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} \psi\left(\frac{1}{x}\right) dx + \frac{1}{2} \int_0^1 (x^{\frac{s}{2}-\frac{3}{2}} - x^{\frac{s}{2}-1})\, dx$$

$$\Rightarrow \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} \psi\left(\frac{1}{x}\right) dx + \frac{1}{2} \left[\frac{1}{\frac{s}{2}-\frac{1}{2}} x^{\frac{s}{2}-\frac{1}{2}} - \frac{1}{\frac{s}{2}} x^{\frac{s}{2}}\right]_0^1$$

$$\Rightarrow \int_0^1 x^{\frac{s}{2}-\frac{3}{2}} \psi\left(\frac{1}{x}\right) dx + \frac{1}{s(s-1)}$$

we shall replace $x = \frac{1}{u}$ and $dx = -\frac{1}{u^2}\, du$ and change the boundary the integral from $|_0^1$ to $|_0^\infty$

$$= \int_1^\infty \left(\frac{1}{u}\right)^{\frac{s}{2}-\frac{3}{2}} \psi(u) \left[\frac{-du}{u^2}\right] + \frac{1}{s(s-1)}$$

we will replace u by x

$$= \int_1^\infty \left(\frac{1}{u}\right)^{\frac{s}{2}-\frac{3}{2}} \psi(u) \left[\frac{-dx}{x^2}\right] + \frac{1}{s(s-1)}$$

then

$$\int_0^1 x^{\frac{s}{2}-1} \psi(x) dx = \int_1^\infty x^{-\frac{s}{2}-1} \psi(x) dx + \frac{1}{s(s-1)} \tag{5.12}$$

we dividing the integral

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(\mathrm{x})dx = \int_1^\infty x^{\frac{s}{2}-1}\psi(\mathrm{x})dx + \int_0^1 x^{\frac{s}{2}-1}\psi(\mathrm{x})dx \qquad (5.13)$$

we putting (5.12) in (5.13)

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(\mathrm{x})dx = \int_1^\infty x^{\frac{s}{2}-1}\psi(\mathrm{x})dx + \int_0^1 x^{-\frac{s}{2}-1}\psi(\mathrm{x})dx + \frac{1}{s(s-1)}$$

$$\int_0^\infty x^{\frac{s}{2}-1}\psi(\mathrm{x})dx = \int_1^\infty \left[x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right]\psi(\mathrm{x})dx + \frac{1}{s(s-1)} \qquad (5.14)$$

from (5.7) $\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \int_0^\infty x^{\frac{s}{2}-1}\,\psi(\mathrm{x})\,dx$

we apply (5.14) in (5.7)

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{S}{2}\right)\zeta(s) = \int_1^\infty \left[x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}}\right]\psi(\mathrm{x})dx + \frac{1}{s(s-1)}$$

$$= \int_1^\infty \left[x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right]\frac{\psi(\mathrm{x})}{x}dx - \frac{1}{s(s-1)}$$

this holds for $\alpha > 1$. this integral on the right-hand side is absolutely convergent for $\alpha > 1$ and when we replace the right-hand side s by $1 - s$ is unchanged and this formula gives the analytic continuation. Then we have

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{S}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

Theorem 5.12. [4]

For all s we have

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

equivalently,

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\ sin\left(\frac{\pi s}{2}\right)\zeta(1-s)$$

Proof. We start of Legendre duplication for the gamma function this we can put the functional equation is a simple form

$$2\sqrt{x}2^{-2s}\Gamma(2s) = \Gamma(s)\Gamma\left(s+\frac{1}{2}\right)$$

we replace s by $\frac{s}{2}$

$$\Gamma\left(2\frac{S}{2}\right) = \frac{2^{2\frac{S}{2}-1}}{\sqrt{\pi}}\Gamma\left(\frac{S}{2}\right)\Gamma\left(\frac{S}{2}+\frac{1}{2}\right)$$

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}}\Gamma\left(\frac{S}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

$$\frac{\sqrt{\pi}}{2^{s-1}}\Gamma(s) = \Gamma\left(\frac{S}{2}\right)\Gamma\left(\frac{s+1}{2}\right)$$

we will looking

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin\pi s}$$

we replace s by $\frac{s+1}{2} = \frac{s}{2} + \frac{1}{2}$

$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(1-\frac{s+1}{2}\right)=\frac{\pi}{\sin\left(\frac{\pi s}{2}+\frac{\pi}{2}\right)}$$
$$\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1-s}{2}\right)=\frac{\pi}{\cos\frac{\pi s}{2}}$$
now we will use the Riemann functional equation
$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$
multiply $\Gamma\left(\frac{s+1}{2}\right)$ in both side
$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\zeta(1-s) \tag{5.15}$$
now we shall use Legendre duplication in the left side and Euler reflection formula in right side we obtain
$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$
$$\pi^{-\frac{s}{2}}\frac{\sqrt{x}}{2^{s-1}}\Gamma(s)\zeta(s)=\pi^{\frac{-1-s}{2}}\frac{\pi}{\cos\frac{\pi s}{2}}\zeta(1-s)$$
then
$$\zeta(1-s)=\frac{2}{(2\pi)^s}\cos\frac{\pi s}{2}\Gamma(s)\,\zeta(s)$$
we replace $1-s$ with s
$$\zeta(s)=\frac{2}{(2\pi)^{1-s}}\cos\frac{\pi(1-s)}{2}\Gamma(1-s)\,\zeta(1-s)$$
then we obtain
$$\zeta(s)=2^s\pi^{s-1}\sin\frac{\pi s}{2}\Gamma(1-s)\,\zeta(1-s)$$

Corollary 5.13. [14] *let*
$$\xi(s)=\frac{1}{2}s(s-1)\pi^{-\frac{1}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$
is entire function and $\xi(s)=\xi(1-s)$.

6 - Generalization of The Riemann Zeta Function

There are diverse way in the Riemann Zeta Function can be generalized. This section will present the generalization of Riemann Zeta Function to the Dirichlet L-Function. the Dirichlet L-Function is function of complex variables, similar to the Riemann Zeta Function. Now we will define the function is useful for character.

Definition 6.1. [12]
Let q be positive integer, an arithmetical function is called a Dirichlet character modulo m such that:

χ(n) = 0 if and only if (n, q) = 1
χ(nq) = χ(n)χ(q) for all n, q ∈ N
χ(n + q) = χ(n) for all n ∈ N

The special kind of character which is very important is the primitive character.

Definition 6.2. [1]

let χ is primitive if $\chi(n)$ the function restricted by the condition $(n, q) = 1$ and $\chi(n)$ may have a period less than q other wise we say that χ is primitive.

The very important property is the character χ_1

Lemma 6.3. [1]

$let\ \chi_1 the\ Dirichlet\ character\ such\ that$

$$\chi(n) = \begin{cases} \chi_1(n) & if\ (n, q) = 1 \\ 0 & if\ (n, q) > 1 \end{cases}$$

This character χ_1 is primitive.

Definition 6.4. [13] $(p.g.49)$

Let k be the natural number and let $\chi(n)$ be the character modulo k. The function $L(s, \chi)$ is called the Dirichlet L- Function or the Dirichlet L-Series and is defined by the Dirichlet series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \qquad R(e) > 1$$

Theis series converges for every $s \in \mathbb{C}$ with $Re(s) > 1$, the $L(s, \chi)$ is analytic function in half-plane $Re(s) > 1$.

The second way of defining the Dirichlet L- Function is through the Euler product for $L(s, \chi)$.

Lemma 6.5. [14] $(p.g.110)$

$$L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \quad , for\ Re(s) > 1 \qquad (6.1)$$

Proof. For $X > 1$ we define the function

$$\Phi(s, X) = \prod_{p \le x} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \qquad (6.2)$$

since $Re(s)$ greater than 1 we have

$$\left(1 - \frac{\chi(p)}{p^s}\right)^{-1} = 1 + \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} \ldots$$

we can using the multiplicatively of $\chi(n)$ and into the prime factor and we using the unique decomposition of a natural number in to the prime factors. we get

$$\Phi(s, X) = \prod_{p \le x} \left\{1 + \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{p^{2s}} + \cdots\right\} = \sum_{n \le X}^{\infty} \frac{\chi(n)}{n^s} + R(s, X) \qquad (6.3)$$

Furthermore, $R(s, X)$ is summation over those natural number $n > X$ whose prime divisors all $\le X$. Now for this summation we give a upper bounded

$$|R(s,X)| \leq R(s,X) = \sum_{n>X} \frac{1}{n^{\sigma}} < \int_X^{\infty} \frac{du}{u^{\sigma}} = \frac{1}{\sigma - 1} X^{(1-\sigma)}$$

where $\sigma = Re(s)$ greater than 1, passing to the limit in (6.3) as X goes to $+\infty$ this complete the proof.

From (6.2) we get

$$\left|\frac{1}{L(s,\chi)}\right| = \left|\prod_p \left(1 + \frac{\chi(p)}{p^s}\right)\right| \leq \sum_{n-1}^{\infty} \frac{1}{n^{\sigma}} < 1 + \int_1^{\infty} \frac{du}{u^{\sigma}} = 1 + \frac{1}{\sigma - 1},$$

$$L(s,\chi) > \frac{\sigma - 1}{\sigma}$$

for for $Re(s) > 1$ let $L(s,\chi)$ does not equal 0 then if the character χ modulo k , then the Dirichlet L- Function differs from $\zeta(s)$ by only a simple factor.

Lemma 6.6. [14]

If $\chi(n) = \chi 0(n)$ *is the principal character modulo* k*.Then,*

$$L(s,\chi_0) = \zeta(s) \prod_{p \backslash X} \left(1 - \frac{1}{p^s}\right) \qquad , Re(s) > 1$$

Lemma 6.7. [14] $(p.g. 112)$

For $Re(s) > 1$, let χ_1 be primitive character modulo q_1 and χ be the imprimitive character modulo q_1. When χ_1 is primitive character corresponding to χ.

$$L(s,\chi) = L(s,\chi_1) = \prod_{p \backslash q, p x q_1} \left(1 - \frac{\chi_1(p)}{p^s}\right)$$

Proof. By the Euler's for $L(s,\chi)$. we present

$$L(s,\chi) = \prod_p (1 - \chi(p) p^{-s})^{-1}$$

and

$$L(s,\chi_1) = \prod_p (1 - \chi_1(p) p^{-s})^{-1}$$

for $s \in \mathbb{C}$ with $Re(s) > 1$ and product of $L(s,\chi)$ and $L(s,\chi_1)$ are convergent. noting that for each prime let

$$\begin{cases} \chi_1(\mathrm{p}) = \chi(\mathrm{p}) & \text{if p is aprime not dividing q} \\ \chi(\mathrm{p}) = 0 & \text{if p is aprime dividing q} \end{cases}$$

hence,

$$L(s,\chi_1) = \prod_p (1 - \chi_1(p) p^{-s})^{-1}$$

$$= \prod_{p \nmid q} (1 - \chi_1(p) p^{-s})^{-1}$$

$$= \prod_{p \nmid q} (1 - \chi_1(p) p^{-s})^{-1} \prod_{p | q} (1 - \chi_1(p) p^{-s})^{-1}$$

from definition of $L(s,\chi)$ by Euler's product.
we obtain

$$L(s,\chi_1) = L(s,\chi) = \prod_{p|q}(1-\chi_1(p)p^{-s})^{-1}$$

6.1 Continuation of $L(s,\chi)$

We will introduce the important property of the Dirichlet L-Function in terms of analytic continuation to $Re(s) > 0$. Note that the next theorems and remark are thaken from Steuding [15] ,Davenport [1] and Forster [13] .

Theorem 6.8.
let χ be a Dirichlet character modulo q and $L(s,\chi)$ is Dirichlet L-function such that if $\chi \neq \chi_0$ where χ is the principle character modulo q the series $\sum \frac{\chi(n)}{n^s}$ ns converge in $\sigma > 0$. The function $L(s,\chi)$ is analytic in half plane $\sigma > 0$.

Theorem 6.9. let χ a character mod q with $\chi \neq \chi_0$ and let q be positive integer greater than or equal 2. Then, the Dirichlet L-Function has an analytic continues to $\mathbb{C}$.

Theorem 6.10. let χ is principle character and $\chi = \chi_0$, then

$$L(s,\chi_0) = \left(\prod_{p|m}\frac{1}{1-p^s}\right)\zeta(s)$$

hence $L(s,\chi_0)$ is analytic continues with a single pole at $s=1$. Then $L(s,\chi_0)$ can be analytically continue to the whole plane $\mathbb{C}$.

Remark 6.11. We present in Lemma (6.7) the relation between the imprimitive character χ and the primitive character χ_1. By using the Euler's product, we can see that the lemma implies a simple relation between the corresponding Dirichlet L-Functions. Then,

$$L(s,\chi_1) = L(s,\chi)\prod_{p|q}(1-\chi_1(p)p^{-s})^{-1}$$

is analytic continuation for $Re(s) > 0$.

Lemma 6.12. [14] (p.g. 112)
For Re(s) > 0 , when $\chi = \chi_1$, let $S(x) = \sum_{n \le x}\chi(n)$. we have

$$L(s,\chi) = s\int_1^{\infty} S(x)x^{-1}dx$$

Proof. Let $N \geq 1$ and $Re(s) > 1$, Now applying lemma (partial summation) where $C(x) = \sum_{1<n<x}\chi(n)$ this is $C(x) = S(x)$, $c_n = \chi(n)$. Let $f(x) = \frac{1}{x^s}$ which is continuous and differentiable on $[1,N]$. Then,

$$\sum_{1<n\le N}\chi(n)\frac{1}{x^s} = -\int_1^N\left(\sum_{1<n\le N}\chi(n)\right)\frac{-s}{x^{s+1}}dx + C(N)\frac{1}{N^s}$$

$$= s\int_1^N S(x)\frac{1}{x^{s+1}}dx + C(N)\frac{1}{N^s}$$

$$= s\int_1^N ((S(x)-1)+1)\frac{1}{x^{s+1}}dx + C(N)\frac{1}{N^s}$$

$$= s\int_1^N (S(x)-1)\frac{1}{x^{s+1}}dx + s\int_1^N \frac{1}{x^{s+1}}dx + C(N)\frac{1}{N^s}$$

$$= s\int_1^N c(x)\frac{1}{x^{s+1}}dx + s\left[-\frac{1}{s}\frac{1}{x^s}\right]_1^N + C(N)\frac{1}{N^s}$$

$$= s\int_1^N c(x)x^{-s-1}dx + \frac{-1}{N} + 1 + C(N)\frac{1}{N^s}$$

$$= s\int_1^N c(x)x^{-s-1}dx + 1 + (C(N)-1)\frac{1}{N^s}$$

Thus

$$= \sum_{n=1}^{N}\frac{\chi(n)}{n^s} = 1 + s\int_1^N c(x)x^{-s-1}dx + C(N)\frac{1}{N^s}$$

Now when $N \rightarrow \infty$ we get

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} = \text{s}\int_1^{\infty} S(x)x^{-1}dx$$

Since

$$S(x) \leq \phi(x)$$

Therefore, the integral above converges in the half plane $Re(s) > 0$. then define there an analytic function.

6.2 The Functional equation of the Dirichlet L-Function

In 1882, Hawritz presented the first functional equation for the Dirichlet L-Function. Before introducing the functional equation of L(s, χ), we must to present the Gauss sums.

Gauss Sums

let χ be Dirichlet character modulo q. To prove the functional equation for $L(s,\chi)$ we need to express $n \rightarrow \chi(n)$ as linear combination of imaginary exponentials $n \rightarrow e\left(\frac{mn}{q}\right)$ for m mod q.

Definition 6.13.

For any Dirichlet character modulo q , The Gaussian sum $\tau\ (\chi)$ is defined by

$$\tau\,(\chi) = \sum_{m=1}^{q}\chi(\text{m})\,\text{e}\left(\frac{m}{q}\right)$$

If $(n,q) = 1$ then let the inverse of n is n^{-1}

$$\chi(n)\tau(\bar{\chi}) = \chi(n)\sum_{m=1}^{q}\bar{\chi}(m)\, e\left(\frac{m}{q}\right)$$

$$= \sum_{m=1}^{q}\bar{\chi}(n^{-1}m)\, e\left(\frac{m}{q}\right)$$

Then we have

$$= \sum_{m=1}^{q}\bar{\chi}(h)\, e\left(\frac{hn}{q}\right)$$

Lemma 6.14. *The primitive Dirichlet character* χ *modulo* q*. For every* $n \in \mathbb{Z}/q\mathbb{Z}$ *we have*

$$\sum_{h \in \mathbb{Z}/q\mathbb{Z}}\bar{\chi}(h)\, e\left(\frac{hn}{q}\right) = 0$$

Then we have

$$\chi(n)\tau(\bar{\chi}) = \sum_{h \in \mathbb{Z}/q\mathbb{Z}}\bar{\chi}(h)\, e\left(\frac{hn}{q}\right) = 0$$

Lemma 6.15. let the primitive Dirichlet character χ modulo q then

$$|\tau(\chi)| = q^{\frac{1}{2}}$$

Theorem 6.16. let χ be a primitive character modulo $q > 3$. Then the $L(s,\chi)$ has analytic continuation to an entire function. let

$$\begin{cases} a = 0 & if\ \chi(-1) = 1 \\ a = 1 & if\ \chi(-1) = -1 \end{cases}$$

The Dirichet L-function satisfies the following the functional equation

$$\xi(1 - s, \bar{\chi}) = \frac{i^a\ q^{\frac{1}{2}}}{\tau(\chi)}\ \xi(s,\chi)$$

when

$$\xi(s,\chi) = \left(\frac{\pi}{q}\right)^{-\frac{1}{2}(s+a)}\Gamma\left(\frac{1}{2}(s + a)\right) L(s,\chi)$$

The function $\xi(s,\chi)$ is entire.

Proof. suppose that $\chi(-1) = 1$. we start in the gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} \quad e^{-t}\, dt \qquad (6.4)$$

we replace s to $\frac{s}{2}$ in (6.4)

then we get

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty t^{\frac{s}{2}-1}\ e^{-t}\, dt$$

substituting $t = \frac{\pi n^2 x}{q}$ $\qquad dt = q\pi n^2\, dx$

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty t^{\frac{s}{2}-1}\ e^{-t}\, dt$$

$$= \int_0^\infty \left(\frac{\pi n^2 x}{q}\right)^{\frac{s}{2}-1} e^{-\frac{\pi n^2 x}{q}}\ q\pi n^2\, dx$$

$$= \int_0^\infty \pi^{\frac{s}{2}-1} n^{s-2} x^{\frac{s}{2}-1} q^{-\frac{s}{2}+1} q\pi n^2 \, e^{-\frac{\pi n^2 x}{q}} \, dx$$

then we get

$$\Gamma\left(\frac{s}{2}\right) q^{\frac{s}{2}} \pi^{-\frac{s}{2}} n^{-s} = \int_0^\infty x^{\frac{s}{2}-1} \, e^{-\frac{\pi n^2 x}{q}} \, dx$$

we multiply $\sum \chi(n)$ in both side

$$\pi^{-\frac{s}{2}} q^{\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \sum_{n=1}^{\infty} \int_0^\infty x^{\frac{1}{2}s-1} e^{-\frac{\pi n^2 x}{q}} \chi(n) \, dx$$

$$= \sum_{n=1}^{\infty} \chi(n) \int_0^\infty x^{\frac{1}{2}s-1} e^{-\frac{\pi n^2 x}{q}} \, dx$$

$$= \int_0^\infty x^{\frac{1}{2}s-1} \left(\sum_{n=1}^{\infty} \chi(n) \, e^{-\frac{\pi n^2 x}{q}} \right) dx$$

let first assume $\chi(-1) = 1$ then we have $\chi(-n) = n$ for all $n \in \mathbb{Z}$, $\chi(0) = 0$ we write last formula

$$\pi^{-\frac{s}{2}} q^{\frac{1}{2}s} \Gamma\left(\frac{s}{2}\right) \mathrm{L}(\mathrm{s}, \chi) = \frac{1}{2} \int_0^\infty x^{\frac{s}{2}-1} \, \psi(\mathrm{x}, \chi) \mathrm{dx} \qquad (6.5)$$

where,

$$\psi(\mathrm{x}, \chi) = \sum_{n=-\infty}^{\infty} \chi(n) \, e^{-\frac{\pi n^2 x}{q}} \qquad (x > 0)$$

A symmetry relation between $\psi(x, \chi)$ and $\psi(x^{-1}, \chi)$ can be deduced form Lemma (6.14)

$$\chi(n) \tau\,(\bar{\chi}) = \sum_{h \in \mathbb{Z}/q\mathbb{Z}} \bar{\chi}(h) \, e\left(\frac{hn}{q}\right)$$

and Theorem

$$\sum_{n=-\infty}^{\infty} e^{-(\mathrm{n}+\mathrm{a})^2 \frac{\pi}{x}} = x^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\mathrm{n}^2 \pi \, x + 2\pi i \alpha}$$

with x replaced by $\frac{x}{q}$ from lemma (6.14) we have

$$\tau\,(\bar{\chi}) \, \psi(x, \chi) = \sum_{n=-\infty}^{\infty} \left(\sum_{m=1}^{q} \bar{\chi}(m) \, e\left(\frac{mn}{q}\right) \right) e^{-n^2 \pi \frac{x}{q}}$$

$$= \sum_{m=1}^{q} \bar{\chi}(m) \sum_{n=-\infty}^{\infty} e^{-\frac{n^2 \pi x}{q} + \frac{2\pi i m n}{q}}$$

then by theorem (6.15)

$$= \sum_{m=1}^{q} \bar{\chi}(m) \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{n=-\infty}^{\infty} e^{-\left(n+\frac{m}{q}\right)^2 \pi q / x}$$

$$= \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{m=1}^{q} \bar{\chi}(m) \sum_{n=-\infty}^{\infty} e^{-(nq+m)^2 \frac{\pi}{qx}}$$

let $l = nq + m$ then

$$= \left(\frac{q}{x}\right)^{\frac{1}{2}} \sum_{m=1}^{\infty} \bar{\chi}(l) \sum_{n=-\infty}^{\infty} e^{-(l)^2 \frac{\pi}{qx}} = \left(\frac{q}{x}\right)^{\frac{1}{2}} \psi(x^{-1}, \bar{\chi})$$

$$\tau(\bar{\chi})\, \psi(x,\chi) = \left(\frac{q}{x}\right)^{\frac{1}{2}} \psi(x^{-1}, \bar{\chi}) \tag{6.6}$$

Now, we will spilt the integral (6.5)

$$\xi(s,\chi) = \pi^{-\frac{1}{2}s} q^{\frac{1}{2}s} \Gamma\left(\frac{1}{2}s\right) L(s,\chi) = \frac{1}{2}\int_1^{\infty} x^{\frac{1}{2}s-1} \psi(x,\chi)dx + \frac{1}{2}\int_1^{\infty} x^{-\frac{1}{2}s-1} \psi(x^{-1},\chi)dx$$

from (6.5) we get

$$= \frac{1}{2}\int_1^{\infty} x^{\frac{1}{2}s-1} \psi(x,\chi)dx + \frac{1}{2}\frac{q^{\frac{1}{2}}}{\tau(\bar{\chi})}\frac{1}{2}\int_1^{\infty} x^{-\frac{1}{2}s-\frac{1}{2}} \psi(x,\bar{\chi})dx \tag{6.7}$$

this clear there are present an everywhere analytic function of s . Also, this expression gives the analytic continuation of $L(s,\chi)$ over the whole pane. we observe that that $L(s,\chi)$ is entire function since $\Gamma\left(\frac{1}{2}s\right)$ is never 0.

Now we will replace s by $1 - s$ and χ by χ in above formula we obtain:

$$\xi(1-s,\bar{\chi}) == \frac{1}{2}\frac{q^{\frac{1}{2}}}{\tau(\chi)}\int_1^{\infty} x^{\frac{1}{2}s-1} \psi(x,\chi)dx + \frac{1}{2}\int_1^{\infty} x^{-\frac{1}{2}s-\frac{1}{2}} \psi(x,\bar{\chi})dx \tag{6.8}$$

from last formula is equal to (6.7) multiplied by $\frac{q^{\frac{1}{2}}}{\tau(\chi)}$ since $\tau(\chi)\,\tau(\bar{\chi}) = q$ this relation is consequence of Lemma (6.14) and $\chi(-1) = 1$ since implies that

$$\tau(\bar{\chi}) = \sum_{m=1}^{q} \overline{\chi(m)}\, e\left(\frac{m}{q}\right) = \overline{\sum_{m=1}^{q} \chi(m)\, e\left(\frac{-m}{q}\right)} = \overline{\sum_{m=1}^{q} \chi(-m)\, e\left(\frac{-m}{q}\right)} = \overline{\tau(\chi)}$$

hence, we have obtained the functional equation

$$\xi(1-s,\bar{\chi}) == \frac{q^{\frac{1}{2}}}{\tau(\chi)}\, \xi(s,\chi)$$

we have proved in the case $\chi(-1) = 1$ we next prove the case $\chi(-1) = -1$ the previous argument fails. since the $\psi(x,\chi)$ simply vanishes. Now we change the procedure by replacing s with $s + 1$ in the gamma function

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} t^{\frac{s}{2}-1} e^{-t}\, dt$$

$$\Gamma\left(\frac{1}{2}(s+1)\right) = \int_0^{\infty} t^{\frac{s}{2}-\frac{1}{2}} e^{-t}\, dt$$

we sub $t = \frac{\pi n^2 x}{q}$ $\quad dt = q\pi n^2 dx$

$$\Gamma\left(\frac{1}{2}(s+1)\right) = \int_0^{\infty} \left(\frac{\pi n^2 x}{q}\right)^{\frac{s}{2}-\frac{1}{2}} e^{-\frac{\pi n^2 x}{q}}\ q\pi n^2 dx$$

$$= \int_0^{\infty} (\pi n^2 x q^{-1})^{\frac{s}{2}-\frac{1}{2}} e^{-\pi n^2 \frac{x}{q}}\ q\pi n^2 dx$$

$$= \int_0^\infty \pi^{\frac{s}{2}-\frac{1}{2}} n^2 \;\; n^{-1}\, x^{\frac{s}{2}-\frac{1}{2}}\, q^{-\frac{s}{2}+\frac{1}{2}} e^{-\pi n^2 \frac{x}{q}} \;\; q\pi n^2 dx$$

then we obtain

$$\pi^{-\left(\frac{1}{2}s+\frac{1}{2}\right)} q^{\frac{s}{2}+\frac{1}{2}} \;\; n^{-s}\, \Gamma\left(\frac{1}{2}(s+1)\right) = \int_0^\infty n x^{\frac{s}{2}-\frac{1}{2}} \;\; e^{-\pi n^2 \frac{x}{q}}\, dx$$

In the same way as before, when $\sigma > 1$.
we add sum $\sum \chi(\mathrm{n})$

$$\pi^{-\left(\frac{1}{2}s+\frac{1}{2}\right)} q^{\frac{s}{2}+\frac{1}{2}} \;\; \sum_{n=1}^{\infty} \frac{\chi(\mathrm{n})}{n^s}\, \Gamma\left(\frac{1}{2}(s+1)\right) = \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \left(\sum_{n=1}^{\infty} \mathrm{n}\,\chi(\mathrm{n})\, e^{-\pi n^2 \frac{x}{q}} \right) dx$$

then,

$$\pi^{-\left(\frac{1}{2}s+\frac{1}{2}\right)} q^{\left(\frac{s}{2}+\frac{1}{2}\right)} \;\; \sum_{n=1}^{\infty} \frac{\chi(\mathrm{n})}{n^s}\, \Gamma\left(\frac{1}{2}(s+1)\right) = \int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \left(\sum_{n=1}^{\infty} \mathrm{n}\,\chi(\mathrm{n})\, e^{-\pi n^2 \frac{x}{q}} \right) dx$$

then,

$$\pi^{-\left(\frac{1}{2}s+\frac{1}{2}\right)} q^{\left(\frac{s}{2}+\frac{1}{2}\right)} \;\; L(s,\chi)\Gamma\left(\frac{1}{2}(s+1)\right) = \frac{1}{2}\int_0^\infty x^{\frac{s}{2}-\frac{1}{2}} \psi_1(x,\chi)\;\; dx \qquad (6.9)$$

where $\psi_1(x,\chi) = \sum_{-\infty}^{\infty} \mathrm{n}\,\chi(\mathrm{n})\, e^{-\pi n^2 \frac{x}{q}}$ when $x > 0$

to prove a symmetry relation for $\psi(x,\chi)$ we use the differentiated of the theorem (6.13) with respect to α and written y in place of x

$$-\frac{2\pi}{y} \sum_{-\infty}^{\infty} (n+\alpha) e^{-(n+\alpha)^2 \frac{\pi}{y}} = 2y^{\frac{1}{2}} \pi i \sum_{n=-\infty}^{\infty} n e^{-n^2 \pi y + 2\pi i n \alpha}$$

we are setting $y = \frac{x}{q}$ and $\alpha = \frac{m}{q}$ we obtain

$$2\left(\frac{x}{q}\right)^{\frac{1}{2}} \pi i \sum_{n=-\infty}^{\infty} n e^{-n^2 \pi \left(\frac{x}{q}\right) + 2\pi i n \frac{m}{q}} = -2\pi \frac{q}{x} \sum_{n=-\infty}^{\infty} \left(n + \frac{m}{q}\right) e^{\left(n+\frac{m}{q}\right)^2 \frac{\pi q}{x}}$$

then

$$\sum_{n=-\infty}^{\infty} n e^{-n^2 \pi \frac{x}{q} + 2\pi i n \frac{m}{q}} = \frac{-2\pi \frac{q}{x}}{2\left(\frac{x}{q}\right)^{\frac{1}{2}} \pi i} \sum_{n=-\infty}^{\infty} \left(n + \frac{m}{q}\right) e^{-\left(n+\frac{m}{q}\right)^2 \pi \frac{q}{x}}$$

$$= -\frac{q}{x} \frac{q^{\frac{1}{2}}}{x^{\frac{1}{2}}} \mathrm{i} \sum_{n=-\infty}^{\infty} \left(n + \frac{m}{q}\right) e^{-\left(n+\frac{m}{q}\right)^2 \pi \frac{q}{x}}$$

$$= -\frac{q^{\frac{3}{2}}}{x^{\frac{3}{2}}} \mathrm{i} \sum_{n=-\infty}^{\infty} \left(n + \frac{m}{q}\right) e^{-\left(n+\frac{m}{q}\right)^2 \pi \frac{q}{x}}$$

we obtain

$$\sum_{n=-\infty}^{\infty} n e^{-n^2 \pi \frac{x}{q} + 2\pi i n \frac{m}{q}} = -\mathrm{i} \left(\frac{q}{x}\right)^{\frac{3}{2}} \sum_{n=-\infty}^{\infty} \left(n + \frac{m}{q}\right) e^{-\pi\left(n+\frac{m}{q}\right)^2 \frac{q}{x}} \qquad (6.10)$$

now using this we can now carry out a computation similar as before we have

$$\tau\,(\bar{\chi})\,\psi_1(x,\chi) = \sum_{m=1}^{q} \bar{\chi}(m) \sum_{n=-\infty}^{\infty} n e^{-n^2\pi\frac{x}{q}+2\pi i n\frac{m}{q}}$$

from (6.9)

$$= \mathrm{i}\left(\frac{q}{x}\right)^{\frac{3}{2}} \sum_{m=1}^{q} \bar{\chi}(m) \sum_{n=-\infty}^{\infty} \left(n+\frac{m}{q}\right) e^{-\pi\left(n+\frac{m}{q}\right)^2\frac{q}{x}}$$

$$= \mathrm{i}q^{\frac{1}{2}}x^{-\frac{3}{2}} \sum_{m=1}^{q} \bar{\chi}(m) \sum_{n=-\infty}^{\infty} (nq+m)e^{-\pi\left(n+\frac{m}{q}\right)^2/qx}$$

$$= \mathrm{i}q^{\frac{1}{2}}x^{-\frac{3}{2}} \sum_{m=1}^{q} \bar{\chi}(l)\ e^{-\pi l^2/qx}$$

then we have

$$\tau\,(\bar{\chi})\,\psi_1(x,\chi) = \mathrm{i}q^{\frac{1}{2}}x^{-\frac{3}{2}}\ \psi_1(x^{-1},\bar{\chi}) \qquad (6.11)$$

now we will use symmetry relation in (6.9) we get

$$\xi(s,\chi) = \pi^{-\frac{1}{2}(s+1)} q^{\frac{1}{2}(s+1)}\,\Gamma\left(\frac{1}{2}(s+1)\right) L(s,\chi)$$

we will use (6.11)

$$= \frac{1}{2}\int_0^{\infty} x^{\frac{s}{2}-\frac{1}{2}}\psi_1(x,\chi)\ dx + \frac{1}{2}\frac{iq^{\frac{1}{2}}}{\tau\,(\bar{\chi})}\psi(x^{-1},\bar{\chi})x^{-\frac{1}{2}s}dx$$

this gives the analytic continuation of $\xi(x,\chi)$ and $L(s,\chi)$ to entire function. Now if we replace χ by χ and s by $s-1$ in above formula furthermore, using the fact when $\chi(-1) = -1$ since $\tau(\chi)\,\tau(\chi) = -q$.

The proof is exactly as for $\tau(\chi)\,\tau(\chi) = -q$ that we have $\chi(-m) = -\chi(m)$ we get

$$\xi(1-s,\bar{\chi}) = \frac{iq^{\frac{1}{2}}}{\tau\,(\chi)}\xi(s,\chi)$$

now we proved $\xi(1-s,\bar{\chi}) = \frac{iq^{\frac{1}{2}}}{\tau\,(\chi)}\xi(s,\chi)$ in the case $\chi(1) = -1$.

Note all the informatin in this section from [1] Davenport.

7 Zeros of the Riemann zeta function

As seen in the section related to the Riemann zeta function and the Euler's state for $Re(s) > 1$ the $\zeta(s)$ can be defined as an $\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$ and we see that $\zeta(s)$ does not vanish when $Re(s) > 1$.

In terms of obtaining the information about the location of zeros of the $\zeta(s)$, we see from the functional equation (Theorem 5.11) of $\zeta(s)$ that

$$\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

therefore,

$$\zeta(s) = \frac{\pi^{\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)}{\Gamma\left(\frac{s}{2}\right)}$$

let for $Re(s) < 0$ we observe that:

(i) $\frac{1}{\Gamma(\frac{s}{2})}$ is equal zero at $s = -2, -4, -6, \ldots$.

(ii) the $\zeta(s)$ is not equal to zero at $s = 0$

(iii) $\Gamma\left(\frac{s}{2}\right)$ has no zeros.

since the zero of $\frac{1}{\Gamma(\frac{s}{2})}$ cancels the pole of $\zeta(1-s)$ these are called the trivial zeros. Also, the $\zeta(s)$ in $0 \leq Re(s) \leq 1$ has infinitely many non trivial zeros is called critical strip and the real part of every zero of $\zeta(s)$ is equal $\frac{1}{2}$.

Theorem 7.1. the $\zeta(s)$ has infinite number of zeros in the critical strip

The above-mentioned information we take from Karatsuba [14], Stein [6] and Bateman [23]. Following this, the next theorems and lemmas in this section are taken from Stein [6].

Theorem 7.2. the only zeros of the Riemann zeta function are at negative even integers $-2, -4, -6, \ldots$. if the zeros are outside the strip $0 \leq Re(s) \leq 1$.

Theorem 7.3. $\zeta(s)$ has no zeros on the line $Re(s) = 1$

Note: from Theorem (7.3) we can observe the following: we know that the Riemann zeta function has a pole at $s = 1$ and there are no zeros in the neighbourhood of the point $s = 1$. However, we want is the deeper property that $\zeta(1 + it) \neq 0$, $t \in R$.

Before proofing the theorem (7.3) we need present certain properties:

Lemma 7.4. let

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s} \quad , c_n \geq 0$$

where $Re(s) > 1$

Lemma 7.5. Let

$$3 + 4\cos\theta + \cos 2\theta \geq 0, \qquad \theta \in R$$

from the simple observation this follow

$$3 + 4\cos\theta + \cos 2\theta = 2(1 + \cos\theta)^2$$

Corollary 7.6. [6]

let

$$\log|\zeta^3(\sigma)\,\zeta^4(\sigma + it)\,\zeta(\sigma + 2it)| \geq 0$$

where $\sigma > 1$ and t is real.

Proof. let $s = \sigma + it$ and we have

$$Re(n^{-s}) = Re\left(e^{-(\sigma+it)\log n}\right) = e^{-\sigma \log n}\cos(t\ \log n) = n^{-\sigma}\cos(t \log n)$$

therefore, we see

$$log|\zeta^3(\sigma)\,\zeta^{\,4}(\sigma + it)\,\zeta(\sigma + 2it)|$$

which gives,

$$= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma + it)| + \log|\zeta(\sigma + 2it)|$$

Now,

$$= 3Re[log\zeta(\sigma)]d + 4Re[log\,\zeta(\sigma + it)] + Re[log\,\zeta(\sigma + 2it)]$$

from the lemma (7.5) we have

$$\sum c_n n^{-\sigma}\ (3 + 4\,cos\,\theta_n + cos\,2\theta_n)$$

where $\theta_n = t\log n$ The positively follows from lemma (7.5) and we see that $c_n \geq 0$. Now we can finish the proof of our theorem, which comes we take the proof from [6] (p.g. 187)

In proving the theorem (7.3)

we assume that $\zeta(1 + i\,t_0) = 0$ for some $t_0 \neq 0$ since the Riemann zeta function is analytic at $1 + it_0$ it should vanish at least to order 1 at this point.

hence, for some constant $C > 0$ let

$$|\zeta(\sigma + it0)|^4 \leq C(\sigma - 1)^4 \qquad as\,\sigma \to 1$$

In addition,

we know that $s = 1$ is simple pole for the Riemann zeta function, for some constant $C' > 0$ we have

$$|\zeta(\sigma)|^3 \leq C'(\sigma - 1)^{-3} \qquad as\,\sigma \to 1$$

lastly, since the Riemann zeta function is analytic at the points $\sigma + 2it_0$, for σ goes to 1 then the quantity $|(\sigma + 2it_0)|$ remains bounded. Now we will put these facts together.

we obtain

$$|\zeta^3(\sigma)\,\zeta^4(\sigma + it)\,\zeta(\sigma + 2it)| \to 0 \qquad as\ \sigma \to 1$$

Now, from the corollary (7.6) we observe the logarithm of real numbers between 0 and 1 is negative. Hence,

The $\zeta(s)$ has no zero on the real line $Re(s) = 1$

Theorem 7.7. [4] $(theorem$ 3) If ρ_n are the non trivial zeros of the Riemann zeta function and B_0 is an absolute constant. Then we have

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s - 1} + \sum_{n=1}^{\infty}\left(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{2n}\right) + B_0$$

Corollary 7.8. [4] $(corollary$ 2) The zeros of $\zeta(s)$ in the critical strip are distributed symmetrically with respect to the lines $Re(s) = \frac{1}{2}$ and $Im(s) = 0$.

Theorem 7.9. [4] $(theorem$ 4)

We have

$$\sum_{n=1}^{\infty}\frac{1}{1 + (\mathrm{T} - \gamma_{\mathrm{n}})^2} \leq c\log T$$

where if $\rho_n = \beta_n + i\gamma_n\,,\ n = 1, 2, 3\ldots\ldots$ be non trivial zeros of the Riemann zeta function and if $T \geq 2$.

Remark 7.10. [4]

From the functional equation (Theorem 5.12)

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)$$

taking $s = -2k$
then

$$\zeta(-2k) = 2^{-2k}\pi^{-2k-1} \sin \pi(-k)\ \Gamma(1+2k)\ \zeta(1+2k)$$

from $\Gamma(1\ +\ 2k) = 2k!$ we have

$$= -2^{-2k}2k!\,\pi^{-2k-1} \sin \pi \mathrm{k}\ \zeta(1+2k) = 0$$

then $\zeta(-2k)\ =\ 0\ for\ \ k = 1,2,3\ldots.$ is called trivial zero of $\zeta(s)$.
the same in

$$\zeta(1-s) = 2(2\pi)^{-s}\,\Gamma(s) \cos\left(\frac{\pi s}{2}\right)\ \zeta(s)$$

we taking $s\ =\ 2k\ +\ 1$ where $k\ =\ 1,2,3\ldots.$. then the factor $\cos\left(\frac{\pi s}{2}\right)$ is vanishing and we obtained $\zeta(-2n) = 0$ is called trivial zero of the Riemann zeta function.

7.1 The infinite products for ξ(s) and ξ(s, χ)

7.1.1 The infinite products for ξ(s)

We have defined

$$\xi(s) = \frac{1}{2}\ s(s\ -\ 1)\,\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

this function is an entire function which satisfies $\xi(1-s) = \xi(s)$ and the zeros of entire function is located in the open critical strip, $\{0\ <\ \sigma\ <\ 1\}$, indeed these zeros are placed symmetrically with respect to the real axis and with respect to the central line $\sigma = \frac{1}{2}$.
We shall apply the Weierstrass factorization theorem to $\xi(s)$. Following this, we can determine the order of $\xi(s)$. In this section, the theorems and propositions are taken from [1]

Proposition 7.11. When $|s|$ is large let $|\xi(s)|\ <\ e^{C|s|\,log|s|}$ for some constant $C\ >\ 0$ furthermore, there exists no any choice of $C_1 > 0$ such that,

$$|\xi(s)|O\left(e^{C_1|s|}\right) \qquad as \quad |s| \to \infty$$

then $|\xi(s)|$ has order 1

The next sequence of theorems, lemma and proposition gathers necessary ingredients for proposition (7.11)

Theorem 7.12. The Weierstrass factorization theorem: let $f(z)$ be an entire function with $f(0)\ \neq 0$ and f has finite order $\rho\ \geq\ 0$ and let all zeros of f is $z_1, z_2 \ldots\ldots..$
then, there is a polynomial $g(z)$ of degree $\leq \rho$ such that

$$f(z)\ = e^{\,g(z)} \prod_n \left\{\left(1-\frac{z}{z_n}\right) exp\left(\frac{z}{z_n}+\frac{1}{2}\left(\frac{z}{z_n}\right)^2+\cdots+\frac{1}{k}\left(\frac{z}{z_n}\right)^k\right)\right\}$$

if $k \in Z$ satisfying $0\ \leq k\ \leq\ \rho$ and $\sum_n |z_n|^{-1-k} <\ \infty$ there exist at least one such k.

Lemma 7.13. let

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} - \sum_{n=1}^{\infty}\left(\frac{1}{z+n} - \frac{1}{n}\right)$$
where the sum is absolutely convergent for every $z \in \mathbb{C}\backslash 0, -1, -2. \ldots.$

Theorem 7.14. ($Stirling's\ formula$)
We have

$$log\Gamma(z) = \left(z - \frac{1}{z}\right)\log z - z + \log\sqrt{2\pi} + O(|z|^{-1}), \qquad \epsilon > 0$$

for all z with $|z| \geq 1$ and $|arg\ z| \leq \pi - \epsilon$.

Proposition 7.15. If $0 < \delta < 1$
we have

$$|\zeta(s)| \ll \log t \quad , \qquad \forall\, \sigma \geq 1\,, t \geq 2$$

and

$$|\zeta'(s)| \ll (\log t)^2 \quad , \qquad \forall\, \sigma \geq 1\,, t \geq 2$$

and

$$|\zeta(s)| \ll \sigma\, t^{1-\sigma} \quad , \qquad \forall\, \sigma \geq \delta\ , t \geq 1$$

Now we will proof proposition (7.11) since by the functional equation $\xi(1 - s) = \xi(s)$ it suffices to prove

$$|\xi(s)| < e^{C|s|\log|s|} \qquad when\ \sigma = R(s) > 1$$

clearly,

$$\left|\frac{1}{2}s(s - 1)\pi^{-\frac{1}{2}s}\right| e^{C_1|s|} \qquad \text{when } |s| \text{ is large}$$

by the Stirling's formula theorem which is applicable since

$$-\frac{1}{2}\pi < arg\left(\frac{1}{2}s\right) < \frac{1}{2}\pi \qquad because\ of \quad \sigma \geq \frac{1}{2} > 0$$

this gives

$$\left|\Gamma\left(\frac{1}{2}s\right)\right| < e^{C_2|s|\log|s|} \quad \text{when } |s| \text{ is large}$$

Lastly, by the proposition (7.15) we see that $\zeta(s) \ll |t|^{\frac{1}{2}}$ for all s with $\sigma > \frac{1}{2}$ and the Riemann zeta function is bounded in the half plane $\{\sigma \geq 2\}$, since $|\zeta(s)| \leq \sum_{n=1}^{\infty} n^{-2}$ whem $\sigma \geq \frac{1}{2}$ and |s| is large , hence

$$|\zeta(s)| \leq e^{C_3|s|}$$

Now, $|\zeta(s)| < e^{C|s|\ log|s|}$ follow by multiplying our three bounds.
Now to prove that $|\xi(x)|\ O(e^{C_1|s|})$ cannot holds as $|s|$ goes to ∞ it suffices to look real s tending to ∞ by Stirling's formula we have $\zeta(s) \to 1$ for such s while $\log\Gamma(s)\ s\ \log s$

Corollary 7.16. [19] The Riemann zeta function has infinity many zero where $0 \leq Re(s) \leq 1$
$Proof$. The zero of the Riemann zeta function in the stated region are exactly those of $\xi(s)$ then the $\xi(s)$ would be polynomial and has order zero. If there were finitely zeros. which is not the case. The next corollary are taken from [1] (p. g. 80) and [14] (p. g. 57).

Corollary 7.17. we have the formula

$$\xi(s) = e^{A+B}\prod_{\rho}^{\infty}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} \qquad \text{A, B are constant} \tag{7.1}$$

if $\xi(s)$ has infinitely of zeros $\rho_1,\ \rho_2,\ldots\ldots\ldots$ such that $0 \le Re(s) \le 1$ these have $\sum|\rho n|^{-1-\epsilon}$ converges for any $\varepsilon > 0$ and $\sum|\rho n|^{-1}$ diverges. the zeros of $\xi(s)$ are the nontrivial zeros of the Riemann zeta function $\zeta(s)$

Proof. the zeros of entire function are the non-trivial zeros of the Riemann zeta function for in

$$\xi(s) = \frac{1}{2}\, s(s-1)\,\pi^{-\frac{s}{2}}\,\Gamma\left(\frac{1}{2}s\right)\zeta(s) \tag{7.2}$$

the trivial zeros of ζ(s) are cancelled by simple poles of $\Gamma\left(\frac{s}{2}\right)$ and $\frac{1}{2}s\Gamma\left(\frac{1}{2}s\right)$ has no zeros and the zero $(s-1)$ is cancelled by the pole of the Riemann zeta function $\zeta(s)$. Hence, the Riemann zeta function has infinitely of non-trivial zero ρ in $0 \le \rho \le 1$ the critical strip. the product formula

$$\xi(s) = e^{A+B}\prod_{\rho}^{\infty}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$$

leads to an expression of $\frac{\zeta'(s)}{\zeta(s)}$ as sum of partial fractions and logarithmic differentiation of (7.1) we get

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \tag{7.3}$$

now we write (7.2) in the form

$$\zeta(s) = \left(\frac{1}{2}s\right)^{-1}(s-1)^{-1}\,\pi^{\frac{1}{2}s}\,\Gamma\left(\frac{1}{2}s\right)^{-1}\xi(s) = (s-1)^{-1}\,\pi^{\frac{1}{2}s}\,\Gamma\left(\frac{1}{2}s+1\right)^{-1}\xi(s) \tag{7.4}$$

we take log logarithmic derivative of (7.4) we get

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1}+\frac{1}{2}\log\pi-\frac{1}{2}\frac{\Gamma'\left(\frac{1}{2}s+1\right)}{\Gamma\left(\frac{1}{2}s+1\right)}+\frac{\xi'(s)}{\xi(s)}$$

then we will combine the last formula with (7.3) we obtain

$$\frac{\zeta'(s)}{\zeta(s)} = B-\frac{1}{s-1}+\frac{1}{2}\log\pi-\frac{1}{2}\frac{\Gamma'\left(\frac{1}{2}s+1\right)}{\Gamma\left(\frac{1}{2}s+1\right)}+\sum_{\rho}\left(\frac{1}{s-\rho}+\frac{1}{\rho}\right) \tag{7.5}$$

the formula show the pole of $\zeta(s)$ at $s = 1$ and the non-trivial zeros at $s = \rho$. The trivial zeros at $s = -2, -4\ldots$.. are include in the gamma term.

sine, by the Weierstrass formula

$$\frac{1}{s\Gamma(s)} = e^{\gamma s}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

we have

$$-\frac{1}{2}\frac{\Gamma'\left(\frac{1}{2}s+1\right)}{\Gamma\left(\frac{1}{2}s+1\right)} = \frac{1}{2}\gamma+\frac{1}{s+2}+\sum_{n=1}^{\infty}\left(\frac{1}{s+2+2n}-\frac{1}{2n}\right) \tag{7.6}$$

$$= \frac{1}{2}\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right)$$

the representation of $\frac{\zeta'(s)}{\zeta(s)}$ in (7.5) will be essential for much of the later work on the Riemann zeta function.

The A and B is constant and it not very important can be evaluated. by the formula (7.2) notice that

$$\xi(1) = \lim_{s\to 1} s(s-1)\,\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(\mathrm{s})$$

$$= \tfrac{1}{2}\,\pi^{-\frac{1}{2}}\Gamma\left(\tfrac{1}{2}\right)\lim_{s\to 1}(s-1)\,\zeta(\mathrm{s})$$

$$= \frac{1}{2}\,\pi^{-\frac{1}{2}}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}$$

therefore, $\xi(0) = \frac{1}{2}$ and $e^A = \frac{1}{2}$ by (7.1) to evaluate B we have

$$B = \frac{\xi'(0)}{\xi(0)} = -\frac{\xi'(1)}{\xi(1)}$$

from the funcunal equation $\xi(s) = \xi(s-1)$ and (7.3) and by (7.2) we have

$$\frac{\xi'(\mathrm{s})}{\xi(\mathrm{s})} = \frac{\zeta'(\mathrm{s})}{\zeta(\mathrm{s})} + \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{1}{2}\frac{\Gamma'\left(\frac{1}{2}s+1\right)}{\Gamma\left(\frac{1}{2}s+1\right)} \qquad (7.7)$$

now, we have from (7.6) and the series for log 2 that

$$-\frac{1}{2}\frac{\Gamma'\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} = \frac{1}{2}\gamma - \sum_{n=1}^{\infty}\left(\frac{1}{1+2n} - \frac{1}{2n}\right) = \frac{1}{2}\gamma - 1 + \log 2 \qquad (7.8)$$

since,

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = -\log 2$$

we combined (7.8) in (7.7) this gives

$$\frac{\xi'(\mathrm{s})}{\xi(\mathrm{s})} = \frac{\zeta'(\mathrm{s})}{\zeta(\mathrm{s})} + \frac{1}{s-1} - \frac{1}{2}\log\pi - \frac{1}{2}\gamma + 1 - \log 2$$

thus

$$B = -\frac{\xi'(1)}{\xi(1)} = \frac{1}{2}\gamma - 1 + \frac{1}{2}\log 4\pi - \lim_{s\to 1}\left[\frac{\zeta'(\mathrm{s})}{\zeta(\mathrm{s})} + \frac{1}{s-1}\right]$$

Now, we calculate the limit

$$\zeta(\mathrm{s}) = \frac{s}{s-1} - s\int_1^{\infty}(x-[x])x^{-s-1}\,dx$$

then $\zeta(s) = \frac{s}{s-1} - s\,I(s)$ where $I(s) = \int_1^{\infty}(x-[x])x^{-s-1}\,dx$

where $[x]$ is integral part

then

$$I(s) = \int_1^{\infty}(x-[x])x^{-s-1}\,dx$$

$$= \int_1^2 \frac{x-1}{x^2}dx + \int_2^3 \frac{x-2}{x^2}dx + \int_3^4 \frac{x-3}{x^2}dx + \cdots$$

$$= \int_1^2\left(\frac{1}{x} - \frac{1}{x^2}\right)dx + \int_2^3\left(\frac{1}{x} - \frac{2}{x^2}\right)dx + \int_3^4\left(\frac{1}{x} - \frac{3}{x^2}\right)dx + \cdots$$

$$= \left(\int_1^2\frac{1}{x}dx + \int_2^3\frac{1}{x}dx + \int_3^4\frac{1}{x}dx\right) - \left(\int_1^2\frac{1}{x^2}dx + 2\int_2^3\frac{1}{x^2}dx + 3\int_3^4\frac{1}{x^2}dx\ \ldots\ldots\right)$$

$$= \lim_{N\to\infty}\left[\int_1^N\frac{1}{x}dx - \sum_{n=1}^{N-1} n\int_n^{n+1}\frac{dx}{x^2}\right]$$

Now,

$$I(1) = \int_1^{\infty} (x - [x])x^{-2}dx$$
$$= \lim_{N\to\infty}\left[\log N + \sum_{n=1}^{N-1} n\int_n^{n+1}\frac{dx}{x^2}\right]$$
$$= \lim_{N\to\infty}\left[\log N + \sum_{n=1}^{N-1} n\left(\frac{1}{n} - \frac{1}{n+1}\right)\right]$$
$$= \lim_{N\to\infty}\left(\log N + \sum_{n=1}^{N}\frac{1}{n} + 1\right) = 1 - \gamma$$

so that

$$\lim_{s\to 1}\left\{\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right\} = 1 - I(1)$$

then

$$B = \frac{\gamma}{2} - 1 + \frac{1}{2}\log 4\pi - (1 - (1-\gamma))$$
$$= \frac{\gamma}{2} - 1 + \frac{1}{2}\log 4\pi - \gamma$$

hence,

$$B = \frac{-\gamma}{2} - 1 + \frac{1}{2}\log 4\pi$$

for B we can give different interpretation. Although, $\sum|\rho|^{-1}$ is diverges series. $\sum \rho^{-1}$ is converges series, as long as one groups together the terms from ρ and $\bar{\rho}$.
Furthermore, if $\rho = \beta + i\gamma$ then

$$\frac{1}{\rho} + \frac{1}{\rho} = \frac{1}{\beta + i\gamma} + \frac{1}{\beta + i\gamma}$$
$$\frac{\beta + i\gamma + \beta + i\gamma}{B^2 + \gamma^2} = \frac{2\beta}{\beta^2 + \gamma^2} \le \frac{2}{|\rho|^2}$$

we know that the series $\sum|\rho|^{-2}$ converges.

It follows from the functional equation for $\xi(s) = \xi(1-s)$ and from (7.3) we obtain

$$B + \sum_{\rho}\left(\frac{1}{1-s-\rho} + \frac{1}{\rho}\right) = -B - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

and the terms cancel the $1 - s - \rho$ and $s - \rho$.

since if the zeros is ρ then so is $1 - \rho$ thus $B = -\sum_{\rho}\frac{1}{\rho} = -2\sum_{\gamma>0}\frac{B}{B^2+\gamma^2}$ since $B \approx -0.023$ from this can be seen $|\gamma| > 6$ for all zeros.

7.1.2 The infinite products for ξ(s, χ)

Next we apply similar study to the Dirichlet L-function let χ be primitive character to the modulo q and we have defined

$$\xi(s,\chi) = \left(\frac{q}{\pi}\right)^{\frac{1}{2}s+\frac{1}{2}a}\Gamma\left(\frac{1}{2}s + \frac{1}{2}a\right)L(s,\chi)$$

where a = 0 if $\chi(-1) = 1$
a = 1 if $\chi(-1) = -1$

and $\xi(s,\chi)$ is entire function the zeros of $\xi(s,\chi)$ are all situated in the critical strip $\{0 \le \sigma \le 1\}$ and no zero at $s = 0$ or $s = 1$. these zeros coincide with non-trivial zeros of L(s, χ)

we have

$$\xi(1-s,\bar{\chi}) = \frac{i^a q^{\frac{1}{2}}}{\tau\,(x)}\xi(s,\chi)$$

where $\left|\frac{i^a q^{\frac{1}{2}}}{\tau\,(x)}\right|$ is equal 1

Theorem 7.18. $\xi(s,\chi)$ is entire function has an infinity of zeros say $\rho_1, \rho_2 \ldots\ldots$ such that $\sum |\rho_n|^{-1-\varepsilon}$ converges for any $\varepsilon > 0$ but $\sum |\rho_n|^{-1}$ diverges. furthermore, there exist constant $A = A(\chi), B = B(\chi)$ such that,

$$\xi(s,\chi) = e^{\,A+B_s} \prod_{n=1}^{\infty}\left(1 - \frac{s}{\rho_n}\right) e^{\frac{s}{\rho_n}}$$

$$\frac{\xi'(s,\chi)}{\xi(s,\chi)} = B + \sum_{n=1}^{\infty}\left(\frac{1}{s-\rho_n} + \frac{1}{\rho_n}\right)$$

This section will take from [1].

8 Zero free region for the Riemann zeta function

The final part of this dissertation involves application of the Riemann zeta function. This section will put forth the theorem of zero free region on $\zeta(s)$ and $L(s,\chi)$. This section will prove the zero free region on the Riemann zeta function, although the theorem of zero free region on $L(s,\chi)$ will not be proved here. The aforementioned proof is available in [1] (p.g. 88). Note that we take the information in this section from [1].

In 1896 Hadamard and de la Vallee Poussin proved that $\zeta(s) \neq 0$ on $\sigma = 1$. This was the fundamental step in proving the prime number theorem. This step remains vital in all subsequence proofs until in 1948 when Selberg and Erdos discovered of elementary proof.

For $\sigma > 1$, we have

$$log\,\zeta(s) = \textstyle\sum_p \sum_{m=1}^{\infty} m^{-1}\; p^{-m\sigma}\; e^{-itm\,log\,p} \qquad (8.1)$$

If the Riemann zeta function had zero at $1 + it$ then from the right in formula (8.1) $Re\,log\,\zeta(\alpha + it)$ would tend to $-\infty$ as $\sigma \to 1$. this indicate that the numbers $cos(tm\,log\,p)$ would be negative. Therefore, the numbers $cos(2tm\,log\,p)$ we should expect it to be predominantly positive and would contradict the fact that $Re\,log\,\zeta(\sigma + 2it)$ keep bounded above as $\sigma \to 1$.

Theorem 8.1. There exist a constant which is greater than zero such that $\zeta(s)$ has no zero in the region

$$\sigma \le 1 - \frac{c}{log(|t| + 2)}$$

Proof. From lemma (7.5) we have

$$3 + 4\cos\theta + 2\theta \ge 0 \qquad (8.2)$$

the left side is equal $2(1 + cos\theta)^2$.

Now we apply to

$$Re\, log\, \zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-m\sigma} \cos(t\, log\, p^m)$$

we replace t to $0, t, 2t$ in succession, its gives

$$3 \log \zeta(\sigma) + 4\mathrm{Re} \log \zeta(\sigma + \mathrm{it}) + \mathrm{Re} \log \zeta(\sigma + 2\mathrm{it}) \geq 0$$

then we obtain

$$\zeta^3(\sigma)\, |\zeta^4(\sigma + it)\, \zeta(\sigma + 2it)| \geq 1$$

therefore, we can make a sharper argumentation since we want to access the infinite product formula for the Riemann zeta function. With this in mind, it is more convenient to work with $\frac{\zeta'(s)}{\zeta(s)}$ than with $log\, \zeta(s)$. This is because the analytic continuation of the latter to the left of σ s equal 1 is clearly difficult. Since,

$$\frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \Lambda(n)\, n^{-\sigma}$$

Thus

$$-Re\, \frac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \Lambda(n)\, n^{-\sigma} \cos(t \log n)$$

for $\sigma > 1$. Hence using (8.2)

$$3\left[\frac{\zeta'(\sigma)}{\zeta(\sigma)}\right] + 4\left[-Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)}\right] + \left[-Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)}\right] \geq 0 \qquad (8.3)$$

for σ goes to 1 the behavior of $-\frac{\zeta'(\sigma)}{\zeta(\sigma)}$ from the right presents no difficulty. Since $-\frac{\zeta'(\sigma)}{\zeta(\sigma)}$ has simple pole of $\zeta(s)$ at $\sigma = 1$ with residue 1.

We have

$$-\frac{\zeta'(\sigma)}{\zeta(\sigma)} < \frac{1}{\sigma - 1} + A$$

near $\sigma = 1$ the behavior of the other two functions is clearly significantly affected by any zero that the Riemann zeta function may have just to the left of σ is equal 1, at height near to t and $2t$.

This effect is rendered clear by the formula from (7.5)

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - B - \frac{1}{2}\log\pi - \frac{1}{2}\frac{\Gamma'\left(\frac{1}{2}s + 1\right)}{\Gamma\left(\frac{1}{2}s + 1\right)} - \sum_{\rho}\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

Here the Gamma Γ term is bounded by $A\ log\ t\ if\ t \geq 2$ and $1 \leq \sigma \leq 2$.

Therefore, in this region

$$-Re\frac{\zeta'(s)}{\zeta(s)} < A\ log - \sum_{\rho} Re\left(\frac{1}{s-\rho} + \frac{1}{\rho}\right)$$

if we write $\rho = \beta + i\gamma \quad (0 \leq \beta \leq 1),\ \gamma \in \mathbb{R}$ then

$$-Re\frac{1}{s-\rho} = \frac{\sigma - \beta}{|s-\rho|^2} \quad and \quad Re\frac{1}{\rho} = \frac{\beta}{|\rho|^2}$$

we obtain when $s = \sigma + 2it$

$$-Re\frac{\zeta'(\sigma+2it)}{\zeta(\sigma+2it)} < A\ \log t$$

As regard $s = \sigma + 2it$ if we choose t to coincide with the imaginary part γ of a zero $\beta + i\gamma$ with $\gamma \geq 2$

from the sum now we take the one term $\frac{1}{(s-\rho)}$ which corresponds to the zero

$$-Re\frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} < A\ \log t - \frac{1}{\sigma-\beta}$$

now we are replacing our upper bounds in (8.3).

we get

$$\frac{4}{\sigma - \beta} < \frac{3}{\sigma - 1} + A\ log\ t$$

now we are taking $\sigma = 1 + \delta\ log\ t$ where $\delta > 0$

then

$$\frac{4}{1+\frac{\delta}{\log t} - \beta} < \frac{3}{1+\frac{\delta}{\log t} - 1} + A\ log\ t$$

$$1+\frac{\delta}{\log t} - \beta > \frac{4}{\frac{3}{1+\frac{\delta}{\log t} - 1} + A\ log\ t}$$

$$\beta < 1 + \frac{\delta}{\log t} - \frac{4\delta}{(3+A\delta)\log t}$$

we make the choice δ, in the order to obtain positive numinator the last expression in the last expression. These yields

$$\beta < 1 + \frac{c}{\log t}$$

thus, we have proved that there exists a positive numerical constant c such that the Riemann zeta function does not have any zero in the region

$$\sigma \geq 1 - \frac{c}{\log t} \qquad , t \geq 2$$

Finally, the Riemann zeta function has no zero arbitrarily near $\sigma = 1$ with $|t| \leq 2$.

As such we can say there exists c greater than zero meaning that $\zeta(s)$ does not have any zero in the region

$$\sigma \geq 1 - \frac{c}{\log(|t|+2)}$$

8.1 Zero Free Regions for L(s, χ)

Theorem 8.2. [1] $(p.g.93)$

There exist a absolute constant $c > 0$ such that if χ is complex character modulo q.
Then $L(s,\chi)$ does not has any zero in the region

$$\sigma \geq \begin{cases} 1 - \frac{c}{\log q|t|} & if\ |t| \geq 1 \\ 1 - \frac{c}{\log q} & if\ |t| \leq 1 \end{cases}$$ N

If χ is real non principal character, then $L(s,\chi)$ has at most a single (simple) real zero.

9 Conclusion

The Riemann zeta function is one of the oldest functions in mathematics. It has a long history for this function and there are many studies about this function. Over the duration of this dissertation of concerning the Riemann zeta function, we have seen the significant aspect of the Riemann zeta function, including its analytic continuation, functional equation and application. we will now review the information presented. This dissertation has presented the historical background of the Riemann zeta function. In addition, we proved some properties of the Riemann zeta function such as:

1) $\sum_n \delta_n(s) = \zeta(s) - \frac{1}{s-1}$ is analytic where $Re(s) > 0$.
2) $\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + s\int_N^\infty \frac{\rho(u)}{u^{s+1}}du$ is analytic where $Re(s) > 0$. Then, we considered the functional equation of $\zeta(s)$. Firstly, we proved this equation depend on the gamma and theta functions, and that the $\zeta(s)$ satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)$$

Secondly, we proved that the two equations are equivalent by using Legendre duplication for the gamma function such as

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

equivalently,

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)$$

The Dirichlet L- function has been discussed and we have concluded that this can be analytically continued, and the functional equation is obtained by using methods similar to the Riemann zeta function. Moreover, the zero of $\zeta(s)$ has been considered, in particular the trivial zero and the critical strip; we proved $\zeta(s)$ does not has zero when $Re(s) > 1$ and the application was then presented.

Overall, it appears in this dissertation, there are many parts on the analytic continued of $\zeta(s)$ which are still to be considered. Moreover, a number of important limitations need be considered that the Riemann hypothesis.

References

[1] H. Davenport, Multiplicative number theory, Vol. 74. (Springer Science Business Media 2013).

[2] A. Hassen, and M. Knopp, The Riemann Zeta Function and Its Application to Number Theory, (2007) Available: [Online] http://users.rowan.edu/ hassen/Papers/distprime.pdf

[3] A.A. Karatsuba, and S.M. Voronin, The Riemann zeta-function, (Walter de Gruyter 1992).
[4] T.M. Apostol, Introduction to analytic number theory, (Springer 2013)
[5] G.A. Jones, and J.M. Jones, Elementary number theory. Springer Science,1998.
[6] E.M. Stein, and R. Shakarchi, Complex analysis. Princeton Lectures in Analysis, II2003.
[7] p Garrett, poisson summation and converges of Fourier series (August 2013) available [online] : http://www.math.umn.edu/ garrett/
[8] E. Segarra, An Exploration of the Riemann Zeta Function and its Application to the Theory of Prime Number Distribution,Harvey Mudd College, 2006.
[9] A Steiger, Riemanns second proof of the analytic continuation of the Riemann Zeta function Seminar on Modular Forms, Winter term 2006
[10] W. Rudin, Real and complex analysis. Tata McGraw-Hill Education, 1987.
[11] P. Sebah, and X. Gourdon, Introduction to the Gamma Function, at numbers. computation. free. fr. Constants/constants. html, 2002.
[12] A. Vrilly, Dirichlets Theorem on Arithmetic Progressions.Harvard University, Cambridge.
[13] O, Forste.,r Analytic Number Theory , LMU Munich, Germany .Winter 2001/02
[14] A.A. Karatsuba, Basic analytic number theory. Springer Science,Business Media,2012. [15] J. Steuding, Theory of L-functions. 2005/06
[16] H. Iwaniec, and E. Kowalski, Analytic number theory (Vol. 53). American Mathematical Soc, 2004.
[17] I. Stewart,and D. Tall,Complex analysis. Cambridge University Press.1983.
[18] Analytic continuation,[Online]. Available: http://www.math.Manchester.ac.uk
[19] M. Ram Murty, Problem in Analytic Number Theory,Queen’s University .Canada,(2000).
[20] G. Everest, and T. Ward, An introduction to number theory (Vol. 232). Springer Science,2006.
[21] P. Biane, J.Pitman and M. Yor, Probability laws related to the Jacobi theta and Riemann zeta functions, and Brownian excursions. 2001. Bulletin of the American Mathematical Society, 38(4), pp.435-465.
[22] F. Neubrander, Lecture Note for Complex Analysis fall 2003.
[23] P. Bateman, and H. Diamond, Analytic Number Theory: An Introductory Course (Reprinted 2009) (Vol.1). World Scientific, 2004.