Original Research Article

An A- Stable block integrator scheme for the solution of first order system of IVP of ordinary differential equations

Abstract

This paper considers deriving four blocks numerical scheme of block backward differentiation formula for the approximate solutions of initial value problem of the first order system of ordinary differential equations. The block scheme at a single integration step produces four approximate solution values of y_{n+1}, y_{n+2}, y_{n+3} and y_{n+4} at point x_{n+1}, x_{n+2}, x_{n+3} and x_{n+4} respectively. The work investigated the order and stability property of the scheme, the method is zero, A–Stable and of order 6. Test problems are solved with the scheme and the comparison shows that the proposed block scheme has advantages over some existing methods in terms of accuracy, minimum error and less CPU time.

Keyword: Block, IVPs, Order, Ordinary Differential equation, Stiff

Introduction

A number of real life issues that we encounter, especially in the field of engineering, sciences both physical, social and life sciences can be modeled in Mathematics as differential equations. Considering the vast application of differential equations, analytical and numerical methods are being developed to find solutions.

This study consider a method for solving system of first order initial value problem of ordinary differential equation of the form

$$y' = f\left(x,\ \hat{Y}\right), \qquad \hat{Y}(a) = \eta, \qquad a \le x \le b \qquad (1)$$
$$\hat{Y} = (y_1, y_2, y_3, \dots \dots \dots y_n), \qquad \hat{\eta} = (\eta_1, \eta_2, \eta_3, \dots \dots \dots \eta_n)$$

Ordinary differential equations can be solved by analytical and numerical methods. The solutions generated by the analytical method are generally exact values, whereas with the numerical method an approximation is given as a solution approaching the real value (Fatokun *et al* 2005). Implicit numerical schemes proved to be more efficient in solving problems than explicit ones. Most common implicit algorithms are based on Backward Differentiation Formula (BDF). The BDF first appeared in the work of (Curtiss and Hirschfelder 1952). Researchers continued to improve on the BDF methods. Such improvements include the Extended Backward Differential Formula by (Cash, 1980), modified extended backward differential formula by (Cash, 2000), 2 point diagonally implicit super class of backward differentiation formula (Musa *et al.*, 2016), an order five implicit 3-step block method for solving ordinary differential equation (Yahaya and Sagir, 2013), Implicit r-point block backward differentiation formula for solving first- order stiff ODEs (Ibrahim *et al.*, 2007), a new variable step size block backward differentiation formula for solving stiff initial value problems (Suleiman *et al.*, 2013), a new fifth order implicit block method for solving first order stiff ordinary differential equations by (Musa *et al* 2014), an accurate computation of block hybrid method for solving stiff ODEs (Sagir, 2012), One-leg Multistep Method for first Order Differential Equations (Fatunla, 1984), Sagir (2014), Numerical Treatment of Block Method for the Solution of Ordinary Differential Equations. All the schemes mentioned above developed by different scholars possesses various sort of accuracy, minimum error and less computation time at one step or the other. However, there is need of developing a numerical algorithm that will solve system of ODEs with minimal computational time and converge faster, hence the motivation for this research.

Formulation of the Methods

Consider the general k-step linear multistep method of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_{k}f_{n+k} \qquad (2)$$

This study consider adding a future point in (2), with three step backward, to came-up with the formula of the form

$$\sum_{j=0}^{7} \alpha_j y_{n+j-3} = h\beta_{k}f_{n+k-3} \qquad k = 1,2,3,4 \qquad (3)$$

The implicit four point method is constructed using a linear operator L_i. To derive the four point, define the linear operator L_i associated with (3) as

$$L_i[y(x_n),h)]: \alpha_0\, y_{n-3} + \alpha_1\, y_{n-2+}\alpha_2\, y_{n-1} + \alpha_3\, y_n + \alpha_4\, y_{n+1} + \alpha_5\, y_{n+2} + \alpha_6\, y_{n+3} + \alpha_7\, y_{n+4} - h\beta_k f_{n+k-3} = 0 \qquad k = i = 1,2,3,4 \qquad (4)$$

To derive the first, second, third, and fourthpoints as $y_{n+1}, y_{n+2}, y_{n+3}$ *and* y_{n+4} respectively Using Taylor series expansion in (4) and normalizing $\alpha_3 = 1$, $\alpha_4 = 1$, $\alpha_5 = 1$and$\alpha_6 = 1$ as coefficient's of the four points, $k = 1$, $k = 2$, $k = 3$ and $k = 4$respectively. To obtain

$$y_{n+1} = -\frac{1298881}{341643939}y_{n-3} + \frac{341643939}{569406565}y_{n-2} - \frac{72003623}{113881313}y_{n-1} + \frac{426060731}{341643939}y_n + \frac{6274637}{16268759}y_{n+2} - \frac{143998979}{1708219695}y_{n+3} + \frac{1847955}{113881313}y_{n+4} - \frac{9603792}{113881313}f_{n-2}$$

$$y_{n+2} = -\frac{79696}{845265}y_{n-3} + \frac{41929759}{9861425}y_{n-2} - \frac{68414023}{3944570}y_{n-1} + \frac{189894686}{5916855}y_n - \frac{7210474}{394457}y_{n+1} + \frac{21582821}{59168550}y_{n+3} + \frac{14016}{1972285}y_{n+4} + \frac{19789614}{1972285}f_{n-1} \qquad (5)$$

$$y_{n+3} = \frac{70450}{1797393}y_{n-3} - \frac{1295843}{1198262}y_{n-2} + \frac{5593225}{599131}y_{n-1} + \frac{676840}{105729}y_n - \frac{11495780}{599131}y_{n+1} + \frac{6496015}{1198262}y_{n+2} + \frac{42690}{599131}y_{n+4} + \frac{845710}{46087}f_n$$

$$y_{n+4} = -\frac{338687}{348237}y_{n-3} - \frac{353855969}{77076456}y_{n-2} + \frac{2326014617}{19269114}y_{n-1} - \frac{4938738481}{115614684}y_n + \frac{1117145237}{19269114}y_{n+1} - \frac{11296250177}{77076456}y_{n+2} + \frac{495749336}{28903671}y_{n+3} + \frac{951570371}{3211519}f_{n+1}$$

Order of the Method

In this section, we derive the order of the methods (5). It can be transform to a general matrix form as follows

$$\sum_{j=0}^{1} C_j^* Y_{m+j-1} = h \sum_{j=0}^{1} D_j^* Y_{m+j-1}, \qquad (6)$$

Let C_0^*, C_1^*, D_0^* and D_1^* be block matrices defined by

$$C_0^* = [C_{0,} C_1, C_2, C_3], C_1^* = [C_4, C_5, C_6, C_7], \quad D_0^* = [D_0, D_1, D_2, D_3], \mathrm{D}_1^* = [D_4, D_5, D_{6,}, D_7]$$

Where C_0^*, C_1^*, D_0^* and D_1^* are square matrices and $\mathrm{Y}_{\mathrm{m}-1}, \mathrm{Y}_{\mathrm{m},} \mathrm{F}_{\mathrm{m}-1}$ and F_{m} are column vectors defined by

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \\ y_{3m+4} \end{bmatrix}, \; Y_{m-1} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \\ y_{3(m-1)+4} \end{bmatrix}, \; F_{m-1} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_{3(m-1)+1} \\ f_{3(m-1)+2} \\ f_{3(m-1)+3} \\ f_{3(m-1)+4} \end{bmatrix}$$

$$F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} = \begin{bmatrix} f_{3m+1} \\ f_{3m+2} \\ f_{3m+3} \\ f_{3m+4} \end{bmatrix} \qquad (7)$$

Thus, equations (5) can be rewritten as

$$\begin{bmatrix} -\frac{1298881}{341643939} & \frac{341643939}{569406565} & -\frac{72003623}{113881313} & \frac{426060731}{341643939} \\ -\frac{79696}{845265} & \frac{41929759}{9861425} & -\frac{68414023}{3944570} & \frac{189894686}{5916855} \\ \frac{70450}{1797393} & -\frac{1295843}{1198262} & \frac{5593225}{599131} & \frac{676840}{105729} \\ -\frac{338687}{348237} & -\frac{353855969}{77076456} & \frac{2326014617}{19269114} & -\frac{49388481}{115614684} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} +$$

$$\begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{1129625017\ 7}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = h$$

$$\begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{845710}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \qquad (8)$$

From the (8) we have

$$C_0^* = \begin{bmatrix} -\frac{1298881}{341643939} & \frac{341643939}{569406565} & -\frac{72003623}{113881313} & \frac{426060731}{341643939} \\ -\frac{79696}{845265} & \frac{41929759}{9861425} & -\frac{68414023}{3944570} & \frac{189894686}{5916855} \\ \frac{70450}{1797393} & -\frac{1295843}{1198262} & \frac{5593225}{599131} & \frac{676840}{105729} \\ -\frac{338687}{348237} & -\frac{353855969}{77076456} & \frac{2326014617}{19269114} & -\frac{49388481}{115614684} \end{bmatrix}$$

$$C_1^* = \begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{1129625017\ 7}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix}$$

$$D_0^* = \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{845710}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} D_1^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix}$$

Where

$$C_0 = \begin{bmatrix} -\frac{1298881}{341643939} \\ -\frac{79696}{845265} \\ \frac{70450}{1797393} \\ -\frac{338687}{348237} \end{bmatrix} C_1 = \begin{bmatrix} \frac{341643939}{569406565} \\ -\frac{41929759}{9861425} \\ -\frac{1295843}{1198262} \\ -\frac{353855969}{348237} \end{bmatrix} C_2 = \begin{bmatrix} -\frac{72003623}{113881313} \\ -\frac{68414023}{3944570} \\ \frac{5593225}{599131} \\ \frac{2326014617}{19269114} \end{bmatrix} C_3 = \begin{bmatrix} \frac{426060731}{341643939} \\ \frac{1898894686}{5916855} \\ \frac{676840}{105729} \\ -\frac{49388481}{115614684} \end{bmatrix}$$

$$C_4 = \begin{bmatrix} 1 \\ \frac{7210474}{394457} \\ \frac{11495780}{599131} \\ -\frac{1117145237}{19269114} \end{bmatrix}, C_5 = \begin{bmatrix} -\frac{6274637}{16268759} \\ 1 \\ -\frac{6496015}{1198262} \\ \frac{1129625017\ 7}{77076456} \end{bmatrix}, C_6 = \begin{bmatrix} \frac{143998979}{1708219695} \\ -\frac{21582821}{59168550} \\ 1 \\ -\frac{495749336}{28903671} \end{bmatrix}, C_7 = \begin{bmatrix} -\frac{9603792}{113881313} \\ -\frac{14016}{1972285} \\ -\frac{42690}{599131} \\ 1 \end{bmatrix}$$

$$D_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, D_1 = \begin{bmatrix} \frac{9603792}{113881313} \\ 0 \\ 0 \\ 0 \end{bmatrix}, D_2 = \begin{bmatrix} 0 \\ \frac{19789614}{1972285} \\ 0 \\ 0 \end{bmatrix}, D_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{845710}{46087} \\ 0 \end{bmatrix}, D_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{951570371}{3211519} \end{bmatrix},$$

$$D_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{951570371}{3211519} \end{bmatrix}, D_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, D_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, D_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Definition: (Order of the method) the order of the block method (5) and its associated linear operator are given by

$$L[y(x); h] = \sum_{j=0}^{7} [C_j y(x + jh)] - h \sum_{j=0}^{7} [D_j y'(x + jh)] \quad (9)$$

where p is unique integer such that

$E_q = 0, \; q = 0,1, \ldots \; p$ and $E_{p+1} \neq 0$, where the E_q are constant Matrix.
With

$$E_0 = \sum_{J=0}^{7} C_J = 0,$$

$$E_1 = \sum_{J=0}^{7} \left[jC_j - 2D_j \right] = 0,$$

$$E_2 = \sum_{J=0}^{7} \left[\frac{1}{2!} j^2 C_j - 2jD_j \right] = 0,$$

$$E_3 = \sum_{J=0}^{7} \left[\frac{1}{3!} j^3 C_j - 2\frac{1}{2!} j^2 D_j \right] = 0,$$

$$E_4 = \sum_{J=0}^{7} \left[\frac{1}{4!} j^4 C_j - 2\frac{1}{3!} j^3 D_j \right] = 0,$$

$$E_5 = \sum_{J=0}^{7} \left[\frac{1}{5!} j^5 C_j - 2\frac{1}{4!} j^4 D_j \right] = 0,$$

$$E_6 = \sum_{J=0}^{7} \left[\frac{1}{6!} j^6 C_j - 2\frac{1}{5!} j^5 D_j \right] = 0$$

$$E_7 = \sum_{J=0}^{7} \left[\frac{1}{7!} j^7 C_j - 2\frac{1}{6!} j^6 D_j \right] \neq 0$$

Therefore, the method is of order 6, with error constant as: $E_7 = \begin{bmatrix} -\frac{210}{8293585} \\ \frac{324}{3184255} \\ -\frac{981}{6926402} \\ \frac{563}{5947583} \end{bmatrix}$ (10)

Zero Stability of the Method

Definition: (Zero-Stable) A linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple.

The method (5) is converted into matrix form as:

$$
\begin{bmatrix}
1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\
\frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\
\frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\
-\frac{1117145237}{19269114} & \frac{1129625017 7}{77076456} & -\frac{495749336}{28903671} & 1
\end{bmatrix}
\begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = h
$$

$$
\begin{bmatrix}
\frac{1298881}{341643939} & -\frac{341643939}{569406565} & \frac{72003623}{113881313} & -\frac{426060731}{341643939} \\
\frac{79696}{845265} & -\frac{41929759}{9861425} & \frac{68414023}{3944570} & -\frac{189894686}{5916855} \\
-\frac{70450}{1797393} & \frac{1295843}{1198262} & -\frac{5593225}{599131} & -\frac{676840}{105729} \\
\frac{338687}{348237} & \frac{353855969}{77076456} & -\frac{2326014617}{19269114} & \frac{49388481}{115614684}
\end{bmatrix}
\begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_{n} \end{bmatrix} + h
$$

$$
\begin{bmatrix}
0 & \frac{9603792}{113881313} & 0 & 0 \\
0 & 0 & \frac{19789614}{1972285} & 0 \\
0 & 0 & 0 & \frac{845710}{46087} \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{bmatrix} + h
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{951570371}{3211519} & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \qquad (11)
$$

The equation above can be written in matrix form as:

$$A_0 Y_m = A_1 Y_{m-1} + h(B_0 F_{m-1} + B_1 F_m) \qquad (12)$$

Where

$$
A_0 = \begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{1129625017\,7}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix},
$$

$$
A_1 = \begin{bmatrix} \frac{1298881}{341643939} & -\frac{341643939}{569406565} & \frac{72003623}{113881313} & -\frac{426060731}{341643939} \\ \frac{79696}{845265} & -\frac{41929759}{9861425} & \frac{68414023}{3944570} & -\frac{189894686}{5916855} \\ -\frac{70450}{1797393} & \frac{1295843}{1198262} & -\frac{5593225}{599131} & -\frac{676840}{105729} \\ \frac{338687}{348237} & \frac{353855969}{77076456} & -\frac{2326014617}{19269114} & \frac{49388481}{115614684} \end{bmatrix}
$$

$$
B_0 = \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{845710}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix}
$$

Y_{m-1} , Y_m , F_{m-1} and F_m are column vectors defined by

$$
Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \\ y_{3m+4} \end{bmatrix}, Y_{m-1} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \\ y_{3(m-1)+4} \end{bmatrix}, F_{m-1} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_{3(m-1)+1} \\ f_{3(m-1)+2} \\ f_{3(m-1)+3} \\ f_{3(m-1)+4} \end{bmatrix}
$$

$$
F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} = \begin{bmatrix} f_{3m+1} \\ f_{3m+2} \\ f_{3m+3} \\ f_{3m+4} \end{bmatrix}
$$

Substituting scalar test equation $y' = \lambda y$ ($\lambda < 0$, λ complex) into (12) and using $\lambda h = \bar{h}$ gives

$$
A_0 Y_m = A_1 Y_{m-1} + \bar{h}(B_0 Y_{m-1} + B_1 Y_m) \tag{13}
$$

The stability polynomial of (5) is obtained by evaluating

$$
det\left[(A_0 - \bar{h}B_1)t - (A_1 + \bar{h}B_0)\right] = 0 \tag{14}
$$

to get

$$R(\bar{h},t) = \frac{996090316807547559435103303 3}{1329753259363668203573654 60}h - \frac{446737709680296868675844731429}{106380260749093456285892368}t - \frac{267896832298507582085725524 9}{664876629681834101786827 3}t^4h - \frac{193733184956304804387420096}{6648766296818341017868273}t^3h^2 - \frac{256081694624308812616426 08}{6648766296818341017868273}h^2 + \frac{638171663697310966422440976921}{106380260749093456285892368}t^2 - \frac{121326420211773039356715312 7537}{199462988904550230536048190}th - \frac{567630082507188660567238515954 1}{398925977809100461072096380}t^2h - \frac{1567043388639268347778339112886}{33243831484091705089341365}t^3h + \frac{636761357124782350302383414 4}{51144356129371853983602 1}t^2h^2 - \frac{283264387512266327961835113 6}{6648766296818341017868273}th^2 + \frac{305957123435183189886671552 47}{531901303745467281429461840} - \frac{921565130889493728528082 09}{511443561293718539836021 0}t^4 - \frac{153560628738985568751170538 3}{835009895989744554834320}t^3 \quad (15)$$

By putting $\bar{h} = 0$ in (14), we obtain the first characteristic polynomial as

$$R(0,t) = -\frac{446737709680296868675844731429}{106380260749093456285892368}t + \frac{638171663697310966422440976921}{106380260749093456285892368}t^2 - \frac{1535606287389855687511705383}{835009895989744554834320}t^3 - \frac{92156513088949372852808209}{5114435612937185398360210}t^4 + \frac{305957123435183189886671552 47}{531901303745467281429461840} \quad (16)$$

Since, the roots of (16) are $t_1 = 1$ *and* $t_2, t_3, t_4 \leq 0$
Therefore, the method (5) is zero Stable.

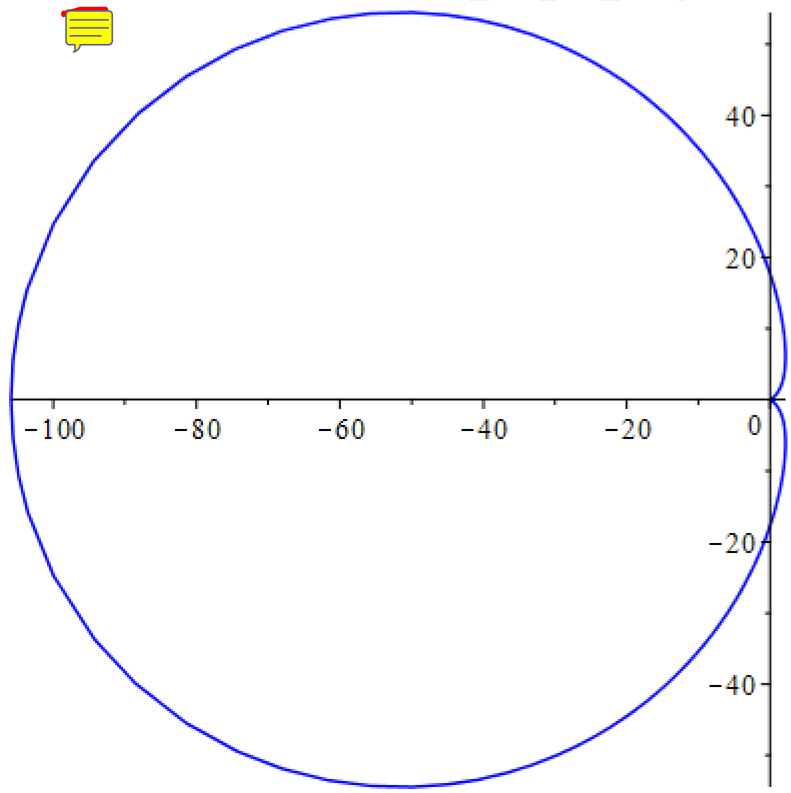

Image 1 . The A-stability region of the proposed scheme ABISBDF

Test Problems

To validate the method developed, the following stiff IVPs are solved.

Problem 1 : $y_1' = y_2$ $y_1(0) = 1$

$y_2' = y_1$ $y_2(0) = 1$ $\quad 0 \le x \le 100$

Exact Solution

$y_1(x) = e^x$

$y_2(x) = e^x$

Source: (Bronson, 1973)

$Problem$ 2: $y_1' = 198y_1 + 199y_1$ $y_1(0) = 1$ $\quad 0 \le x \le 10$

$y_2' = -398y_1 - 399y_2$ $\quad y_2(0) = -1$

Exact solution

$y_1(x) = e^{-x}$

$y_2(x) = -e^{-x}$

Eigen values −1 and −200

Source: (Ibrahim *et al.*, 2007);

$Problem$ 3: $y_1' = y_2$ $y_1(0) = 0$ $\quad 0 \le x \le 20$

$y_2' = -y_1$ $y_2(0) = 1$

Exact solution

$y_1(x) = sinx$

$y_2(x) = cosx$

Source: (shampine*et al*, 1975)

Problem 4 : $y_1' = y_2$ $y_1(0) = 0$

$y_2' = -2y_2$ $y_2(0) = 0$ $\quad 0 \le x \le 4л$

$y_3' = y_2 + 2y_3$ $y_3(0) = 1$

Exact Solution

$y_1(x) = 2cosx + 6sinx - 6x - 2$

$y_2(x) = -2sinx + 6cosx - 6$

$y_3(x) = 2sinx - 2cosx + 3$

Source: (Sulaiman,1989)

Numerical Results

The problems sampled in this research are solved using the developed scheme. The results are tabulted, compared; and the graphs highlighting the performance of these methods are plotted. The acronyms below are used in the tables.

h = step-size;

MAXE = Maximum Error;

T=Time in second;

3ESBBDF = 3 Point enhanced fully implicit Super Class of Block Backward Differentiation
$F_1$3ESBBDF = Family of block 3 Super class of Block Backward Differentiation
ABISBDF = A-stable block integrator scheme of Backward Differentiation Formula for solving Stiff IVPs.

Table 1: Comparison of Errors between Proposed Method and Some of the Existing Methods for Problem 1 & 2

Numerical Result for Problem 1				Numerical Result for Problem 2			
h	Method	MAXE	TIME	h	Method	MAXE	TIME
10^{-2}	$F_1$3SBBDF	3.30736e-002	4.23434e-1	10^{-2}	$F_1$3SBBDF	3.23032e-002	3.77590e-002
	3ESBSBDF	3.51456e-002	3.52416e-4		3ESBSBDF	3.98707e-002	2.63337e-002
	ABISBDF	5.83217e-004	4.23441e-5		ABISBDF	5.83217e-003	5.68676e-002
10^{-3}	$F_1$3SBBDF	5.41853e-003	1.81850e-3	10^{-3}	$F_1$3SBBDF	4.76165e-003	5.66636e-001
	3ESBSBDF	5.20191e-003	2.50367e-3		3ESBSBDF	4.40956e-003	2.60816e-001
	ABISBDF	6.95338e-005	4.65467e-4		ABISBDF	6.05338e-005	5.64515e-001
10^{-4}	$F_1$3SBBDF	5.44701e-005	1.71443e-2	10^{-4}	$F_1$3SBBDF	4.66516e-004	5.64385e-001
	3ESBSBDF	5.20417e-005	2.36918e-2		3ESBSBDF	5.08942e-005	2.60725e-001
	ABISBDF	6.95692e-007	4.48433e-3		4BSBDF	6.26692e-007	5.68143e+000
10^{-5}	$F_1$3SBBDF	5.44971e-007	1.70042e-1	10^{-5}	$F_1$3SBBDF	4.68707e005	5.63788e+000
	3ESBSBDF	5.25030e-007	2.34808e-1		3ESBSBDF	5.21534e-007	2.60597e+000
	ABISBDF	6.959740e-009	4.58687e-2		ABISBDF	6.32740e-009	5.59821e+001
10^{-6}	$F_1$3SBBDF	5.44998e-009	1.70308e0	10^{-6}	$F_1$3SBBDF	4.69123e-006	5.65356e+001
	3ESBSBDF	5.25648e-009	2.35791e0		3ESBSBDF	5.89872e-009	2.60700e+001
	ABISBDF	7.186362e-011	4.23434e-1		ABISBDF	6.33362e-011	5.53567e+002

Table 2: Comparison of Errors between Proposed Method and Some of the Existing Methods for Problem 3 & 4

Numerical Result for Problem 3				Numerical Result for Problem 4			
h	Method	MAXE	TIME	h	Method	MAXE	TIME
10^{-2}	$F_1$3SBBDF	2.07208e-002	1.37500e-2	10^{-2}	$F_1$3SBBDF	2.83032e-002	3.67590e-002
	3ESBSBDF	2.54347e-002	1.20394e-3		3ESBSBDF	2.48705e-002	2.63337e-002
	ABISBDF	2.83117e-004	7.36289e-2		ABISBDF	3.83217e-003	5.58676e-002
10^{-3}	$F_1$3SBBDF	3.20160e-004	2.72200e-2	10^{-3}	$F_1$3SBBDF	3.76163e-003	8.56636e-002
	3ESBSBDF	3.02893e-004	1.25972e-2		3ESBSBDF	3.40956e-003	2.60816e-001
	ABISBDF	4.05338e-006	5.81512e-2		ABISBDF	4.05338e-005	5.54515e-001
10^{-4}	$F_1$3SBBDF	3.20233e-006	2.02700e-1	10^{-4}	$F_1$3SBBDF	3.76514e-005	8.54385e-001
	3ESBSBDF	3.09895e-006	1.25148e-1		3ESBSBDF	3.48942e-005	2.60725e+000
	ABISBDF	4.26592e-008	5.81491e-1		ABISBDF	4.26690e-007	5.58143e-001
10^{-5}	$F_1$3SBBDF	3.20261e-008	1.92600e0	10^{-5}	$F_1$3SBBDF	3.70705e005	8.53788e+000
	3ESBSBDF	3.10157e-008	1.25471e0		3ESBSBDF	3.58532e-005	2.60597e+001
	ABISBDF	4.32640e-010	5.81122e0		ABISBDF	4.32740e-009	5.49821e+000
10^{-6}	$F_1$3SBBDF	3.20263e-010	1.91700e1	10^{-6}	$F_1$3SBBDF	3.71121e-007	8.53356e+001
	3ESBSBDF	3.41129e-010	1.24892e1		3ESBSBDF	3.69872e-007	2.60700e+002
	ABISBDF	4.33262e-012	5.79987e1		ABISBDF	4.3335e-009	5.43567e+001

From the numerical problems solved in the Table 1 and Table 2 above, it has been shown that the proposed scheme, ABISBDF outperformed both the 3ESBSDF and $F_1$3SBBDF in terms of minimum error and less computational time. However, $F_1$3SBBDF has advantage over 3ESBSBDF in the entire tested problems with regard to errors. But, 3ESBSBDF competes closely with the $F_1$3SBBDF in terms of execution time, with 3ESBSBDF favored in most of the problem considered in this paper. Finally, the accuracy, computation time of the new methods seems to be better in comparison with the other two methods for all the problems solved.

Similarly, to highlight the performance of the proposed methods, ABISBDF in relation to the other methods, 3ESBSBDF and $F_1$3SBBDF. The graphs of *Log10(MAXE)* against the step size, *h* for the 4 problems are plotted accordingly as shown below:

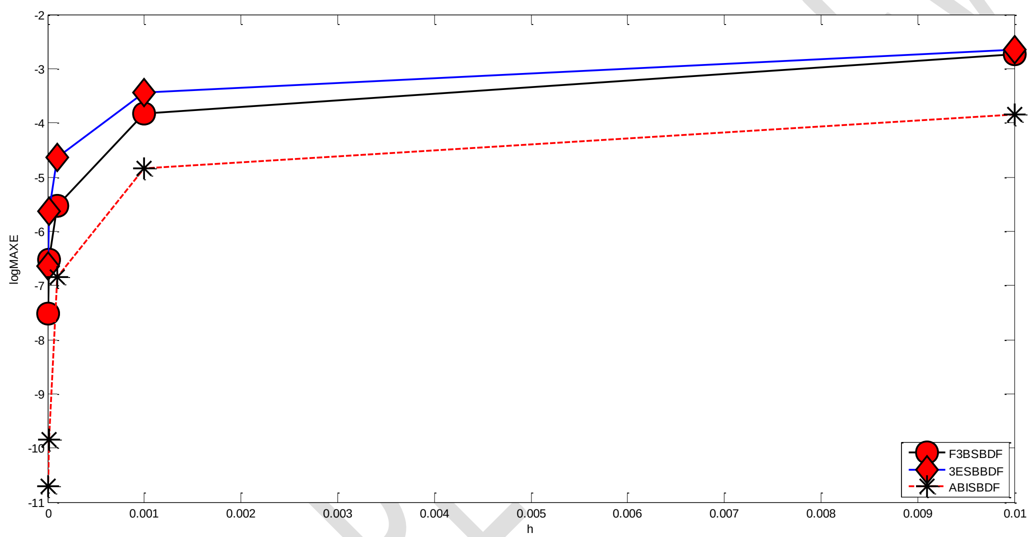

Figure 1: Graph of $\text{Log}_{10}(MAXE)$ against h for problem 1

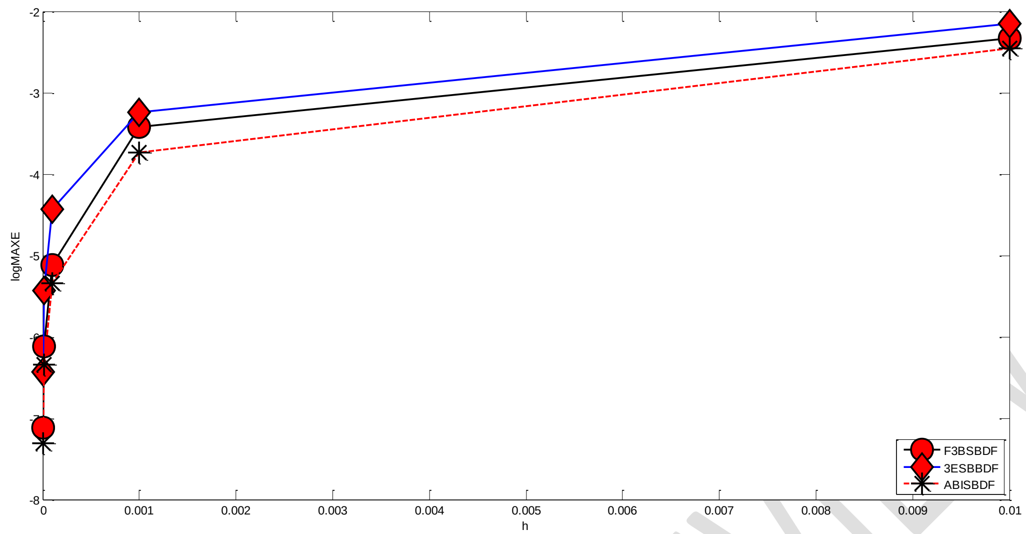

Figure 2: Graph of $\text{Log}_{10}(MAXE)$ against h for problem 2

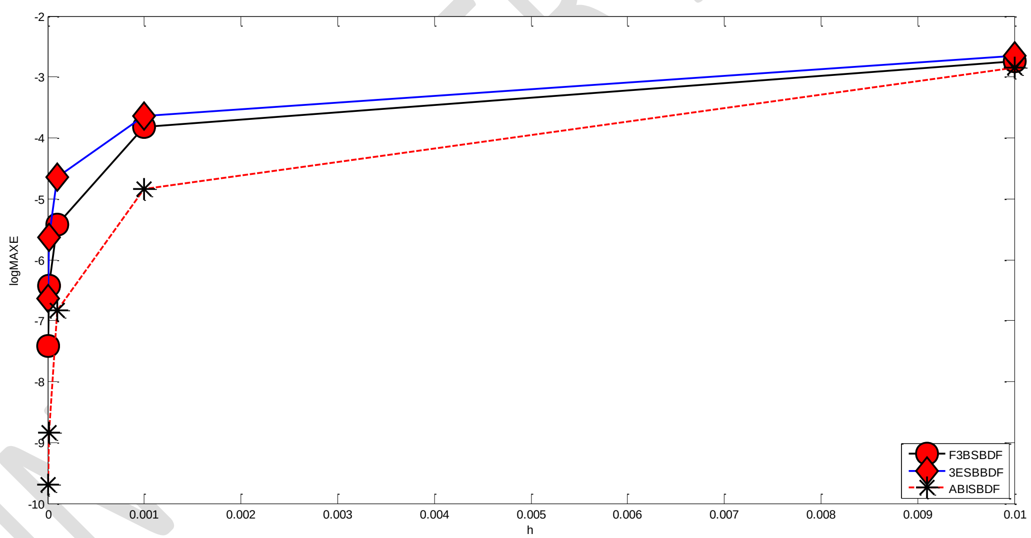

Figure 3: Graph of $\text{Log}_{10}(MAXE)$ against h for problem 3

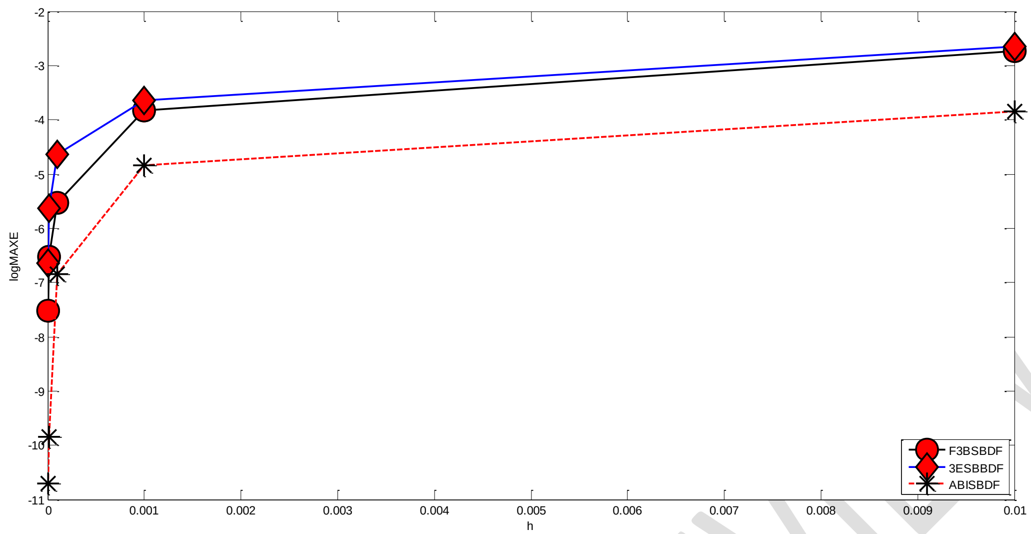

Figure 4: Graph of $\text{Log}_{10}(MAXE)$ against h for problem 4

Conclusion

An order sixth block numerical scheme has been proposed, the scheme have good stability properties. The developed methods are fully implicit block methods, can computes, four solution values at a time per step, concurrently. The tested problem's results shows advantage in terms of accuracy of the scaled error and computational time when compared with the 3ESBSBDF and $F_1$3SBBDF methods. The proposed scheme can be used in solving a system of first order initial value problem of ordinary differential equations.

References

Abdullahi M, Musa, H(2021); Enhanced 3 point fully implicit super class of block backward differentiation formula for solving first order stiff initial value problems. Fudma journal of science (FJS) **5**(2): 120-127.

Cash, J. R. (1980). On the integration of stiff systems of ODEs using extended backward differentiation formulae. *NumerischeMathematik.* **34**: 235-246.

Cash, J. R. (2000). Modified extended backward differentiation formula for the numerical solution of stoff IVPs in ODE and DAEs." *Computational and Applied Mathematics* **125**: 117-130.

Curtiss C.F. and Hirschfelder J.O. (1952). Integration of Stiff Equations. Proceedings of the National Academy of Sciences, 38, 235-243.

Fatokun, J., Onumanyi, P. &Sirisena U.W. (2005). Solution of Ordinary System of Ordinary Differential Equations by Continuous Finite DifferenceMethods with Arbitrary Basis Functions.J.Nig.Math. Society **24**: pp.31 –36.

Suleiman, M.B. (1989). Solving Higher Order ODEs Directly by the Direct Integration Method. Applied Mathematics and Computation **33**: 197-219.

Bronson, R (1973), Modern Introductory Differential Equation: Schaum's Outline Series. USA, McGraw-Hill Book Company

Ibrahim, Z. B., Othman, K.,& Suleiman, M.B. (2007). Implicit r-point block backward differentiation formula for solving first- order stiff ODEs. *Applied Mathematics and Computation, 186*, 558-565.

Musa, H., Suleiman, M. B., Ismail, F.,Senu, N, Majid and Z. A., Ibrahim, Z. B. (2014). A new fifth order implicit block method for solving first order stiff ordinary differential equations. *Malaysian Journal of Mathematicam Sciences* 8(S): 45-59.

Musa, H., Bature, B., & Ibrahim, L. K. (2016). Diagonally implicit super class of block backward differentiation formula for solving Stiff IVPs. *Journal of the Nigerian Association of Mathematical Physics, **36***: 73 – 80.

Shampine, L.F. and Gordon, M.K.(1975), Computer Solution of Ordinary Differential Equations. The Initial Value Problem, W. H. Freeman and company, San Francisco.

Fatunla, S.O. (1984). One-leg Multistep Method for first Order Differential Equations. Computer and Mathematics with Applications, **10**: 1-4.

Sagir (2012), An accurate computation of block hybrid method for solving stiff ODEs IOSR Journal of Mathematic (IOSR-JM) **4**(4): 18-21.

Sagir (2014), Numerical Treatment of Block Method for the Solution of Ordinary Differential Equations. International Journal of Mathematical, Computational Science and Engineering, 8(2): 16-20.

Yahaya, Y.A and A.M. Sagir (2013); an order five implicit 3-step block method for solving system of ODEs.Pacific journal of science and Technology, 14(1): 176-181.

Musa H, M.A.Unwala (2019); Extended 3-point super class of block backward differentiation formula for solving initial value problem. 38th conference of National Mathematical Science university of Nigeria.Nsukka.