Original Research Article

Analytical Solution of Linear Fractional Partial Differential Equation of Order $0 < \alpha \leq 1$ by Improved Adomian Decomposition Method

Abstract

The paper aims to obtain exact analytical solution of linear nonhomogeneous space-time fractional order partial differential equation by improved Adomian decomposition method coupled with fractional Taylor expansion series. The solution of these equations are in series form may have rapid convergence to a closed-form solution. The effectiveness and sharpness of this method is shown by obtaining the exact solution of these equations with suitable initial conditions(ICs). With the help of this method, it is possible to investigate nature of solutions when we vary order of the fractional derivative. Behaviour of the solution of these equations are represented by graphs using Matlab software.

Keywords : Improved Adomian decomposition method; Fractional Taylor expansion series; Mittage-Leffler function; Caputo fractional derivative

1 Introduction

The study of nonlinear partial differential equations(PDEs) is a well developed research area. There are physical models governed by nonlinear fractional partial differential equations(FPDEs) in various sciences such as sciences of Mathematics, sciences of Physics, sciences of Chemistry, and sciences of Biology as well as in technologies [16, 25, 26, 29, 32, 33]. Several researchers have focused on the study of physical models directed by FPDEs. The difficulty of getting the exact solution of equations in such models is an important and attractive area of research. Not all nonlinear equations in physical models have an exact solution, therefore, many researchers have developed various methods of solving nonlinear FPDEs. The Adomian decomposition method (ADM)[6, 7] is a powerful weapon to determine solution of fractional differential equations.In a present years, a numerous attention has been devoted to the study of the Adomian's Decomposition Method (ADM)[1, 6, 7]. Cherrualt et al.[11], Adomian and

Rach [8], Duan et al. [17], which permit us to survey the properties and solutions of a huge variety of ordinary and partial differential equations, as well as of fractional partial differential equations(FPDEs), which express several mathematical problems, or can be used to mathematically model diverse physical processes. From a historic view, the ADM was first introduced, and enormously used in the 1980's [2, 3, 4, 5], and afterward many mathematicians and scientists have constantly modified the ADM in an attempt to upgrade its accuracy and/or to widen the applications of the initial method. Duan, Rach and Wazwaz [21, 22, 23, 35, 36, 37, 39, 43] put in a lot of effort to improve ADM. The ADM procedure to solve linear or nonlinear boundary value problems using the Duan-Rach recursion[18, 19, 20], which is intrinsic to the ADM, does not itself require Green's functions, Dirac delta functions, and discretization techniques such as a finite-difference method or a finite element method. Also, it does not invoke the shooting method, special basis functions, guessing the starting term, linearization, perturbation, and so on. Importantly, fast, efficient, cost-effective and accurate solutions can be found without the need to resort to high performance computing. As the nonlinear terms are not ignored or crudely linearized, a much better appreciation of the physics of a particular problem ensues. This aspect of simulation is often lost in numerical methods. A key concept is that the Adomian decomposition series is any rearrangement of the Banach-space analog of the Taylor expansion series about the initial solution component function that permits solution by recursion, in which the aforesaid rearrangement is accomplished through the choice of the recursion scheme. The ADM yields a rapidly convergent sequence of analytic functions as the approximate solutions of the original mathematical model. The most important work about convergence has been carried out by Cherruault [10]. Further remarks about the convergence of the decomposition method are in [11]. Historical view of ADM given by Rach in [34].Thus the ADM subsumes even the classic power series method while extending the class of amenable nonlinearities to include any analytic nonlinearity.

While Shawagfeh [40] has employed ADM for solving nonlinear fractional partial differential equations, Daftardar-Gejji and Jafri have obtained solution of numerous problems [27] by using ADM. Also Dhaigude and Birajdar [14, 15] extended the discrete ADM for obtaining the numerical solution of system of FPDEs. Chitalkar-Dhaigude and Bhadgaonkar in [12] have shown that the ADM is more convenient than the Charpit's method to solve first-order nonlinear PDEs. Sontakke and Pandit [41, 42] investigates the iterative solution of linear and NFPDEs using fractional ADM. Bhadgaonkar and Sontakke [9] obtained exact Solution of Space-Time FPDEs by ADM.

Rach et al. [38] created a new modification of the ADM for solving ordinary differential equations (ODEs) using the Taylor expansion series for a nonhomogeneous term. N. Khodabakshi et al. [28], discussed the basic ADM method and extended the proposed method in [38] to solve time-fractional ODEs. As a result of these ideas, Dhaigude and Bhadgaonkar in [13] combined the ADM with a fractional Taylor expansion series and obtain an almost analytical solution of physical models such as Gas

dynamics model, Advection model, Wave model, and Klein-Gordon model in nonlinear nonhomogeneous space-time fractional PDEs. The fractional Taylor expansion series used in this method is represented as a differential transform in the differential transform method [31]. This method is developed specifically for nonhomogeneous differential equations. The main aim of this paper is to implement improved ADM to solve the linear nonhomogeneous space-time fractional partial differential equation so we write proposed method in [13] for linear fractional PDE. The solution of this equation is calculated in the form of convergent series with easily computable components. The space-time fractional derivatives are described in the Caputo sense.

The paper is structured in this way: in section 2 few basic results about fractional calculus and related properties are given which are used in this paper, while in section 3 we clarify the steps of the improved ADM for solving nonlinear nonhomogeneous space and time fractional order PDEs in section 4. Section 5 is conclusions.

2 Basic Definitions

In this section, basic definitions on fractional calculus are discussed which are useful for further discussion.

Definition 2.1 *[33] Let $f \in \mathcal{C}_\alpha$ and $\alpha \geq -1$, then Riemann-Liouville fractional integral operator(RLFIO) of $w(x,t)$ with respect to t of order α is indicated by $I_t^\alpha w(x,t)$ and is explained as*

$$I_t^\alpha w(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{(\alpha-1)} w(x,\tau) d\tau, \quad t > 0, \alpha > 0. \tag{2.1}$$

Definition 2.2 *[33] Let $m-1 < \alpha < m$, $t \in R$ and $t > 0$. The Caputo fractional derivative operator(CFDO) for the function $f \in H^1([a,b], \mathbb{R}_+)$ with order $\alpha \geq 0$ is explained as*

$$D_t^\alpha w(x,t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m w}{\partial \tau^m} d\tau, \\ \dfrac{\partial^m w}{\partial t^m}, \qquad \alpha = m \in N. \end{cases} \tag{2.2}$$

We have following properties of RLFIO and CFDO

$$D_t^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)} t^{(\mu-\alpha)}, \tag{2.3}$$

$$I_t^\alpha t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} t^{(\mu+\alpha)}, \quad \alpha > 0, \ \mu > -1. \tag{2.4}$$

Note that the relation between RLFIO and CFDO is given by:

$$I_t^\alpha D_t^\alpha w(x,t) = w(x,0) - \sum_{k=0}^{m-1} w^{(k)}(x,0) \frac{t^k}{k!}, \quad m-1 < \alpha \leq m. \tag{2.5}$$

Mittage-Leffler function(MLF) : The MLF for one parameter and two parameter is explained as follows

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbf{C}, Re(\alpha) > 0),$$
$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta \in \mathbf{C}, Re(\alpha, \beta) > 0).$$

When we apply CFDO on MLF we get

$$D_t^\alpha E_\alpha(at^\alpha) = aE_\alpha(at^\alpha), \tag{2.6}$$

where a is constant.

3 Analysis of method

Consider the initial value problem (IVP) for space-time FPDE of order $0 < \alpha \leq 1$,

$$D_t^\alpha w(x,t) + D_x^\alpha w(x,t) = g(x,t), \tag{3.1}$$
$$w(x,0) = h(x), \tag{3.2}$$

or equivalently

$$L(w(x,t)) = g(x,t), \tag{3.3}$$
$$w(x,0) = h(x), \tag{3.4}$$

where $w(x,t)$ is unrecognized function which we want to determined, t is time variable, x is the space coordinate, $L(w(x,t))$ is fractional differential operator and $g(x,t)$ is nonhomogeneous function.
Now, applying the RLFIO I_t^α on both side of equation(3.1) and use the IC (3.2), we attain:

$$w(x,t) = w(x,0) + I_t^\alpha \big[g(x,t) - D_x^\alpha w(x,t))\big]. \tag{3.5}$$

The unrecognized function $w(x,t)$ can be expressed as an infinite series of the form

$$w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) \tag{3.6}$$

Suppose that $g(x,t)$ is analytic.Its fractional Taylor expansion series [24, 30, 31] is:

$$g(x,t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h) x^{k\alpha} t^{h\alpha}, \tag{3.7}$$

where

$$G_{\alpha,\alpha}(k,h) = \frac{1}{\Gamma(k\alpha+1)\Gamma(h\alpha+1)} (D^\alpha)_x^k (D^\alpha)_t^h g(x,t)\big|_{x=t=0}$$
$$\text{and} \quad (D^\alpha)_x^k = D_x^\alpha D_x^\alpha ... D_x^\alpha, \quad k \quad times.$$

By using (3.6) and (3.7) in (3.5) we attain

$$\sum_{n=0}^{\infty} w(x,t) = h(x) + I_t^{\alpha}\Big[\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h)x^{k\alpha}t^{h\alpha} - D_x^{\alpha}\sum_{n=0}^{\infty} w_n(x,t)\Big], \tag{3.8}$$

$$\begin{aligned}\sum_{n=0}^{\infty} w(x,t) = h(x) &+ I_t^{\alpha}\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} + \cdots\Big] \\ &- I_t^{\alpha}\Big[D_x^{\alpha}\sum_{n=0}^{\infty} w_n(x,t)\Big].\end{aligned} \tag{3.9}$$

Taking term by term comparison on both side of equation (3.9), we set recursion scheme like:

$$\begin{aligned} w_0(x,t) &= h(x), \\ w_1(x,t) &= I_t^{\alpha}\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha} - D_x^{\alpha}w_0\Big], \\ w_2(x,t) &= I_t^{\alpha}\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} - D_x^{\alpha}w_1\Big], \\ w_3(x,t) &= I_t^{\alpha}\Big[\sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha} - D_x^{\alpha}w_2\Big], \end{aligned}$$

and so forth. Then the solution $w(x,t)$ of IVP $(3.1)-(3.2)$ is

$$\phi_{m+1} = \sum_{n=0}^{m} w_n(x,t) \tag{3.10}$$

which gives

$$\lim_{m\to\infty} \phi_{m+1} = w(x,t). \tag{3.11}$$

4 Numerical Applications

The effectiveness and sharpness of the improved ADM can be demonstrated by applying it to space-time fractional nonhomogeneous linear partial differential equation.

Example 4.1 *Consider the linear nonhomogeneous space-time fractional PDE,*

$$\frac{x^{\alpha}}{\Gamma(\alpha+1)}D_x^{\alpha}w(x,t) + D_t^{\alpha}w(x,t) = \frac{x^{\alpha}}{\Gamma(\alpha+1)}t^{\alpha}E_{2\alpha,1+\alpha}(t^{2\alpha}) + w, \qquad 0 < \alpha \le 1 \tag{4.1}$$

with initial condition

$$w(x,0) = \frac{x^{\alpha}}{\Gamma(\alpha+1)}. \tag{4.2}$$

Applying I_t^α on both side of equation (4.1) and use initial condition (4.2), we have

$$w(x,t) = w(x,0) + I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha w(x,t) + \frac{x^\alpha}{\Gamma(\alpha+1)}t^\alpha E_{2\alpha,1+\alpha}(t^{2\alpha}) + w\Big],$$
$$w(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha w(x,t) + \frac{x^\alpha}{\Gamma(\alpha+1)}t^\alpha E_{2\alpha,1+\alpha}(t^{2\alpha}) + w\Big]. \tag{4.3}$$

Here

$$g(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)}t^\alpha E_{2\alpha,1+\alpha}(t^{2\alpha}).$$

By using (3.6) and (3.7) in (4.3) we have

$$\sum_{n=0}^{\infty} w_n(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha\sum_{n=0}^{\infty} w_n + \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} G_{\alpha,\alpha}(k,h)x^{k\alpha}t^{h\alpha} + \sum_{n=0}^{\infty} w_n\Big],$$
$$\sum_{n=0}^{\infty} w_n(x,t) = \frac{x^\alpha}{\Gamma(\alpha+1)} + I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha\sum_{n=0}^{\infty} w_n + \sum_{n=0}^{\infty} w_n + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha}$$
$$+ \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha} + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha} + \cdots\Big]. \tag{4.4}$$

Here first few coefficients of $G_{\alpha,\alpha}(k,h)$ are given in Table1.

Table 1: The components of $G_{\alpha,\alpha}(k,h)$

$G_{\alpha,\alpha}(k,h)$ $k-varies\,vertically$ $h-varies\,horizontally$	$G_{\alpha,\alpha}(k,0)$	$G_{\alpha,\alpha}(k,1)$	$G_{\alpha,\alpha}(k,2)$	$G_{\alpha,\alpha}(k,3)$	$G_{\alpha,\alpha}(k,4)$	$\cdots$
$G_{\alpha,\alpha}(0,h)$	0	0	0	0	0	$\cdots$
$G_{\alpha,\alpha}(1,h)$	0	$\frac{1}{\Gamma(\alpha+1)\Gamma(\alpha+1)}$	0	$\frac{1}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}$	0	$\cdots$
$G_{\alpha,\alpha}(2,h)$	0	0	0	0	0	$\cdots$
$G_{\alpha,\alpha}(3,h)$	0	0	0	0	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Taking term by term comparison on both side of equation(4.4), we set recursion scheme as follows:

$$
\begin{aligned}
w_0(x,t) &= \frac{x^\alpha}{\Gamma(\alpha+1)},\\
w_1(x,t) &= I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha w_0 + w_0 + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,0)x^{k\alpha}t^{0\alpha}\Big],\\
&= I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)} + 0\Big],\\
&= I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)}\Big] = 0,\\
w_2(x,t) &= I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha w_1 + w_1 + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,1)x^{k\alpha}t^{\alpha}\Big],\\
&= I_t^\alpha\Big[-0+0+\frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^\alpha}{\Gamma(\alpha+1)}\Big],\\
&= \frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
w_3(x,t) &= I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha w_2 + w_2 + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,2)x^{k\alpha}t^{2\alpha}\Big],\\
&= I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha\big[\frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\big] + \frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 0\Big],\\
&= I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big] = 0,\\
w_4(x,t) &= I_t^\alpha\Big[-\frac{x^\alpha}{\Gamma(\alpha+1)}D_x^\alpha w_3 + w_3 + \sum_{k=0}^{\infty} G_{\alpha,\alpha}(k,3)x^{k\alpha}t^{3\alpha}\Big],\\
&= I_t^\alpha\Big[-0+0+\frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\Big],\\
&= \frac{x^\alpha}{\Gamma(\alpha+1)}\frac{t^{4\alpha}}{\Gamma(4\alpha+1)},
\end{aligned}
$$

and so on.Then the exact solution of IVP (4.1)-(4.2) is

$$
\begin{aligned}
w(x,t) = \sum_{n=0}^{\infty} w_n(x,t) &= \frac{x^\alpha}{\Gamma(\alpha+1)}\Big[1+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}+\cdots\Big],\\
&= \frac{x^\alpha}{\Gamma(\alpha+1)}\sum_{k=0}^{\infty}\frac{t^{2\alpha k}}{\Gamma(2\alpha k+1)},\\
w(x,t) &= \frac{x^\alpha}{\Gamma(\alpha+1)}E_{2\alpha}(t^{2\alpha}). \qquad (4.5)
\end{aligned}
$$

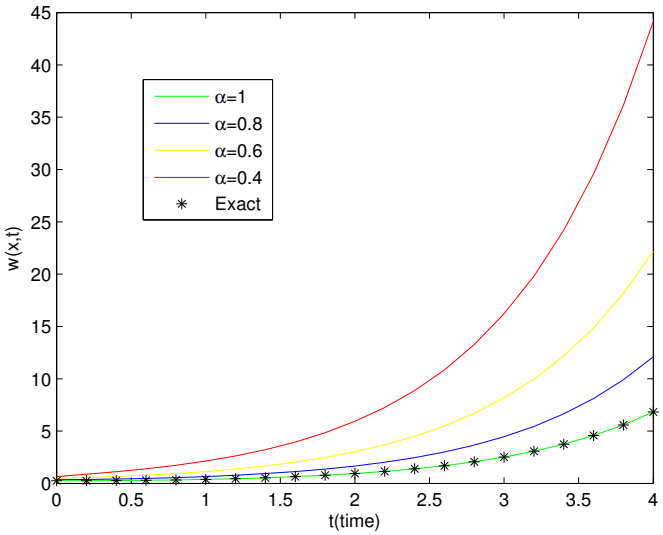

Figure 1: 2D Graphical representation of solution (4.5) of IVP (4.1)-(4.2) for different values of α such as $\alpha = 1, 0.8, 0.6, 0.4$ and exact when $x = 0.25$.

If $\alpha = 1$ then IVP (4.1)-(4.2) is

$$xw_x + w_t = x\sinh(t) + w, \tag{4.6}$$

with IC

$$w(x, 0) = x. \tag{4.7}$$

the exact solution of given IVP is

$$w(x, t) = x\cosh(t). \tag{4.8}$$

If $\alpha = \frac{1}{2}$ then solution of IVP (4.1)-(4.2) is

$$w(x, t) = 2\sqrt{\frac{x}{\Pi}}e^t. \tag{4.9}$$

Remark 4.1 *: Figure* 1 *is the graphical behaviour of improved ADM solution* (4.5) *for different values of* α *such as* $\alpha = 1, 0.8, 0.6, 0.4$ *and exact solution* (4.8) *when* $x = 0.25$*. 3D graphical representation of solution* (4.5) *of IVP* (4.1)-(4.2) *for different values of* α *such as* $\alpha = 1, 0.8, 0.6, 0.4$ *are given in Figure*2 *and Figure*3*. It is clear from Figure* 1*, Figure* 2 *and Figure* 3 *that, when the limit* $\alpha \to 1$*, the solution* (4.5) *approaches to the exact solution* (4.8)*.*

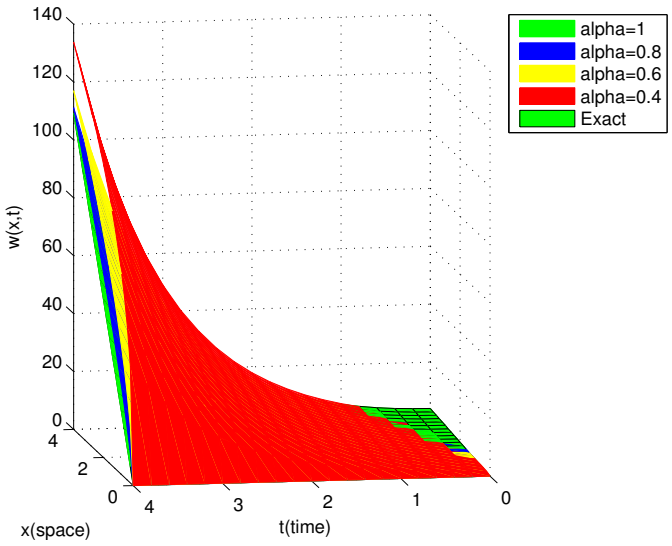

Figure 2

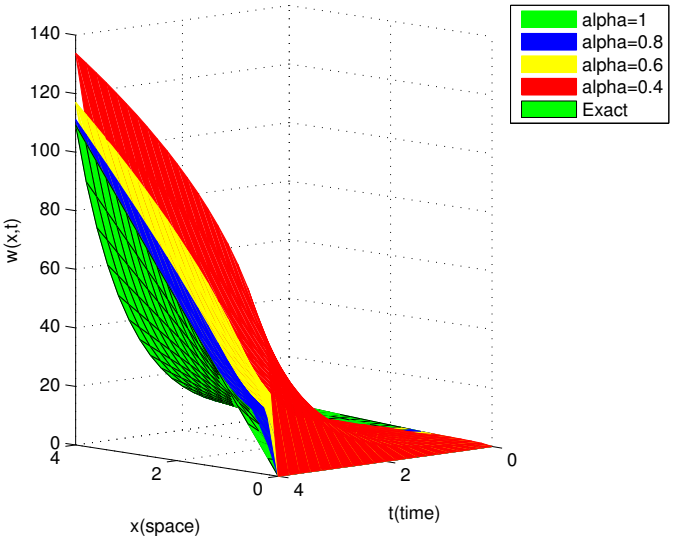

Figure 3

Figure 2 and Figure 3 shows the 3D Graphical representation of solution (4.5) of IVP (4.1)-(4.2) for different values of α such as $\alpha = 1, 0.8, 0.6, 0.4$ and exact solution (4.8).

5 Conclusions

The applicability of improved ADM is demonstrated by some physically significant linear nonhomogeneous fractional partial differential equation of order $0 < \alpha \leq 1$. It returns either a fast convergent series or an exact solution. Another advantage of this method is that we can see where we want to stop the recursion by looking at the coefficient table that is created during the process. The solution of these models are in series form may have rapid convergence to a closed-form solution. It is a more convenient way to solve such types of partial differential equation with the help of improved ADM than general ADM.

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