New Fantastic Curves Discovered from Rectangular Hyperbola

Original Research Article

Abstract

By means of constructing on the hyperbola xy = N a triangle whose shape is changed under the rule that one vertex is fixed at the vertex of the hyperbola, one vertex is moving on the hyperbola and the two laterals with respect to the fixed and the moving vertices respectively keep their directions unchanged, it is discovered that the loci of the triangle's centroid and orthocenter are respectively a hyperbola and a line, the locus of the circumcenter is a new cubic algebraic curve, and those of the incenter and ex-centers are planar curves that have not been reported before. All the loci of the centers form a fantastic graph like a flying insect. Meanwhile, the discovered hyperbola and curves are merely N-dependant and can be used to estimate the distribution of y divided by x with respect to xy = N.

Keywords: Rectangle hyperbola; locus; centroid; circumcenter; incenter; orthocenter

2010 Mathematics Subject Classification:51N20/51-00

1 Introduction

The hyperbola was early studied by Menaechmus, Euclid and Aristaeus, as reviewed in [1]. It was formally and particularly studied systematically in Apollonius's book of conics [2] and [3]. Since the analytic geometry came into being, it has been a primary knowledge for middle school students and even college students of engineering to learn. These can be seen in either "older" textbooks like [4][5] or "newer" textbooks like [6][7]. Everyone who learned the analytic geometry knows that all the textbooks focus a lot on the so-called standard form in the Cartesian coordinate system while another form xy = N has been less exploited because the later is thought to be a transformed form from the standard form via coordinate transformations. Our recent study on integer factorization problem came across the literatures [8] and [9]. Accordingly we were aroused to have a study on the kind of the hyperbola. After the study, we gained windfall benefit of discovering several new fantastic curves. This paper introduces these curves as well as some of their applications. This paper is composed of 6 sections. Section 1 is this introductory part, the section 2 lists and proves some primary and fundamental properties that are cited in later sections, section 3 presents the newly found curves, section 4 shows the key points to draw multiple figures into one graph with Maple software, section 5 shows an application in analyzing divisor-ratio of a semiprime and the last section is the prospect for the future work.

2 Preliminaries

This section presents the necessary preliminaries for later descriptions, including symbols, notations and fundamental geometrical elements that support later researches in this paper.

2.1 Symbols and Notations

In this whole paper, symbol $A \Rightarrow B$ means statement *A* can derive out statement *B*, symbol $|AB|$ means the length of line segment AB, symbol $P : (x, y)$ means *x* and *y* are coordinates of point *P* and symbol $\Gamma : f(x, y) = 0$ means Γ is defined by the equation $f(x, y) = 0$. An odd interval $[a, b]$ means the interval contains odd integer from a to b; for example, $[3, 11]$ is equivalent to the set $3, 5, 7, 9, 11$.

2.2 Necessary Mathematical Foundations

Given a real number $N > 0$, this paper mainly investigates the hyperbola H:$xy = N$ with $x > 0$ and $y > 0$ in the Cartesian coordinate system. For convenience, the capital letter H is sometimes of the same meaning as H: $xy = N$. It is easy to establish the following basic properties, which are utilized in later sections.

Property 1. In the rectangular coordinate system XOY, as shown in Fig. 1, the vertex and focus of the hyperbola H:$xy = N$ are $P : (\sqrt{N}, \sqrt{N})$ and $F : (\sqrt{2N}, \sqrt{2N})$, respectively.The tangent line at P to H is $x + y = 2\sqrt{N}$.

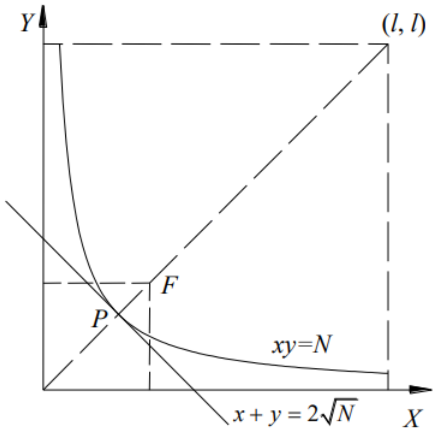

Figure 1: Vertex and Focus of hyperbola H

Property 2. Let Q be the intersection of H with the line $y = 2x$, U and V be respectively the intersections of the horizontal line passing through Q with Y-axis $x = 0$ and the line $y = x$, as shown in Fig. 2; then U is coincided with the focus of H and $|QU| = |QV|$.

Proof. The coordinates of Q, U and V are simply calculated to be

$$Q : (\sqrt{\frac{N}{2}}, \sqrt{2N}),$$
$$U : (\sqrt{2N}, \sqrt{2N})$$

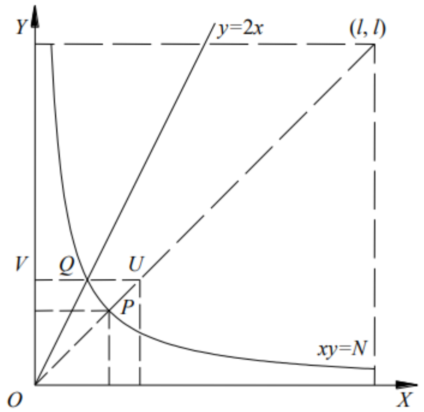

Figure 2: $|QU| = |QV|$

and

$$V : (0, \sqrt{2N}).$$

By Property 1, U is surely coincided with the focus of H and it follows

$$|QU| = \sqrt{2N} - \sqrt{\frac{N}{2}} = \frac{\sqrt{2N}}{2} = |QV|$$

□

Remark 1. The divisor-ratio of a semiprime $N = pq$ with odd prime divisors p and q satisfying $2 < p < q$ is defined to be $\alpha = q/p$. $\alpha > 2$ or $\alpha < 2$ is critical for a very big semiprime like the RSA numbers, as studied in [10] and [11]. Therefore, P and Q are two critical points in analyzing the divisor-ratio of the big semiprimes. Actually, as stated in the introductory section, this whole paper is an extra gain that originated from a research to determine whether the divisor-ratio of a given RSA number is bigger or smaller than 2. For this reason and future citations, we list this property and the following Properties 3 and 4.

Property 3. Given real numbers $\alpha_1 = (\frac{71}{24} + \frac{175\sqrt{6}}{288})^2$, $\alpha_2 = (\frac{71}{24} - \frac{175\sqrt{6}}{288})^2$; assume S_1 and S_2 are respectively the intersections of H with the lines $y = \alpha_1 x$ and $y = \alpha_2 x$. Construct two horizontal lines passing through S_1 and S_2, respectively; then the ellipse Γ_e defined by

$$\Gamma_e : 150x^2 - 276xy + 150y^2 - 71\sqrt{N}x - 71\sqrt{N}y + 159N = 0 \tag{2.1}$$

is tangent to the two horizontal lines, as shown in Fig. 3.

Proof. Direct calculations show the coordinates of S_1 and S_2 are respectively

$$S_1 : (x_1, y_1) = (\sqrt{\frac{N}{\alpha_1}}, \sqrt{\alpha_1 N}) = (\frac{\sqrt{N}}{\frac{71}{24} + \frac{175\sqrt{6}}{288}}, (\frac{71}{24} + \frac{175\sqrt{6}}{288})\sqrt{N})$$

and

$$S_2 : (x_2, y_2) = (\sqrt{\frac{N}{\alpha_2}}, \sqrt{\alpha_2 N}) = (\frac{\sqrt{N}}{\frac{71}{24} - \frac{175\sqrt{6}}{288}}, (\frac{71}{24} - \frac{175\sqrt{6}}{288})\sqrt{N}).$$

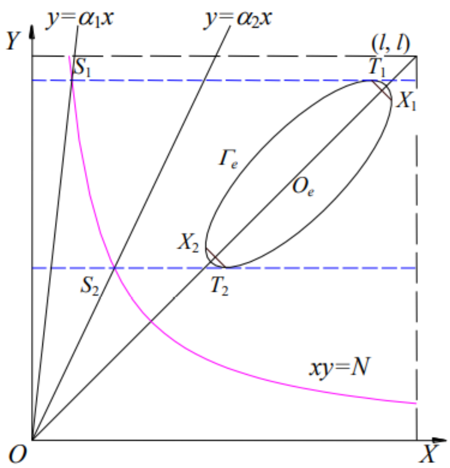

Figure 3: Ellipse tangent to two lines

Substituting y in (2.1) with y_1 yields an equation of x by

$$150x^2 - 276x\sqrt{\alpha_1 N} - 71x\sqrt{N} + 150\alpha_1 N - 71N\sqrt{\alpha_1} + 159N = 0 \tag{2.2}$$

Since the equation 2.2 has a unique solution $x = (\frac{71}{24} + \frac{161\sqrt{6}}{288})\sqrt{N}$, it is known that the point $T_1 : ((\frac{71}{24} + \frac{161\sqrt{6}}{288})\sqrt{N}, (\frac{71}{24} + \frac{175\sqrt{6}}{288})\sqrt{N})$ is a tangent point of Γ_e. Similarly, it can be shown that $T_2 : ((\frac{71}{24} - \frac{161\sqrt{6}}{288})\sqrt{N}, (\frac{71}{24} - \frac{175\sqrt{6}}{288})\sqrt{N})$ is a tangent point of Γ_e.
□

Remark 2. It can verified that, the center of Γ_e is $O_e : (\frac{71}{24}\sqrt{N}, \frac{71}{24}\sqrt{N})$, the length of its major axis is $\frac{35\sqrt{2N}}{12}$, length of its minor axis is $\frac{35\sqrt{3N}}{72}$, and two foci are respectively $(\frac{\sqrt{N}}{\sqrt{2}} \cdot (\frac{71\sqrt{2}}{24} - \frac{35\sqrt{69}}{144}), \frac{\sqrt{N}}{\sqrt{2}} \cdot (\frac{71\sqrt{2}}{24} - \frac{35\sqrt{69}}{144}))$ and $(\frac{\sqrt{N}}{\sqrt{2}} \cdot (\frac{71\sqrt{2}}{24} + \frac{35\sqrt{69}}{144}), \frac{\sqrt{N}}{\sqrt{2}} \cdot (\frac{71\sqrt{2}}{24} + \frac{35\sqrt{69}}{144}))$. Since the major axis is coincided with the line $y = x$, its parametric equation is given by

$$\begin{cases} x = \frac{71}{24}\sqrt{N} + \frac{35\sqrt{N}}{24}\cos t + \frac{35\sqrt{6N}}{288}\sin t \\ y = \frac{71}{24}\sqrt{N} + \frac{35\sqrt{N}}{24}\cos t - \frac{35\sqrt{6N}}{288}\sin t \end{cases}, 0 \leqslant t \leqslant 2\pi \tag{2.3}$$

For convenience to construct the ellipse Γ_e and to calculate with its characteristic points, here choose several important points and mark them with Fig. 4. In the figure, V_1 and V_2 are two vertices on the major axis, U_1 and U_2 are two vertices on the minor axis, M_1 and M_2 are middle points of the half major axis, $J_1J_2 \perp V_1V_2$ and $K_1K_2 \perp V_1V_2$, and X_1 and X_2 that are symmetric to T_1 and T_2 respectively by the major axis. The coordinates of these points are given as follows.

$$V_1 : (\frac{53}{12}\sqrt{N}, \frac{53}{12}\sqrt{N}),$$

$$T_1 : ((\frac{71}{24} + \frac{161\sqrt{6}}{288})\sqrt{N}, (\frac{71}{24} + \frac{175\sqrt{6}}{288})\sqrt{N}),$$

$$J_1 :(\frac{59\sqrt{N}}{16} - \frac{35\sqrt{2N}}{192}, \frac{59\sqrt{N}}{16} + \frac{35\sqrt{2N}}{192}),$$

$$U_1:((\frac{71}{24}-\frac{35\sqrt{6}}{288})\sqrt{N},(\frac{71}{24}+\frac{35\sqrt{6}}{288})\sqrt{N}),$$

$$K_1:(\frac{107\sqrt{N}}{48}-\frac{35\sqrt{2N}}{192},\frac{107\sqrt{N}}{48}+\frac{35\sqrt{2N}}{192}),$$

$$X_2:((\frac{71}{24}-\frac{175\sqrt{6}}{288})\sqrt{N},(\frac{71}{24}-\frac{161\sqrt{6}}{288})\sqrt{N}),$$

$$V_2:(\frac{3}{2}\sqrt{N},\frac{3}{2}\sqrt{N}),$$

$$T_2:((\frac{71}{24}-\frac{161\sqrt{6}}{288})\sqrt{N},(\frac{71}{24}-\frac{175\sqrt{6}}{288})\sqrt{N}),$$

$$K_2:(\frac{107\sqrt{N}}{48}+\frac{35\sqrt{2N}}{192},\frac{107\sqrt{N}}{48}-\frac{35\sqrt{2N}}{192}),$$

$$J_2:(\frac{59\sqrt{N}}{16}+\frac{35\sqrt{2N}}{192},\frac{59\sqrt{N}}{16}-\frac{35\sqrt{2N}}{192}),$$

$$U_2:((\frac{71}{24}+\frac{35\sqrt{6}}{288})\sqrt{N},(\frac{71}{24}-\frac{35\sqrt{6}}{288})\sqrt{N}),$$

$$X_1:((\frac{71}{24}+\frac{175\sqrt{6}}{288})\sqrt{N},(\frac{71}{24}+\frac{161\sqrt{6}}{288})\sqrt{N}).$$

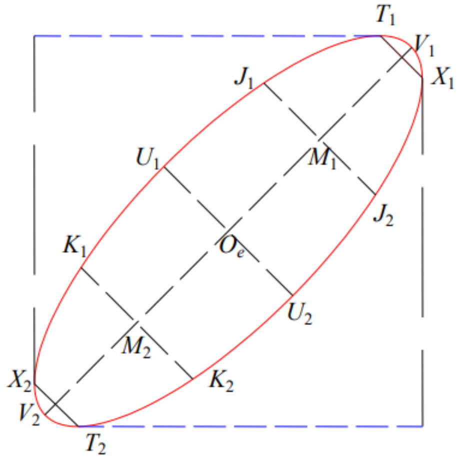

Figure 4: Points on ellipse Γ_e

Property 4. Let S be the intersection of H with the line $l_2 : y = 2x$, l_3 be the line passing through S and perpendicular to l_2 and Γ_e be the ellipse defined by (1); transform Γ_e to be Γ_s such that Γ_s passes through S and its major axis is coincident with l_3, as shown in Fig. 5. Let Γ_S^* be the reflection of Γ_s with respect to l_2. Then equations of Γ_s and Γ_S^* are given by (2.4) and (2.5) respectively.

$$28x^2+92xy+97y^2-\sqrt{2N}(120+\frac{35\sqrt{5}}{6})x-\sqrt{2N}(240-\frac{35\sqrt{5}}{12})y+300N=0 \quad (2.4)$$

$$28x^2 + 92xy + 97y^2 - \sqrt{2N}\left(120 - \frac{35\sqrt{5}}{6}\right)x - \sqrt{2N}\left(240 + \frac{35\sqrt{5}}{12}\right)y + 300N = 0 \tag{2.5}$$

And H has a part lying inside Γ_s and a part lying inside Γ_S^*.

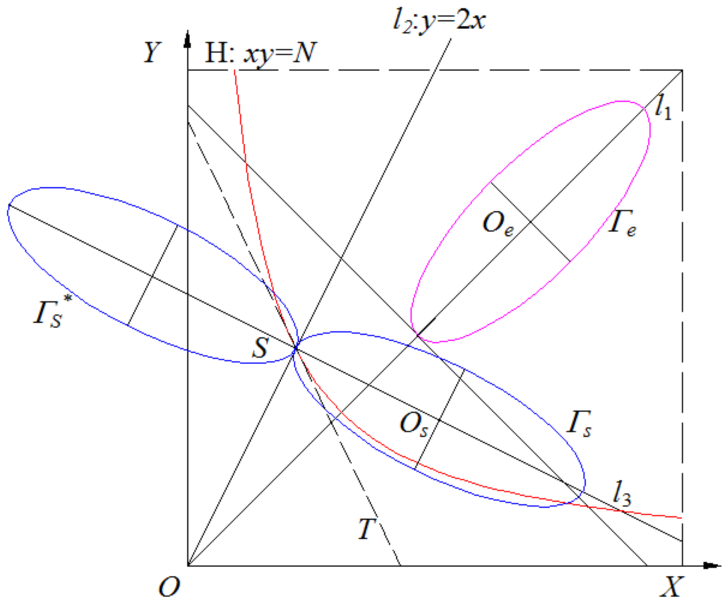

Figure 5: Ellipses Γ_e, Γ_s and Γ_S^*

Proof. The equations (2.4) and (2.5) are easily obtained through coordinate transformations from Γ_e to Γ_s and from Γ_s to Γ_s^*. Here simply show H has a part lying inside of the ellipse Γ_s. Let ST be the tangent line at S of H; then slope of ST is $k_S = -\frac{N}{x_S^2} < 0$, where x_S is the horizontal coordinate of S. Since S is also on l_2 whose slope is 2, there is an angle between ST and l_2. The condition that l_2 is the tangent line at S of the ellipse Γ_s yields that H crosses Γ_s, which means H has a part lying inside of Γ_s. Similarly, H has a part lying inside of Γ_s^*.
□

Remark 3. The ellipses Γ_e to be Γ_s are highly related with H. Referring to Remark 1, it is known that are useful in analyzing the divisor-ratio of a semiprime.

3 Research Method and Results

In the Cartesian coordinate system, consider the hyperbola H:$xy = N$ and two lines $l_1 : y = x$ and $l_\alpha : y = \alpha x$ with $\alpha > 0$; let P and Q be respectively the intersections of H with l_1 and l_α, as shown in Fig. 6. Construct at P a line parallel to l_α, at Q a line parallel to l_1; denote R to be the intersection of the two constructed lines. The triangle PQR is obviously α dependant because its shape changes with the change of α. Since P, Q and R are collinear when $\alpha = 1$, $\alpha \neq 1$ is assumed by default in this section unless particularly mentioned.

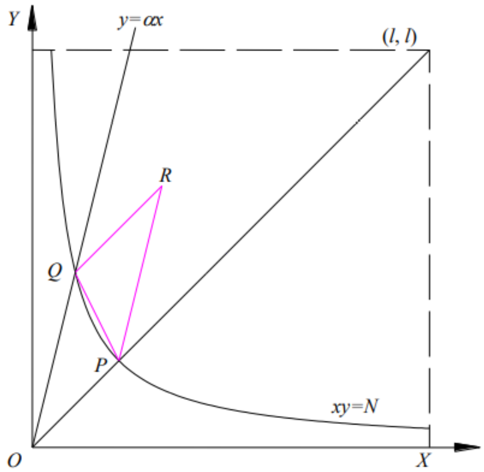

Figure 6: Obtuse triangle PQR

3.1 New Provable Results Induced from the Triangle PQR

Property 5. The triangle PQR is an obtuse one with $\angle Q$ being the obtuse angle.

Proof. Let the coordinates of P and Q be $P:(x_P,y_P)$ and $Q:(x_Q,y_Q)$; direct calculations show that R's coordinate is $R:(x_R,y_R)=(x_P+x_Q,y_P+y_Q)$. That is

$$\begin{cases} x_R = x_P + x_Q = \sqrt{N}(1+\frac{1}{\sqrt{\alpha}}) \\ y_R = y_P + y_Q = \sqrt{N}(1+\sqrt{\alpha}), \end{cases} \tag{3.1}$$

Accordingly,

$$|QR| = \sqrt{x_P^2+y_P^2} = \sqrt{2N}$$
$$|PR| = \sqrt{x_Q^2+y_Q^2} = \sqrt{(\frac{1}{\alpha}+\alpha)N}$$
$$|PQ| = \sqrt{((1-\frac{1}{\sqrt{\alpha}})^2+(\sqrt{\alpha}-1)^2)N}$$

Note that $\sqrt{(\frac{1}{\alpha}+\alpha)}-\sqrt{((1-\frac{1}{\sqrt{\alpha}})^2+(\sqrt{\alpha}-1)^2)} = \frac{2(\frac{1}{\sqrt{\alpha}}+\sqrt{\alpha})-2}{\sqrt{(\frac{1}{\alpha}+\alpha)}+\sqrt{((1-\frac{1}{\sqrt{\alpha}})^2+(\sqrt{\alpha}-1)^2)}}$, $\frac{1}{\alpha}+\alpha>2$ and $\frac{1}{\sqrt{\alpha}}+\sqrt{\alpha}>2$, it knows $|PR|>|QR|$ and $|PR|>|PQ|$, which leads to $\angle Q>\angle P$ and $\angle Q>\angle R$ in ΔPQR.

Since

$$\cos\angle Q = \frac{|QR|^2+|PQ|^2-|PR|^2}{2|PQ||QR|} = \frac{2-1(\frac{1}{\sqrt{\alpha}}+\sqrt{\alpha})}{\sqrt{2((1-\frac{1}{\sqrt{\alpha}})^2+(\sqrt{\alpha}-1)^2)}} < 0,$$

it yields $\angle Q > 90°$

□

Property 6. The vertex R is on hyperbola $(x-\sqrt{N})(y-\sqrt{N})=N$, as illustrated in Fig. 7.

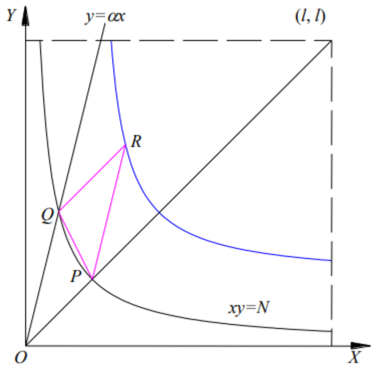

Figure 7: Hyperbola H:$xy=N$ and locus of R

Proof. Direct calculations by (3.1) follows

$$\begin{cases} x_R-\sqrt{N}=\dfrac{\sqrt{N}}{\sqrt{\alpha}} \\ y_R-\sqrt{N}=\sqrt{\alpha N} \end{cases} \Rightarrow (x_R-\sqrt{N})(y_R-\sqrt{N})=N$$

which surely means $R:(x_R,y_R)$ is on the hyperbola $(x-\sqrt{N})(y-\sqrt{N})=N$.
□

Remark 4. Seen from Property 6 , the locus of the vertex R forms a hyperbola when α changes, as described in Fig. 8. For convenience, this hyperbola is denoted by H* and from now on called a *companion hyperbola* of H or simply *companion* and △PQR is called a *companion triangle*.

Property 7. The centroid of △PQR is on the hyperbola $(x-\frac{2}{3}\sqrt{N})(y-\frac{2}{3}\sqrt{N})=\frac{4}{9}N$.

Proof. Let G be the centroid . First consider P, Q and R are not collinear. Then G's coordinates are given by

$$\begin{cases} x_G=(x_P+x_Q+x_R)/3 \\ y_G=(y_P+y_Q+y_R)/3 \end{cases}$$

By (3.1) it follows

$$\begin{cases} x_G=2(x_P+x_Q)/3=2(\sqrt{N}+\sqrt{\dfrac{N}{\alpha}})/3 \\ y_G=2(y_P+y_Q)/3=2(\sqrt{N}+\sqrt{\alpha N})/3 \end{cases} \tag{3.2}$$

That is

$$(x_G-\frac{2}{3}\sqrt{N})(y_G-\frac{2}{3}\sqrt{N})=\frac{4}{9}N \tag{3.3}$$

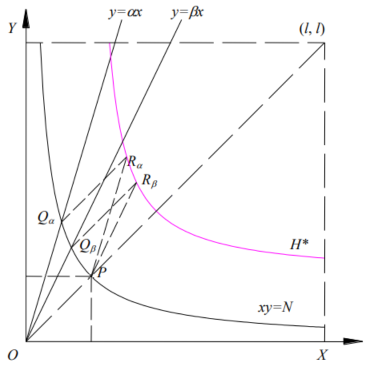

Figure 8: Hyperbola H and its companion H*

which is surely on the hyperbola $(x - \frac{2}{3}\sqrt{N})(y - \frac{2}{3}\sqrt{N}) = \frac{4}{9}N$
When P, Q and R are collinear, it yields $\alpha = 1$. This time by (3.1) it holds

$$\begin{cases} x_G = \dfrac{4}{3}\sqrt{N} \\ \\ y_G = \dfrac{4}{3}\sqrt{N} \end{cases} \Rightarrow \begin{cases} x_G - \frac{2}{3}\sqrt{N} = \frac{2}{3}\sqrt{N} \\ \\ y_G - \frac{2}{3}\sqrt{N} = \frac{2}{3}\sqrt{N} \end{cases} \Rightarrow (x_G - \tfrac{2}{3}\sqrt{N})(y_G - \tfrac{2}{3}\sqrt{N}) = \tfrac{4}{9}N$$

which means G is still on the hyperbola $(x - \frac{2}{3}\sqrt{N})(y - \frac{2}{3}\sqrt{N}) = \frac{4}{9}N$.
□

Remark 5. Seen from (3.2) and (3.3), the centroid G and the vertex R are on the line $y = \sqrt{\alpha}x$ because $\frac{y_R}{x_R} = \frac{y_G}{x_G} = \sqrt{\alpha}$.

Property 8. The orthocenter of ΔPQR is on line $l : x + y = 2\sqrt{N}$,which is tangent to H: $xy = N$ at $P : (\sqrt{N}, \sqrt{N})$.

Proof. Let h be orthocenter. Direct calculations yields its coordinates by

$$\begin{cases} x_h = -\dfrac{\alpha^2 - 2\alpha\sqrt{\alpha} + 1}{(\alpha - 1)\sqrt{\alpha}} \times \sqrt{N} \\ y_h = \dfrac{\alpha^2 - 2\alpha\sqrt{\alpha} + 1}{(\alpha - 1)\sqrt{\alpha}} \times \sqrt{N} + 2\sqrt{N} \end{cases} \tag{3.4}$$

It immediately follows $x_h + y_h = 2\sqrt{N}$, saying that H is on the line l. By Property 1, l is tangent to H at P.
□

Remark 6. Take arbitrary two lines $l_\alpha : y = \alpha x$ and $l_\beta : y = \beta x$ with $1 < \beta \leq \alpha$, construct their companion triangles and draw the orthocenters I_α and I_β respectively, as illustrated in Fig 9; it is seen that I_α and I_β are surely on the tangent line at P of H, provided that P, Q and R are not collinear.

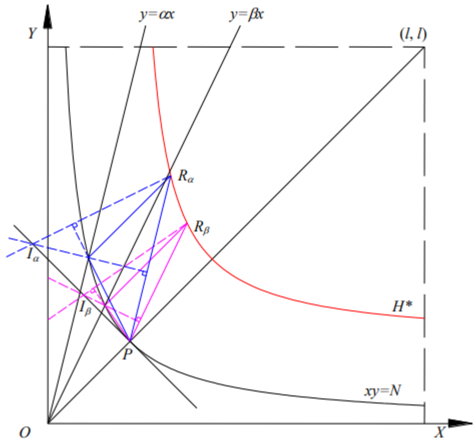

Figure 9: Orthocenters concurrent with tangent line at P

3.2 New Cubic Algebraic Curve Derived from the Triangle PQR

Theorem 1. The locus of the circumcenter of ΔPQR is a new planar algebraic curve whose algebraic equation is given by

$$x^3 + y^3 + (x^2 + y^2)\sqrt{N} - (x + 2\sqrt{N} + y)xy - (x + y)N + 3N\sqrt{N} = 0 \tag{3.5}$$

The curve is symmetric with respect to the line $y = x$ and passes through the point V : $(\frac{3}{2}\sqrt{N}, \frac{3}{2}\sqrt{N})$.

Proof. Let C be circumcenter and Γ_C be the locus formed by C 's changing with the change of ΔPQR. By definition, C is on the intersection of the three perpendicular bisectors of the laterals PQ, QR and RP, as shown in Fig. 10. This means Γ_C passes the point V : $(\frac{3}{2}\sqrt{N}, \frac{3}{2}\sqrt{N})$ because C is on the line $y = x$ in the case that the line OQ is limiting to and coincided with OP.

After mathematical reasoning, C's coordinates are calculated by

$$\begin{cases} x_C = (\dfrac{1}{1+\sqrt{\alpha}} + \dfrac{\alpha+1}{2\sqrt{\alpha}}) \times \sqrt{N} \\ y_C = (\dfrac{\sqrt{\alpha}}{1+\sqrt{\alpha}} + \dfrac{\alpha+1}{2\sqrt{\alpha}}) \times \sqrt{N} \end{cases}, 0 < \alpha < \infty \tag{3.6}$$

With Maple command *implicitize* of the *algcurves* package, the implicit cubic algebraic equation (3.5) is derived out.

□

Remark 7. The parametric equation (3.6) is easily obtained by simple mathematical reasoning. However, its algebraic equation (3.5) is hard to obtain without the help of the Maple software. It took us several months to gain the equation (3.5). Drawing (3.6) with the Maple software yields an ellipse-arc like curve as seen in Fig.11.

Then we tried to find out an ellipse to match it. By taking two values α_1 and α_2 with $1 < \alpha_1 < \alpha_2$,we first calculated four points with $\frac{1}{\alpha_2}$, $\frac{1}{\alpha_1}$, α_1 and α_2. With the calculated four points and the vertex

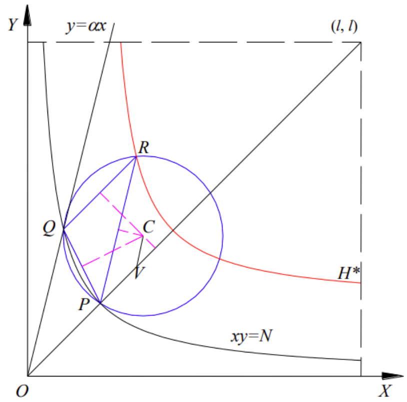

Figure 10: Circumcenter C and lines perpendicular to laterals of ΔPQR

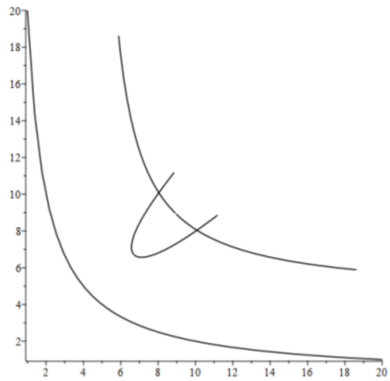

Figure 11: Ellipse-like locus of circumcenter

$V:(\frac{3}{2}\sqrt{N},\frac{3}{2}\sqrt{N})$, we calculated an ellipse with Maple. After many times of experiments, we judged that the calculated ellipse was Γ_e defined by (2.1). When we drew Γ_C with $\frac{1}{\alpha_2}\le\alpha\le\alpha_2$ and half of Γ_e with the Maple software we got Fig. 12. In the figure, the little red squares are points on Γ_e, the black-colored curve is from Γ_C, the pink curve is H and the green curve is H*. It is seen that the head of Γ_C looks highly coincided with Γ_e in $\alpha's$ domain from $\frac{1}{\alpha_2}$ to α_2. Accordingly we took it for granted that Γ_C was a part of the ellipse (1). That is why we spent a lot time on studying Γ_e.

However, drawn with another software Geogebra, the locus is not all coincided with Γ_e, as shown in Fig. 13. It has a head and a tail with the head looking coincided with a part of Γ_e and the tail looking like two lines.

Actually, letting $t=\sqrt{\alpha}$ turns the formula (3.6) into the following (3.7)

$$\begin{cases} x_C=(\frac{1}{1+t}+\frac{1}{2t}+\frac{t}{2})\times\sqrt{N} \\ y_C=(\frac{t}{1+t}+\frac{1}{2t}+\frac{t}{2})\times\sqrt{N} \end{cases},0<t<\infty \tag{3.7}$$

Then it follows

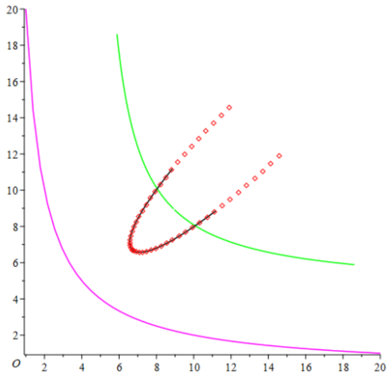

Figure 12: Comparison of ellipse arc with locus of the circumcenter

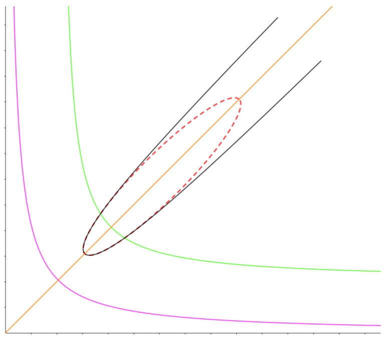

Figure 13: Comparison of Γ_C with Γ_e

$$\begin{cases} x_C{}' = (\frac{1}{2} - \frac{1}{2t^2} - \frac{1}{(1+t)^2}) \times \sqrt{N} \\ y_C{}' = (\frac{1}{2} - \frac{1}{2t^2} + \frac{1}{(1+t)^2}) \times \sqrt{N} \end{cases}$$

and

$$\begin{cases} x_C{}'' = (\frac{1}{t^3} + \frac{2}{(1+t)^3}) \times \sqrt{N} \\ y_C{}'' = (\frac{1}{t^3} - \frac{2}{(1+t)^3}) \times \sqrt{N} \end{cases}$$

The curvature of Γ_C is calculated with

$$\kappa(t) = \frac{|x'y'' - x''y'|}{|x'^2 + y'^2|^{3/2}} = \frac{4\left|t^2 - t + 1\right| \cdot (t+1)^5 t^4 \sqrt{2t}}{(t^8 + 4t^7 + 4t^6 - 4t^5 - 6t^4 - 4t^3 + 4t^2 + 4t + 1)^{3/2}}$$

This yields $\lim_{t\to\infty} \kappa(t) = 0$ and $\lim_{t\to 0} \kappa(t) = 0$, saying that Γ_C is tending to be like a line when α is either bigger or smaller than a certain value. This is surely fit for what is seen in Fig. 13. Looking into the professional books such as books [12][13], there is not a curve matching to the shape of Γ_C. It is surely a new curve.

3.3 New Fantastic Curves Derived from the Triangle PQR

New Curve 1. The locus of the incenter of ΔPQR is a new planar curve passing through the point $V : (\frac{3}{2}\sqrt{N}, \frac{3}{2}\sqrt{N})$. It has two branches symmetric with respect to the line $y = x$ and has a fantastic shape that has never been recorded.

Comment. Let I be the incenter and Γ_I be the locus formed by $I's$ changing with the change of ΔPQR. Consider ΔPQR is over the line $y = x$, as shown in Fig.14. Since by definition $\angle Q > \angle R$ and $\angle Q > \angle P$ in ΔPQR, the distance from I to Q is smaller than that from I to P or that from I to R. As a result, I is infinitesimally close to P when Q is infinitesimally close to P, or ΔPQR is tending to be collinear with the line $y = x$. In another word, P is coincided with I when Q is limiting to P .

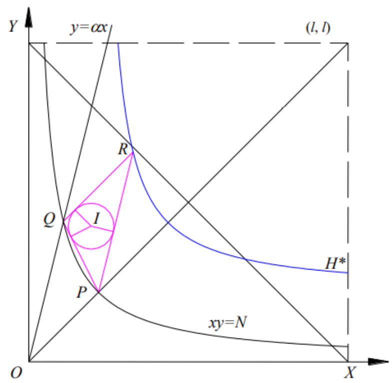

Figure 14: Incenter I and lines perpendicular to of ΔPQR

Denote $a = |PQ|, b = |PR|$ and $c = |QR|$; then it follows

$$\begin{cases} c = \sqrt{x_P^2 + y_P^2} = \sqrt{2N} \\ b = \sqrt{x_Q^2 + y_Q^2} = \sqrt{(\frac{1}{\alpha} + \alpha)N} \\ a = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2} = \sqrt{((1 - \sqrt{\frac{1}{\alpha}})^2 + (\sqrt{\alpha} - 1)^2)N} \end{cases} \tag{3.8}$$

Direct calculations knows that the coordinates of the incenter I are given by

$$\begin{cases} x_I = \dfrac{ax_P + ax_Q + bx_Q + cx_P}{a + b + c} \\ y_I = \dfrac{ay_P + ay_Q + by_Q + cy_P}{a + b + c} \end{cases} \tag{3.9}$$

which is simplified to be

$$\begin{cases} x_I = \sqrt{N}(1 + \sqrt{\dfrac{1}{\alpha}} - \dfrac{\sqrt{1+\alpha^2} + \sqrt{2}}{|\sqrt{\alpha}-1|\sqrt{\alpha+1} + \sqrt{1+\alpha^2} + \sqrt{2\alpha}}) \\ y_I = \sqrt{N}(1 + \sqrt{\alpha} - \dfrac{\sqrt{1+\alpha^2} + \alpha\sqrt{2}}{|\sqrt{\alpha}-1|\sqrt{\alpha+1} + \sqrt{1+\alpha^2} + \sqrt{2\alpha}}) \end{cases}, 0 < \alpha \neq 1 \tag{3.10}$$

Drawing this locus with Maple yields the black curve in Fig.15.

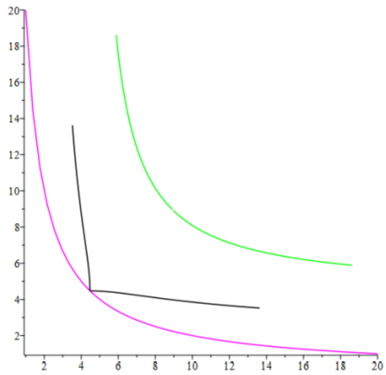

Figure 15: Locus of the incenter I

Enlarged and marked with some of its points, it is an S-shaped curve with two branches coincided at the point V, as Fig. 16 shows. Looking into the professional books such as books [12][13], there is not a curve or even a similar one matching to the shape of Γ_I . So far, there has not been such a record.

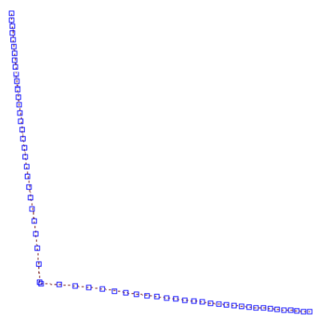

Figure 16: Two branches of an S-shaped curve

Remark 8. We tried to find out the algebraic equation for (3.10) but failed even with Maple and Geogebra. So far we have not known the algebraic equations for (3.10), (3.12), (3.14) and (3.16).

New Curve 2. The locus of the ex-center of the excircle touching PQ of ΔPQR is a new planar curve that has a fantastic shape. It is symmetric with respect to the line $y = x$ and passes through the point V : $(\frac{3}{2}\sqrt{N}, \frac{3}{2}\sqrt{N})$ at which it touches Γ_I defined by (3.10).

Comment. Take the ex-center W_{PQ} that touches PQ as an example. Let Γ_{W-PQ} be the locus formed by W_{PQ} that changes with the change of ΔPQR . Consider ΔPQR is over the line $y = x$, as shown in Fig.17.

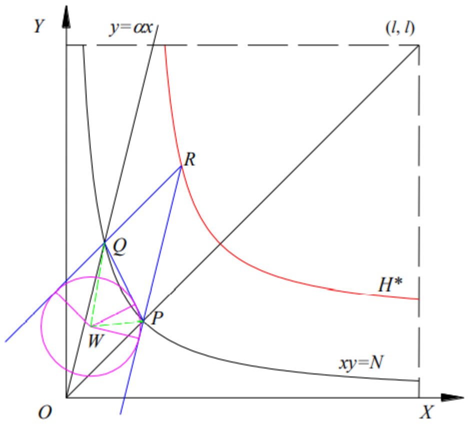

Figure 17: Ex-center of the excircle touching PQ

Since by definition $\angle Q > \angle P$ in ΔPQR, it knows $\angle WQP < \angle QPW$ in ΔQPW and the distance from W to Q is bigger than that from W to P. As a result, W is infinitesimally close to P when Q is infinitesimally close to P, or ΔQPW is tending to be collinear with the line $y = x$. In another word, P is coincided with W when Q is limiting to P .
Let a, b and c be defined as those in (3.8); then the coordinates of W are calculated by

$$\begin{cases} x_W = \dfrac{bx_Q + cx_P - ax_R}{b + c - a} \\ y_W = \dfrac{by_Q + cy_P - ay_R}{b + c - a} \end{cases} \tag{3.11}$$

which is simplified to be

$$\begin{cases} x_W = \sqrt{N}(1 + \dfrac{1}{\sqrt{\alpha}} + \dfrac{\sqrt{1+\alpha^2} + \sqrt{2}}{|\sqrt{\alpha}-1|\sqrt{1+\alpha} - \sqrt{1+\alpha^2} - \sqrt{2\alpha}}) \\ y_W = \sqrt{N}(1 + \sqrt{\alpha} + \dfrac{\sqrt{1+\alpha^2} + \alpha\sqrt{2}}{|\sqrt{\alpha}-1|\sqrt{1+\alpha} - \sqrt{1+\alpha^2} - \sqrt{2\alpha}}) \end{cases} , 0 < \alpha \neq 1 \tag{3.12}$$

Drawing this locus Γ_{W-PQ} together with Γ_I, leads to an interesting figure as Fig. 18 shows. The two loci share the point V and look like an X. By the way, Γ_{W-PQ} is also S-shaped with two branches coincided at V.

New Curve 3. The locus of the ex-center of the excircle touching PR of ΔPQR is a new planar curve that has a fantastic shape. It is symmetric with respect to the line $y = x$.

Comment. The coordinates of the ex-center are calculated by

$$\begin{cases} x = \dfrac{ax_R + cx_P - bx_Q}{a + c - b} \\ y = \dfrac{ay_R + cy_P - by_Q}{a + c - b} \end{cases} \tag{3.13}$$

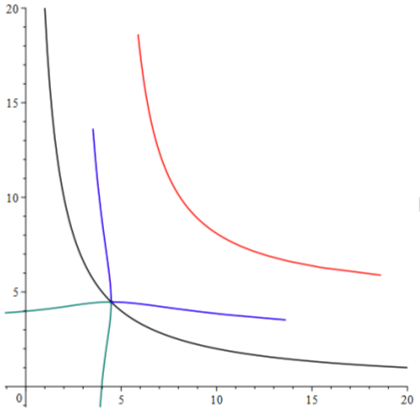

Figure 18: Loci of incenter and ex-center like an X

where a, b and c are defined in (3.8).
Simplifying (3.13) yields

$$\begin{cases} x = \sqrt{N}(1 + \frac{1}{\sqrt{\alpha}} + \frac{\sqrt{1+\alpha^2} - \sqrt{2}}{|\sqrt{\alpha}-1|\sqrt{1+\alpha} - \sqrt{1+\alpha^2} + \sqrt{2\alpha}}) \\ y = \sqrt{N}(1 + \sqrt{\alpha} + \frac{\sqrt{1+\alpha^2} - \alpha\sqrt{2}}{|\sqrt{\alpha}-1|\sqrt{1+\alpha} - \sqrt{1+\alpha^2} + \sqrt{2\alpha}}) \end{cases}, 0 < \alpha \neq 1 \quad (3.14)$$

Drawing the locus yields its shape as the green curve in Fig. 19.

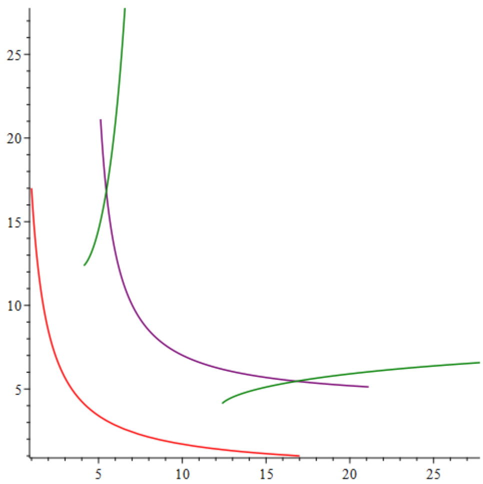

Figure 19: Locus of ex-center of the excircle touching PR

New Curve 4. The locus of the ex-center of the excircle touching QR of ΔPQR is a new planar curve that has a fantastic shape. It is symmetric with respect to the line $y = x$.

Comment. The coordinates of the ex-center are calculated by

$$\begin{cases} x = \dfrac{ax_R + bx_Q - cx_P}{a+b-c} \\ y = \dfrac{ay_R + by_Q - cy_P}{a+b-c} \end{cases} \tag{3.15}$$

where a, b and c are defined in (3.8).
Simplifying (3.15) yields

$$\begin{cases} x = \sqrt{N}(1 + \dfrac{1}{\sqrt{\alpha}} - \dfrac{\sqrt{1+\alpha^2}-\sqrt{2}}{|\sqrt{\alpha}-1|\sqrt{1+\alpha}+\sqrt{1+\alpha^2}-\sqrt{2\alpha}}) \\ y = \sqrt{N}(1+\sqrt{\alpha} - \dfrac{\sqrt{1+\alpha^2}-\alpha\sqrt{2}}{|\sqrt{\alpha}-1|\sqrt{1+\alpha}+\sqrt{1+\alpha^2}-\sqrt{2\alpha}}) \end{cases}, 0 < \alpha \neq 1 \tag{3.16}$$

Drawing the locus yields its shape as the blue curve in Fig. 20.

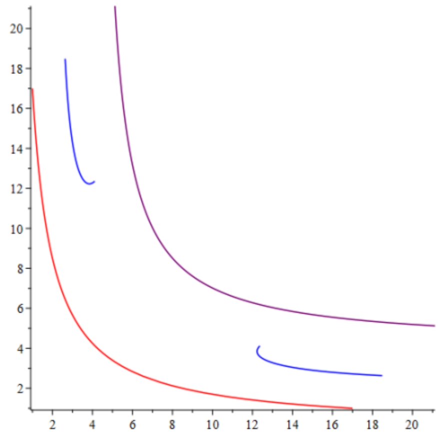

Figure 20: Locus of ex-center of the excircle touching QR

3.4 Fantastic Figure Consisted of the New Curves

Drawing together the hyperbola H:$xy = N$, the companion hyperbola H*:$(x - \sqrt{N})(y - \sqrt{N}) = N$, all the loci of the centers as well as the ellipse Γ_e , we have a very fantastic figure like something of a beetle, as seen in Figs. 21. It looks like a flying insect!

4 Keys to Drawing the Loci with Maple

The loci introduced in the last section can be tested with the software such as Matlab, Mathematica, Geogebra and Maple. It is known that any curve can be drawn in any of the software if the equation of the curve is known in either algebraic form or parametric form. Geogebra is easy to draw several curves in one work sheet no matter what form of the curves' equations are. However, it is better to use the same form for Maple to draw different curves in one graph. Seen in Maple's online help, it is known that Maple command *plot* can plot a list of curves all expressed by parametric equations and command *implicitplot* can plot a list of curves all expressed in algebraic equations. Since merely

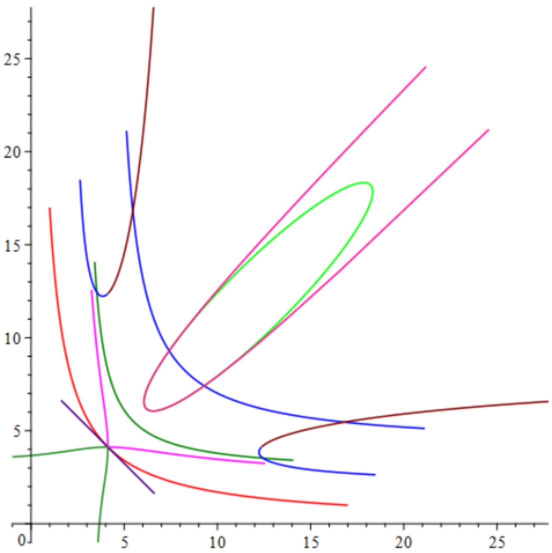

Figure 21: Fantastic pattern consisted of known hyperbolas and new curves

a small part of the new curves introduced in the previous sections have obtained their algebraic equations, it is better to use parametric equations to draw their loci. Considering that the equations (2.3),(3.1),(3.2),(3.4),(3.6),(3.10),(3.12),(3.14) and (3.16) are all in parametric forms, parameterization of the hyperbola H:$xy = N$ is necessary. With Maple command *solve*, parameterized solutions for the hyperbola H, the companion hyperbola H*, the line $x + y = \sqrt{N}$ and the centroid hyperbola $G : (x - \frac{2}{3}\sqrt{N})(y - \frac{2}{3}\sqrt{N}) = \frac{4}{9}N$ can all be parameterized into their parametric equations. For convenience, the following Table 1 lists all the parametric equations related with this paper. Drawing all these curves yields Fig. 21.

5 Application to Analyze Divisor-ratio of Semiprime

Consider N is a composite integer with integers p and q being its divisors with $pq = N$ and $1 < p \le q$. We are intending to know the range of $\alpha = \frac{q}{p}$ before N is factorized. Since $1 < p \le \sqrt{N}$,$\sqrt{N} \le q$ and the point (p, q) is on the hyperbola $xy = N$ and over the line $y = x$. Assume α is to be checked if it is smaller than 2, for example; then referring to Property 4 and the ellipse (2.4), which is for convenience called an α-detecting ellipse, yields the following Theorem 2.

Theorem 2. Let $N = pq$ be a semiprime with $1 < p < q$ and $\alpha = \frac{q}{p}$; then $1 < \alpha < 2$ if and only if p and q are respectively in the integer solutions of x and y of the following inequalities (5.1)

$$\begin{cases} xy = N \\ 1 < x \le \sqrt{N} \le y \\ 28x^2 + 92xy + 97y^2 - (120\sqrt{2N} + \dfrac{35\sqrt{10N}}{6})x - (240\sqrt{2N} - \dfrac{35\sqrt{10N}}{12})y + 300N < 0 \end{cases} \tag{5.1}$$

while $2 < \alpha < \frac{N}{\chi^2}$ if and only if p and q are respectively in the integer solutions of x and y of the following inequalities (5.2)

Table 1: Loci and Their Parametric Equations

	Locus	Parametric Equation	Parameter Scope
1	H	$\begin{cases} x=\sqrt{\frac{N}{t}} \\ y=\sqrt{tN} \end{cases}$	$0<t<\infty$
2	Γ_e	$\begin{cases} x=\frac{71}{24}\sqrt{N}+\frac{35\sqrt{N}}{24}\cos t+\frac{35\sqrt{6N}}{288}\sin t \\ y=\frac{71}{24}\sqrt{N}+\frac{35\sqrt{N}}{24}\cos t-\frac{35\sqrt{6N}}{288}\sin t \end{cases}$	$0\le t\le 2\pi$
3	H*	$\begin{cases} x=\frac{\sqrt{N}(t+1)}{t} \\ y=\sqrt{N}(t+1)) \end{cases}$	$0<t<\infty$
4	Centroid	$\begin{cases} x=\frac{2\sqrt{N}(t+1)}{3t} \\ y=\frac{2\sqrt{N}(t+1)}{3} \end{cases}$	$0<t<\infty$
5	Orthocenter	$\begin{cases} x=\frac{2\sqrt{N}}{t+1} \\ y=\frac{2t\sqrt{N}}{t+1} \end{cases}$	$0<t<\infty$
6	Circumcenter	$\begin{cases} x=(\frac{1}{1+t}+\frac{1}{2t}+\frac{t}{2})\times\sqrt{N} \\ y=(\frac{t}{1+t}+\frac{1}{2t}+\frac{t}{2})\times\sqrt{N} \end{cases}$	$0<t<\infty$
7	Incenter	$\begin{cases} x_I=\sqrt{N}(1+\sqrt{\frac{1}{t}}-\frac{\sqrt{1+t^2}+\sqrt{2}}{\lvert\sqrt{t}-1\rvert\sqrt{t+1}+\sqrt{1+t^2}+\sqrt{2t}}) \\ y_I=\sqrt{N}(1+\sqrt{t}-\frac{\sqrt{1+t^2}+t\sqrt{2}}{\lvert\sqrt{t}-1\rvert\sqrt{t+1}+\sqrt{1+t^2}+\sqrt{2t}}) \end{cases}$	$0<t\neq 1$
8	PQ side ex-center	$\begin{cases} x_W=\sqrt{N}(1+\sqrt{\frac{1}{t}}+\frac{\sqrt{1+t^2}+\sqrt{2}}{\lvert\sqrt{t}-1\rvert\sqrt{1+t}-\sqrt{1+t^2}-\sqrt{2t}}) \\ y_W=\sqrt{N}(1+\sqrt{t}+\frac{\sqrt{1+t^2}+t\sqrt{2}}{\lvert\sqrt{t}-1\rvert\sqrt{1+t}-\sqrt{1+t^2}-\sqrt{2t}}) \end{cases}$	$0<t\neq 1$
9	PR side ex-center	$\begin{cases} x=\sqrt{N}(1+\sqrt{\frac{1}{t}}+\frac{\sqrt{1+t^2}-\sqrt{2}}{\lvert\sqrt{t}-1\rvert\sqrt{1+t}-\sqrt{1+t^2}+\sqrt{2t}}) \\ y=\sqrt{N}(1+\sqrt{t}+\frac{\sqrt{1+t^2}-t\sqrt{2}}{\lvert\sqrt{t}-1\rvert\sqrt{1+t}-\sqrt{1+t^2}+\sqrt{2t}}) \end{cases}$	$0<t\neq 1$
10	QR side ex-center	$\begin{cases} x=\sqrt{N}(1+\sqrt{\frac{1}{t}}-\frac{\sqrt{1+t^2}-\sqrt{2}}{\lvert\sqrt{t}-1\rvert\sqrt{1+t}+\sqrt{1+t^2}-\sqrt{2t}}) \\ y=\sqrt{N}(1+\sqrt{t}-\frac{\sqrt{1+t^2}-t\sqrt{2}}{\lvert\sqrt{t}-1\rvert\sqrt{1+t}+\sqrt{1+t^2}-\sqrt{2t}}) \end{cases}$	$0<t\neq 1$

$$\begin{cases} xy=N \\ 1<x\le\sqrt{N}\le y \\ 28x^2+92xy+97y^2-\sqrt{2N}\left(120-\frac{35\sqrt{5}}{6}\right)x-\sqrt{2N}\left(240+\frac{35\sqrt{5}}{12}\right)y+300N<0 \end{cases} \tag{5.2}$$

where χ is the x-coordinate of the intersection of the hyperbola H with the ellipse (2.5).

Remark 8.
(1) The inequalities (5.1) can be converted into two inequalities of *x* and *y* separately as follows

$$\begin{cases} 28x^4-(120+\frac{35\sqrt{5}}{6})\sqrt{2N}x^3+392Nx^2-(240-\frac{35\sqrt{5}}{12})N\sqrt{2N}x+97N^2<0 \\ 1<x\le\sqrt{N} \end{cases} \tag{5.3}$$

and

$$\begin{cases} 97y^4 - (240 - \frac{35\sqrt{5}}{12})\sqrt{2N}y^3 + 392Ny^2 - (120 + \frac{35\sqrt{5}}{6})N\sqrt{2N}y + 28N^2 < 0 \\ \text{y} \geq \sqrt{N} \end{cases} \tag{5.4}$$

The inequalities (5.3) and (5.4) are called discriminant inequalities of $1 < \alpha < 2$ for semiprime $N = pq$, or simply discriminant. For the case $2 < \alpha < \frac{N}{\chi^2}$, the discriminant can also be defined.

(2) Theorem 2 and its derived process provide a way to construct a geometric object to detecting the range of the divisor-ratio α. For example, we can construct an α-detecting hyperbola by transforming the companion hyperbola H*, or the locus of the centroid.

(3) By Maple, χ is calculated by

$$\chi = -\frac{\sqrt[3]{60801876N\sqrt{2N} + 56796110N\sqrt{10N} + 1260N\sqrt{29718699546N + 12907950048N\sqrt{5}}}}{504} + \frac{504\left(\frac{1283N^2}{127008} + \frac{635N\sqrt{5}}{1512}\right)}{\sqrt[3]{60801876N\sqrt{2N} + 56796110N\sqrt{10N} + 1260N\sqrt{29718699546N + 12907950048N\sqrt{5}}}} + \frac{53\sqrt{2N}}{42} - \frac{5\sqrt{10N}}{72}$$

Example 1. Let $N = 1333$; then odd integer solution of (5.3) is $x \in [27, 35]$ and that for (5.4) is $y \in [35, 51]$. It is seen $x = 31$ and $y = 43$ holds $xy = N$. Consequently, $N = 1333 = 31 \times 43$ and $1 < \alpha < 2$ holds for *N*.

Example 2. Let $N = 4171$; the odd integer solution of (5.3) is $x \in [47, 63]$ and the odd integer solution of (5.4) is $y \in [65, 91]$. Since there is not a pair (x, y) such that $xy = N$, it is known that there is no odd integer solution fit for *N*, which means $\alpha > 2$. Actually, $N = 4171 = 43 \times 97$ and $\alpha > 2$.

6 Conclusions and Feature Work

By means of constructing the companion triangle on the hyperbola xy = N and through study of the loci of the triangle's centroid, incenter, circumcenter and ex-centers, we discover several new planner curves. Except for their fantastic shapes, the loci also can help us to solve certain problems. It is seen that this paper is just a very rough report of the study. There are something worthy of further investigating. For example, the intersections between two curves in Fig. 21 might be an amusing topic related with the divisor-ratio for which we had started the research work. In the future, we will work on finding their properties and solving more problem.

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