UNDER PEER REVIEW

Some Formulae For Integer Sums Of Two Squares

Original Research Article

Abstract

The study of integer sums of two squares is still an open area of research. Much of the recent work done has put more attention on Fermat Sums of two square theorem with little attention given to new formulas of sums of two Squares. This work is set to partially overcome this knowledge gap by introducing new formulas for generating integer sums of two squares.

Keywords: Diophantine Equation; Sums of Two Squares

1 Introduction

Finding the relationship between integers and sums of two squares has become an interesting topic in the recent years. The Theory of sums of two squares was first pioneered by Fermat in 1640, where an odd prime p is expressible as a sum of two squares if and only if $p \equiv 1 mod 4$. Euler succeeded in providing proof for Fermat's theorem on sums of two squares in 1749, The proof majorly relies on infinite descent, and was briefly sketched in a letter. The complete proof consists in five steps and was published in two papers. For reference see [3,4]. Since, Euler gave proof to this somewhat marvelous theorem, a number of researchers have provided alternative proof. For survey of this results reference can be made to [1,2,3,4]. Though a giant method the Fermat formula does not generate all integer sums of two squares since it is purely defined for an odd prime numbers which is congruent to 1 modulo p. Another limitation of Fermat Sums of two square theorem is that one has to determine p before spliting the number into a sum of two square number. This study is set to overcome this challenges by introducing new formulas for integer sums of two squares which has the ability to generate a wide range of integer sums of two squares if not all.

2 Some Formulae For Integer Sums Of Two Squares

In the sequel we provide some formulae for integer sums of two squares.

Proposition 2.1. *Let a and b be any non negative consecutive even or odd integers. Then the equation $ab + b - a = 1^2 + (\frac{a+b}{2})^2$ is a sum of two squares.*

Proof. Case (i) a and b even.
We need to prove that $ab+b-a = 1^2 + (\frac{a+b}{2})^2$. Let $a = 2k$ and $b = 2k+2$. Now, $ab+b-a = 2k(2k+2)+2k+2-2k = 4k^2+4k+2 = 4k^2+4k+2 = 1+(4k^2+4k+1)+1 = 1^2+(2k+1)^2 = 1^2+(\frac{a+b}{2})^2$.
Case (ii) a and b odd.
We need to prove that $ab + b - a = 1^2 + (\frac{a+b}{2})^2$. Let $a = 2k + 1$ and $b = 2k + 3$. Now, $ab + b - a = (2k + 1)(2k + 3) + 2k + 3 - (2k + 1) = (2k + 1)(2k + 3) + 2 = 4k^2 + 6k + 2k + 5 = 4k^2 + 8k + 5 = 1 + 4k^2 + 8k + 4 = 1^+(2k + 2)^2 = 1^2 + (\frac{a+b}{2})^2$.
□

Proposition 2.2. *Let a, b and c be any non negative consecutive even or odd integers. Then the equation $b(a + c) + 2 = (a + 1)^2 + (b + 1)^2$ is a sum of two squares.*

Proof. Case (i) a and b even.
We need to show that $b(a + c) + 2 = (a + 1)^2 + (b + 1)^2$. Let $a = 2k, b = 2k + 2$ and $c = 2k + 4$. Now, $b(a + c) + 2 = (2k + 2)(2k + 2k + 4) + 2 = (2k + 2)(4k + 4) + 2 = 2k(4k + 4) + 2(2k + 4) + 2 = 8k^2 + 16k + 10 = (2k + 1)^2 + (2k + 3)^= (a + 1)^2 + (b + 1)^2$.
Case (ii) a and b odd.
Let $a = 2k + 1, b = 2k + 3$ and $2k + 5$. Now, $b(a + c) + 2 = (2k + 3)(2k + 1 + 2k + 5) + 2 = (2k + 3)(4k + 6) + 2 = 8k^2 + 24k + 20 = (2k + 2)^2 + (2k + 4)^2 = (a + 1)^2 + (b + 1)^2$. □

Proposition 2.3. *Let a, b and c be any non negative consecutive even or odd integers. Then the equation $a(b + c) + 5 = (a + 1)^2 + b^2$ is a sum of two squares.*

Proof. Case (i) a and b even.
Need to prove that $a(b + c) + 5 = (a + 1)^2 + b^2$. Let $a = 2k, b = 2k + 2$ and $c = 2k + 4$. Now, $a(b+c)+5 = 2k(2k+2+2k+4)+5 = 2k(4k+6)+5 = 8k^2+12k+5 = (4k^2+4k+1)+(4k^2+8k+4) = (2k + 1)^2 + (2k + 2)^2 = (a + 1)^2 + b^2$.
Case (i) a and b odd.
Next, let $a = 2k + 1, b = 2k + 3$ and $c = 2k + 5$. We want to show that $a(b + c) + 5 = (a + 1)^2 + b^2$. Proceeding from L.H.S $a(b+c)+5 = (2k+1)(2k+3+2k+5)+5 = (2k+1)(4k+8)+5 = 2k(4k+8)+1(4k+8)+5 = 8k^2+20k+13 = (4k^2+8k+4)+(4k^2+12k+9) = (2k+2)^2+(2k+3)^2 = (a+1)^2+b^2$ proving the result. □

Proposition 2.4. *Let a, b and c be any non negative consecutive even or odd integers. Then the equation $a(b + c) + 17 = (a - 1)^2 + c^2$ is a sum of two squares.*

Proof. Case (i) a and b even.
We wish to prove that $a(b + c) + 17 = (a - 1)^2 + c^2$. Let $a = 2k, b = 2k + 2$ and $c = 2k + 4$. Proving from the L.H.S $a(b + c) + 17 = 2k(2k + 2 + 2k + 4) + 17 = 2k(4k + 6) + 17 = 8k^2 + 12k + 17 = (4k^2 - 4k + 1) + (4k^2 + 16k + 16) = (2k - 1)^2 + (2k + 4)^2 = (a - 1)^2 + c^2$.

Case (ii) a and b odd.
Next, let $a = 2k+1, b = 2k+3$ and $c = 2k+5$. We want to show that $a(b+c)+17 = (a-1)^2+c^2$. Proving from the L.H.S $a(b+c)+17 = (2k+1)(2k+3+2k+5)+17 = (2k+1)(4k+8)+17 = 2k(4k+8)+1(4k+8)+17 = 8k^2+20k+25 = (4k^2)+(4k^2+20k+25) = (2k)^2+(2k+5)^2 = (a-1)^2+c^2$ establishing the proof. □

Proposition 2.5. *Let a, b and c be any non negative consecutive even or odd integers. Then the equation $a(b+c)+65 = (a-4)^2+(c+3)^2$ is a sum of two squares.*

Proof. Case (i) a and b even.
We wish to prove that $a(b+c)+65 = (a-4)^2+(c+3)^2$. Let $a = 2k, b = 2k+2$ and $c = 2k+4$. Proceeding from L.H.S$a(b+c)+65 = 2k(2k+2+2k+4)+65 = 2k(2k+2+2k+4)+65 = 8k^2+12k+65 = (4k^2-16k+16)+(4k^2+28k+49) = (2k-4)^2+(2k+7)^2 = (a-4)^2+(c+3)^2$.
Case (ii) a and b odd.
Next, let $a = 2k+1, b = 2k+3$ and $c = 2k+5$. We want to show that $a(b+c)+65 = (2k+1)(2k+1+2k+5)+65 = (2k+1)(4k+6)+65 = (2k+1)(4k+6)+65 = 2k(4k+6)+1(4k+6)+65 = 8k^2+12k+65 = (4k^2-16k+16)+(4k^2+28k+49) = (2k-4)^2+(2k+7)^2 = (a-4)^2+(c+3)^2$ proving the result.

□

Proposition 2.6. *Let a, b and c be any non negative even or odd integers such that $b-a = c-b = 4$. Then the equation $b(a+c)+8 = (b-2)^2+(c-2)^2$ is a sum of two squares.*

Proof. Case (i) a and b even.
We wish to prove that $b(a+c)+8 = (b-2)^2+(c-2)^2$. Let $a = 2k, b = 2k+4$ and $c = 2k+8$. Proceeding from L.H.S $b(a+c)+8 = (2k+4)(2k+2k+8)+8 = (2k+4)(4k+8)+8 = 2k(4k+8)+4(4k+8)+8 = 8k^2+16k+16k+32+8 = 8k^2+32k+40 = (4k^2+8k+4)+(4k^2+32k+36) = (2k+2)^2+(2k+6)^2 = (b-2)^2+(c-2)^2$ producing the results.
Case (ii) a and b odd.
Next, let $a = 2k+1, b = 2k+5$ and $c = 2k+9$. We want to show that $b(a+c)+8 = (2k+5)(2k+1+2k+9)+8 = (2k+5)(4k+10)+8 = 2k(4k+10)+5(4k+10)+8 = 8k^2+20k+20k+50+8 = 8k^2+40k+58 = (4k^2+12k+9)+(4k^2+28k+49) = (2k+3)^2+(2k+7)^2 = (b-2)^2+(c-2)^2$ giving the result.

□

Conclusion

This study has introduced some new formulas for integer sums of two squares. Up to now, much of the research done in this area is very scanty and we encourage other researchers to give more attention to this particular area of research. For instance there is very little information on the general formula for generating integer sums of two squares since not every multiple of sums of two squares with any number is a sum of two squares.

References

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