Characterization of Compact Operators

Abstract

The concept of a compact operator on a Hilbert space, H is an extension of the concept of a matrix acting on a finite dimensional vector space. In Hilbert space, compact operators are precisely the closure of finite rank operators in the topology induced by the operator norm. In this paper we provide an elementary exposition of compact linear operators in pre-Hilbert and Hilbert spaces. However, whenever advantageous, we may prove a few results in the general context of normed linear spaces. It is well known that strong convergence implies weak convergence but weak convergence does not imply strong convergence. We also show that an operator $T \in B(H)$ is compact if and only if T maps every weakly convergent sequence in H to a strongly convergent sequence.

Keywords: Compact Operator, Normed linear spaces, Strong convergence, Weak convergence.

Mathematics Subject Classification:

1 Introduction

The notion of a compact or a completely continuous linear operator was motivated by the study of integral equations and its systematic theory emerged from the discussions of linear integral equations of the form

$$(T - \lambda I)x(s) = y(s) \text{ where } x(s) = \int_a^b K(s,t)x(t)dt \tag{1}$$

where $\lambda \in \mathbb{K}$ is a parameter which takes nonzero values and the kernel K and y are given functions subject to certain conditions [9]. It was discovered by D.Hilbert in 1912 that certain essential results about the solvability of (1) ("Fredholm's theory") do not depend upon the existence of the integral representation of T in (1) but only on the assumption that T is a compact linear operator [2]. In 1918, F. Riesz put Fredholm's theory in an abstract form. The theory of compact linear operators served as a model for earlier work in Functional analysis [5]. The property of such operators closely resemble those of linear operators in finite-dimensional normed linear spaces [11]. In this paper we provide an elementary exposition of compact linear operators in Pre-Hilbert and Hilbert spaces. However, whenever advantageous, we may prove a few results in the general context of normed linear spaces. Most definitions in this paper can be found in [1], [4], [6], [9], [10], and [12].

2 Definitions and Consequences

Definition 1. Let (X, τ) be a topological space. A subset A of X is said to be **relatively compact** if its closure $\bar{A}$ is compact.

Definition 2. Let (X, τ) be a topological space and A be a subset of X. We say that A is **sequentially compact** if every sequence (x_n) of elements of A has a convergent subsequence with limit in A.

Definition 3. Let (X, τ) be a topological space. A subset A of X is said to satisfy the **Bolzano-Weierstrasss property** (or equivalently, is said to be countably compact) if every infinite subset of A has a limit point in A.

In Metric spaces the following characteristics are equivalent for any subset A of a metric space (X, ρ)

(i) A is compact

(ii) A is sequentially compact

(iii) A is countably compact

(iv) A is precompact and complete (i.e. the metric subspace (X, ρ_A) is complete.

Definition 4. Let (X, ρ) be a metric space. Let $\varepsilon > 0$ be given. A Set E is called an $\varepsilon - net$ with respect to X if for every $x \in X$, there is a $y \in E$ such that $\rho(x, y) \leq \varepsilon$. (If $A \subset X$, we can have an ε-net defined with respect to A to be any set $E \subseteq X$ such that for each $x \in A$ there is a $y \in E$ such that $\rho(x, y) \leqslant \varepsilon$)

Definition 5. Let (X, ρ) be a metric space. A Subset A of X is said to be **precompact** (or **totally bounded**) if for every $\varepsilon > 0$, A has a finite $\varepsilon - net$

Lemma 1. *A is pre-compact implies A is bounded but the converse is not true [3].*

Proposition 1. *Let (X, ρ) be a metric space. A subset A of X is relatively compact if and only if every sequence (x_n) of points of A has a convergent subsequence. (Note that it is not asserted that the limit of the convergent subsequence is in A).*

Proof. Let A be relatively compact, i.e. $\bar{A}$ is compact. Hence $\bar{A}$ is sequentially compact (equivalent to compactness in metric space). Let (x_n) be any sequence of points of A. Since $A \subseteq \bar{A}$, so (x_n) is also a sequence of points of $\bar{A}$. Since $\bar{A}$ is sequentially compact, (x_n) has a subsequence (x_{n_k}) converging to a limit in $\bar{A}$ (so this limit need not be in A).

Conversely, let A have the property that every sequence of elements of A has a convergent subsequence. We must show that A is relatively compact i.e. $\bar{A}$ is compact, i.e. $\bar{A}$ is sequentially compact (which is equivalent to compactness in metric spaces).

Let (x_n) be any sequence of points of A. Since A is dense in $\bar{A}$, for each (x_n), we can find a $y_n \in A$ such that $\rho(x_n, y_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$. Since (y_n) is a sequence of points of A, it follows from the hypothesis, there is a subsequence (y_{n_k}) of (y_n) such that (y_{n_k}) converges to say y. Clearly, $\rho(x_{n_k}, y_{n_k}) < \frac{1}{n_k}$.
Now

$$\rho(x_{n_k}, y) \leq \rho(x_{n_k}, y_{n_k}) + \rho(y_{n_k}, y) < \frac{1}{n_k} + \rho(y_{n_k}, y)$$

as $k \to \infty$, $\frac{1}{n_k} \to 0$ and $y_{n_k} \xrightarrow{\rho} y$ which implies $x_{n_k} \xrightarrow{\rho} y$.

This limit $y \in X$. Clearly, $y \in \bar{A}$, since $\bar{A}$ is closed. Thus every sequence of points of $\bar{A}$ has a subsequence converging to a limit in $\bar{A}$. Therefore, $\bar{A}$ is sequentially compact, i.e. compact. $\square$

Definition 6. Let X, Y be normed linear spaces over $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C}$). A linear transformation $T : X \to Y$ is said to be **compact** (equivalently **completely continous**) if for every bounded subset M of $X, T(M)$ is relatively compact, i.e. $\overline{T(M)}$ is compact.

Proposition 2. *Let X, Y be normed linear spaces over $\mathbb{K}$. A linear transformation $T : X \to Y$ is compact if and only if for every bounded sequence (x_n) of elements of X there is a subsequence (x_{n_k}) such that (Tx_{n_k}) converges (with limit in Y).*

Proof. Let T be compact. Let (x_n) be any bounded sequence of elements of X. Let $E = \overline{\{Tx_n : n \in \mathbb{N}\}}$. Since T is compact, E is compact. {Indeed, since (x_n) is bounded, the set $\{x_n : n \in \mathbb{N}\}$ is bounded so by compactness of T, the set $\overline{T(\{x_n : n \in \mathbb{N})}$ is compact i.e. E is compact }. Hence E is sequentially compact. (Tx_n) is a sequence of elements of E and E is sequentially compact. Hence, there is a subsequence (TX_{n_k}) of (TX_n) such that (Tx_{n_k}) converges to some limit in Y.

Conversely, let for each bounded sequence (X_n) of elements of (x_n) there be a subsequence (x_{n_k}) such that (Tx_{n_k}) converges.
Let M be any bounded subset of X. We must show that $\overline{T(M)}$ is compact. Let (x_n) be any sequence of elements from M, since M is bounded, so (x_n) is bounded. By hypothesis, there exists a subsequence (x_{n_k}) such that (Tx_{n_k}) converges to a limit in Y. Since $Tx_{n_k} \in T(M)$ and Tx_{n_k} is convergent this limit belongs to $\overline{T(M)}$ (and need not be in $T(M)$).

Thus we have shown: For any bounded set M of X, any sequence (x_n) of elements from M has a subsequence (x_{n_k}) such that (Tx_{n_k}) converges. Thus any sequence (y_n) of elements from $T(M)$ has a convergent subsequence. Thus $T(M)$ is relatively compact(by proposition 1) i.e. $\overline{T(M)}$ is compact. Thus for every bounded subset M of X, $\overline{T(M)}$ is compact i.e. T is compact. □

Proposition 3. *Let X, Y be normed linear spaces and $T : X \to Y$ be compact. Then $T \in B(X, Y)$*

Proof. Let M be any bounded subset of X. Since T is compact, so $\overline{T(M)}$ is compact. Compactness implies boundedness (in metric spaces). So $\overline{T(M)}$ is bounded (in Y) and hence $T(M)$ is bounded. We have proved: For every bounded subset M of X, $T(M)$ is also bounded i.e. T is bounded. □

Corollary 1. Let X, Y be normed linear spaces and $T : X \to Y$ be a linear operator. Let $N = N(\bar{0}; 1) \subseteq X$. Then T is compact if and only if $\overline{T(N)}$ is compact.

Proof. Let T be compact. Since N is a bounded subset of X and T is compact, so $\overline{T(N)}$ is compact. Conversely, let $\overline{T(N)}$ be compact. To show that T is compact; Let M be any bounded subset of X. Then there exists a real $k > 0$ such that $\|x\| < k \quad \forall x \in M$. Hence $\left\|\frac{1}{k}x\right\| = \frac{1}{k}\|x\| < 1$ i.e. $\frac{1}{k}x \in N$. We define

$$\frac{1}{k}M = \left\{\frac{1}{k}x : x \in M\right\}$$

Therefore,

$$\frac{1}{k}M \subset N$$

Thus,

$$T\left(\frac{1}{k}M\right) = \frac{1}{K}T(M) \subseteq T(N)$$

and

$$\overline{T(M)} \subseteq \overline{kT(N)} = k\overline{T(N)}(\text{ Trivial steps })$$

Since $\overline{T(N)}$ is compact. So $k\overline{T(N)}$ is also compact. So $\overline{T(M)}$ is a closed subset of a compact set $k\overline{T(N)}$ and hence $\overline{T(M)}$ is compact. □

The last result is seen thus:
Let F be compact E be closed and $E \subseteq F$. F is compact implies F is bounded.
Let $\{G_\alpha : \alpha \in \Lambda\}$ be an open cover for E. Since E is closed, $E^c(= X - E)$ is open. Adjoin E^c to $\{G_\alpha : \alpha \in \Lambda\}$ and we get an open cover for X and hence an open cover for F for $(F \subset X)$.

But F is compact and hence there is a finite subcover (from this cover $\{E^c, G_\alpha = \alpha \in \Lambda\}$ for F). This finite subcover for F is also a finite subcover for E. If this finite subcover contains E^c, delete E^c from it and thus we get a finite subcover from $\{G_\alpha : \alpha \in \Lambda\}$ for E. Thus every open cover for E has a finite subcover. Hence E is compact.

Lemma 2. {*Riesz Lemma*}
Let X be a normed linear space and M be a proper closed linear subspace of X. Let $0 < a < 1$. Then there exists an $x_a \in X$ such that $\|x_a\| = 1$ and dist $(x_a, M) > a$. (clearly $x_a \notin M$).

Proof. Since M is a proper subspace of X, there exists $x_1 \in X - M$. Clearly, $x_1 \neq 0$. Since M is closed, so

$$d = dist\,(x_1, M) > 0$$

{Note: $dist(x_1, M) = \inf \|x_1 - y\|\ y \in M$ }. Let $0 < a > 1$, a given so $\frac{d}{a} > d$. There exists an element $x_0 \in M$ such that $\|x_1 - x_0\| < \frac{d}{a}$ (Assume the contrary, then it would imply $\|x_1 - y\| \geq \frac{d}{a}$ $\forall y \in M$ in which case dist $(x_1, M) \geq \frac{d}{a} > d$, a contradiction!)

Also $x_1 - x_0 \neq \bar{0}$. So $\|x_1 - x_0\ | \neq 0$. Put $x_a = \frac{x_1 - x_0}{\|x_1 - x_0\|}$; so $\|x_a\| = 1$. Also $x_a \notin M$ (Note $x_a \in M \Rightarrow$ since $x_0 \in M$ so $x_a + \frac{1}{\|x_1 - x_0\|}x_0 \in M \Rightarrow \frac{x_1}{\|x_1 - x_0\|} \in M \Rightarrow x_1 \in M$ a contradiction!. Thus $x_a \in M$. Therefore,

$$\text{dist }(x_a, M) = \inf_{y \in M} \|x_a - y\| = \inf_{y \in M} \|\frac{x_1 - x_0}{\|x_1 - x_0\|} - y\|$$
$$= \frac{1}{\|x_1 - x_0\|} \inf \|x_1 - x_0 - \|\, x_1 \overline{\oplus} x_0 \|y\| = \frac{1}{\|x_1 - x_0\|} inf \|x_1 - \{x_0 + \|x_1 - x_0\|\, y\}\,\|$$
$$= \frac{1}{\|x_1 - x_0\|} d > \frac{d}{d/a} = a$$

Thus dist $(x_a, M) > a$. □

There are bounded operators which are not compact.

Proposition 4. *Let X be a normed linear space of infinite Hamel dimension. The identity operator I: $X \to X$ is not Compact.*

Proof. Let X be a normed linear space of infinite Hamel dimension. Pick any countable subset $\{x_n : n \in \mathbb{N}\}$ from an infinite Hamel basis (without loss of generality, we may assume that $\|x_n\| = 1$). Let

$$M_n = [\{x_1, \ldots x_n\}] \quad \text{for all} n \in \mathbb{N}$$

Since each M_n is finite- dimensional, M_n is closed for all $n \in \mathbb{N}$. Since $\{x_1, x_2, \cdots\}$ is linearly independent it follows that

$$M_1 \subset M_2 \subset \ldots \subset M_n \subset M_{n+1} \subset \ldots$$

(strict containment). Since M_1 is strictly contained in M_2, by Riesz's lemma there exists a $y_2 \in M_2$ such that $\|y_2\| = 1$, and dist $(y_2, M_1) > \frac{1}{2}$ Take $y_1 = x_1 \in M_1$, so

$$\|y_2 - y_1\| > \frac{1}{2}. \tag{2}$$

Since M_2 is a proper closed subspace of M_3, by Riesz's lemma there exists a $y_3 \in M_3$ such that $\|y_3\| = 1$ and dist $(y_3, M_2) > \frac{1}{2}$. Since $y_1, y_2 \in M_2$, so

$$\|y_1 - y_3\| > \frac{1}{2}, \|y_2 - y_3\| > \frac{1}{2} \tag{3}$$

Continuing in this manner, we have $y_i \in M_i, \quad i = 1, \dots n$ such that $\|y_i\| = 1$ and $\|y_i - y_j\| > \frac{1}{2} \quad \forall i \neq j, i = 1, \dots n, j = 1 \dots n$ {$M_3 \subset M_4$ properly therefore there exists $y_4 \in M_4$ such that $\|y_4\| = 1$ and dist $(y_4, M_3) > \frac{1}{2}$ but $y_1, y_2 \in M_3$ therefore

$$\|y_4 - y_1\| > \frac{1}{2}\|y_4 - y_2\| > \frac{1}{2}\|y_4 - y_3\| > \frac{1}{2}$$

In the next stage since $M_{n+1} \subset M_n$ properly we would get $y_{n+1} \in M_{n+1}$ such that $\|y_{n+1}\| = 1$. $\|y_{n+1} - y_1\| > \frac{1}{2}$ for all $i = 1, \dots, n$
and thus

$$\|y_i - y_j\| > \frac{1}{2} \quad \text{for all} i \neq j \quad i = 1, \cdots n+1, j = 1, \cdots n+1$$

and the induction is complete.

We thus get a sequence $(y_n)_{n=1}^{\infty}$ of unit vectors in X such that

$$\|y_i - y_j\| > \frac{1}{2} \quad \forall\ i \neq j$$

Since I is the identity operator,

$$Ix = x \quad \forall x \in X$$

So

$$Iy_i = y_i \quad \forall i \in \mathbb{N}.$$

Now $M = \{y_1, y_2, \dots, y_n \dots\}$ is a bounded subset of X and $I(M)$ (image of M under the linear operator I) is also M. The distance between any two distinct points in

$$I(M)\text{is greater than } \frac{1}{2}\left(\|Iy_i - Iy_j\| = \|y_i - y_j\| > \frac{1}{2}\forall i \neq j\right).$$

Hence any sequence of distinct elements of $I(M)$ cannot therefore have a convergent subsequence. Hence (by proposition 1) $I(M)$ is not relatively compact i.e. $\overline{I(M)}$ is not compact. Therefore, I is not compact (when Hamel dimension of X is infinite). □

Remark 1. In the Hilbert space situation, the proof is simpler. Let H be a Hilbert space of infinite orthogonal dimension. Let $\{e_n : n \in \mathbb{N}\}$ be a subset of an orthonormal basis of H. Clearly if $i \neq j \quad \|e_i - e_j\|^2 = \langle e_i - e_j, e_i - e_j \rangle$

$$= \|e_i\|^2 + \|e_j\|^2 \,\text{since}\, \langle e_i, e_j \rangle = 0 \quad \forall i \neq j = 1 + 1 = 2$$

Hence $\|e_i - e_j\| = \sqrt{2}$, whenever $i \neq j$. Let $M = \{e_n : n \in N\}$. Clearly M is bounded and as before $I(M) = M$ is not relatively compact.

Definition 7. Let X and Y be normed linear spaces over $\mathbb{K}$. The set of all compact operators on X into Y is represented by the symbol $B_\infty(X, Y)$ or $(K(X, Y))$. Obviously $B_\infty(X, Y) \subset B(X, Y)$ (proper). If $X = Y$, we use the symbol $B_\infty(X)$ or $K(X)$. Moreover

Proposition 5. *Let X and Y be normed linear spaces over $\mathbb{K}$. Then $K(X, Y)$ is a linear space over $\mathbb{K}$.*

Proof. Suppose $S, T \in K(X, Y)$, i.e. are compact, we shall show that $S + T$ is compact. Let (x_n) be any bounded sequence of elements from X. Since S is compact, by proposition 2 there is a subsequence (x_{n_k}) of (x_n) such that (Sx_{n_k}) converges to some element in Y say

$$Sx_{n_k} \xrightarrow{s} y \tag{4}$$

Since (x_n) is bounded so is (x_{n_k}). Since T is compact $(x_{n_k})_{k=1}^{\infty}$ will have a subsequence $\left(x_{n_{k_r}}\right)_{k=1}^{\infty}$ such that $\left(Tx_{n_{k_r}}\right)_{r=1}^{\infty}$ converges in Y. Since $SX_{n_k} \xrightarrow{s} y \quad$ (by (4)) and $\left(x_{n_{k_r}}\right)$ is a subsequence of (x_{n_k}). So $SX_{n_k} \xrightarrow{s} y$

(If a sequence is convergent, then every subsequence of the sequence is also convergent and the limit to which the subsequence converges is the same as the limit of the sequence). Thus $Tx_{n_{k_r}} \xrightarrow{s} a$ limit in Y. $Sx_{n_{k_r}} \xrightarrow{S} a$ limit in Y.
Consequently,

$$(S+T)x_{n_{k_r}} \xrightarrow{s} a \text{ limit in } Y$$

and $\left(x_{n_{k_r}}\right)$ is a subsequence of a bounded sequence (x_n). Therefore, $S+T$ is compact.
The proof that if $\lambda \in \mathbb{K}$ and T is compact, then λT is compact is similar. Indeed, if (x_n) is any bounded sequence in X, then, since T is compact, there is a subsequence (x_{n_k}) of (x_n) such that $Tx_{n_k} \xrightarrow{s}$ some limit $y \in Y$.
Then (λx_n) is also bounded and

$$(\lambda T)\left(x_{n_k}\right) = T\left(\lambda x_{n_k}\right) = \lambda T\left(x_{n_k}\right) \xrightarrow{s} \lambda y \in Y$$

so λT is also compact. Thus $K(X,Y)$ is a linear space over $\mathbb{K}$. □

Proposition 6. *Let X be a normed linear space over $\mathbb{K}$ and $S \in B(X)$ and $T \in K(X)$. Then ST, TS are also compact i.e. ST, $TS \in K(X)$. (of course $S,T \in K(X) \Rightarrow ST, TS \in K(X)$).*

Proof. Consider ST. Let (x_n) be any bounded sequence of elements of X. Since T is compact, there is a subsequence (x_{n_k}) of (x_n) such that (Tx_{n_k}) converges in X. Since S is bounded, it is continuous and hence $STx_{n_k} = S\left(Tx_{n_k}\right)$ also converges.
Thus every bounded sequence (x_n) has a subsequence (x_{n_k}) such that (STx_{nk}) converges in X. This shows that ST is compact.
Consider TS. If (x_n) is a bounded sequence of elements of X, then so is (Sx_n) for S is bounded there exists $M>0$ such that $\|x_n\| \le M \ \forall n \in \mathbb{N}$. Therefore, $\|Sx_n\| \le \|S\|\|x_n\| \le M\|S\| \ \forall n \in \mathbb{N}$. Since T is compact, there is a subsequence (x_{n_k}) of (x_n) such that (TSx_{n_k}) converges in X which shows that TS is compact. □

Remark 2. Thus the linear space $K(X)$ is a two-sided ideal in the algebra $B(X)$ of all bounded operators.

Example 3. *Let X be a normed linear space and $T_1,\dots,T_n$ be compact linear operators on X; I is the identity operator on X. Define T on X by $I-T=(I-T_1)(I-T_2)\dots(I-T_n)$. Show that T is compact.*

Solution

$$\begin{aligned}
T &= I-(I-T)\\
&= I-(I-T_1)(I-T_2)\dots(I-T_n).\\
&= I-[I-(T_1+T_2)+T_1T_2](I-T_3)\dots(I-T_n).\\
&= I-[I-(T_1+T_2+T_3)+(T_1T_2+T_1T_3+T_2T_3)-T_1T_2T_3](I-T_4)\dots(I-T_n)\\
&= I-\left[I-\left(\sum_{i=1}^{n} T_i\right)+\left(\sum_{i,j=1,\cdots n} T_iT_j\right)\overline{\oplus}\left(\sum_{i<j>k} T_iT_jT_k\right)+\dots+(-1)^n T_1\dots T_n\right]\\
&= \left(\sum_{i=1}^{n} T_i\right)\overline{\oplus}\left(\sum_{i,j=1,\cdots n} T_iT_j\right)+\left(\sum_{i<j>k} T_iT_jT_k\right)+\dots+(-1)^{n+1}T_1\dots T_n \qquad (5)
\end{aligned}$$

Since $T_i \in K(X)$ for $i=1,\cdots n$ and $K(X)$ is a linear space, so $\sum_{i=1}^{n} T_i \in K(X)$ By the proposition 6, ($K(X)$ is an ideal in $B(X)$) it follows that all the rest of the terms in the right hand side of (5) represents compact operator and once again, using the fact that $K(x)$ is a linear space the expression on the right side of (5) represent a compact operator. Thus $T \in K(X)$.

Proposition 7. *Let X be a normed linear space of infinite Hamel dimensional and $T \in B(X)$. If T is invertible then T cannot be compact.*

Proof. Since T is invertible, T^{-1} exists in $B(X)$ and $T^{-1}T = I$. If T were compact, then since $T^{-1} \in B(X)$, so by proposition 6, $T^{-1}T$ must be compact, i.e. I must be compact, which cannot be so since X is of infinite Hamel dimension. Hence T cannot be compact. □

Remark 3. i) Every linear subspace of finite Hamel dimension in a normed linear space is closed

ii) Every bounded sequence in a normed linear space of finite dimension has a subsequence which converges to a limit in the normed linear space.

Proposition 8. *Let X be a normed linear space over $\mathbb{K}$. Then X is of finite dimension if and only if every closed and bounded subset of X is compact.*

Proof. Let X be of finite dimension. If M is a finite set, then clearly every M is closed (finite subsets of metric spaces are always closed). Also it is clear that every open cover for M has a finite subcover.

Suppose M is infinite. Let (x_n) be any sequence of points of M. Since M is bounded so is (x_n). By Remark 3(ii), (x_n) has a subsequence (x_{n_k}) converging to a limit say $x \in X$. Clearly, $x \in \overline{M}$ (for $(x_n \in M \quad$ for all $n \in N)$. But M is closed. So $x \in M$.

Thus every sequence (x_n) of points of M has a subsequence which converges to a limit in M. Thus M is sequentially compact implies M is compact. Thus if X is of finite dimension, M is closed and bounded implies M is compact.

Conversely, let every closed and bounded subset of a normed linear space X be compact. We must show that X is of finite Hamel dimension. Hence the unit sphere $S(x) = \{x \in X : \|x\| = 1\}$ must be compact (it is closed, bounded). Clearly, if we represent by $N\left(x; \frac{1}{2}\right)$ the set $\left\{y \in X : \|y - x\| < \frac{1}{2} : \|x\| = 1\right\}$. Then the family $\left\{N\left(x; \frac{1}{2}\right) : x \in S(x)\right\}$ is an open cover for $S(X)$. By compactness of $S(X)$, there exists a finite subset $\{x_1, \ldots, x_n\} \subset S(X)$ such that

$$\bigcup_{i=1}^{n} N\left(x_i; \frac{1}{2}\right) \supset S(X)$$

.

Let $M = [\{x_1, \cdots, x_n\}]$. Either $M = X$ or M is a proper linear subspace of X. Suppose the later, that is, M is a proper linear subspace of X. Clearly, M is of finite dimension and hence M is closed. Thus M is a proper closed linear subspace of the normed linear space X. Hence, by Riesz lemma, we can find a $y \in X$ such that $\|y\| = 1$ and dist $(y, M) > \frac{2}{3}$. Since $\|y\| = 1$, so $y \in S(X)$. Hence

$$\|y - x_i\| > \frac{2}{3} \quad \forall i = 1, \ldots n$$

since $M = [(x_1, \cdots, x_n)]$ i.e. $y \notin N\left(x_i, \frac{1}{2}\right) \quad \forall i = 1, 2, \cdots, n$ and $y \in S(X)$. This gives a contradiction!! since y is not covered by the finite subcover $\left\{N\left(x_i; \frac{1}{2}\right) : i = 1, \ldots, n\right\}$.
Hence the supposition that M is a proper linear subspace of X is incorrect and hence unacceptable. Therefore,
$M = X$ i.e. $X = [\{x_1, \ldots, x_n\}]$ i.e. X is of finite dimension. □

Corollary 2. Let X be a normed linear space. Then X is of finite dimension if and only if $S(X) = \{x \in X : \|x\| = 1$ or $\bar{N}(x) = \{x \in X : \|x\| \leq 1\}$ is compact.

Proposition 9. *Let X, Y be normed linear space over $\mathbb{K}$ and $T : X \to Y$ be a linear operator.*

(i) If $T \in B(X, Y)$ and $\dim \mathcal{R}_T < +\infty$ (i.e. T is of finite rank) then T is compact.

(ii) If X is of finite Hamel dimension, then T is compact.

Proof. (i) Let M be any bounded subset of X. We must show that $\overline{T(M)}$ is compact. Since T is bounded, and M is a bounded subset, so $T(M)$) is bounded. Hence $\overline{T(M)}$ is bounded. Since T is linear, $\mathcal{R}_T$ is a linear subspace of H. Since $\dim \mathcal{R}_T < +\infty$, so $\mathcal{R}_T$ is closed and $T(M) \subseteq \mathcal{R}_T$. So $\overline{T(M)} \subseteq \overline{\mathcal{R}}_T = \mathcal{R}_T$.

We thus see that $\overline{T(M)}$) is a closed bounded subset of a finite dimensional normed linear space $\mathcal{R}_T$. So $\overline{T(M)}$ is compact (by proposition 8). Hence T is compact.

(ii) If $\dim X < \infty$. Then $\dim T(X) = \dim \mathcal{R}_T < \infty$. Since T is a linear operator on a finite dimensional normed linear space, so T is bounded. Hence by (i) T is compact. □

Proposition 10. *Let X be a normed linear space and Y be a Banach space and (T_n) be a sequence of compact operators on X into Y such that $T_n \to T$ uniformly in $B(X,Y)$, i.e. $\|T_n - T\| \to 0$ as $n \to \infty$ (in the norm of $B(X,Y)$. Then T is compact.*

Proof. Since T_1 is compact, if (x_n) is any bounded sequence of elements of X_1 say $\|x_n\| \leq c\ \forall$ $n \in \mathbb{N}$, then there exists a subsequence $\left(x_n^{(1)}\right)$ of (x_n) such that $\left(T_1(x_n^{(1)})\right)$ converges strongly in Y. Since T_2 is compact , there exists a subsequence $\left(x_n^{(2)}\right)$ of $\left(x_n^{(1)}\right)$ such that $\left(T_2(x_n^{(2)})\right)$ converges strongly in Y. Continuing in this manner we get subsequences $\left(x_n^{(r)}\right)$ of (x_n) such that for any $r, s \in \mathbb{N}, \quad r \leq s$, $\left(x_n^{(r)}\right)$ is a subsequence of $\left(x_n^{(r)}\right)$. We can list these subsequences as below in order

$$(x_n^{(1)}) \subset (x_n^{(2)}) \subset (x_n^{(3)}) \subset \cdots \subset (x_n^{(r)}) \subset \ldots$$

General picture:

$$\begin{array}{cccc} x_1^{(1)} & x_2^{(1)} & x_3^{(1)} & x_4^{(1)} \cdots \\ x_1^{(2)} & x_2^{(2)} & x_3^{(2)} & x_4^{(2)} \cdots \\ x_1^{(3)} & x_2^{(3)} & x_3^{(3)} & x_4^{(3)} \cdots \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Consider the sequence consisting of the principal diagonal in the above array, i.e. the sequence $\left(x_n^{(n)}\right)_{n=1}^{\infty}$. This sequence is bounded since (x_n) is bounded. Since $T_n \to T$ uniformly, for each $\varepsilon > 0$ we can find an $n_\varepsilon \in \mathbb{N}$ such that

$$\|T_n - T\| < \frac{\varepsilon}{3c} \quad \forall n \geq n_\varepsilon.$$

It is clear that each one of the T_n's converges strongly on the diagonal subsequence $\left(x_r^{(r)}\right)$. In particular, Tn_ε converges strongly on $\left(x_r^{(r)}\right)$ i.e. there exists $n_0 \in \mathbb{N}$ such that

$$\left\|T_{n_r} x_s^{(s)} - T_{n_\varepsilon} x_r^{(r)}\right\| < \frac{\varepsilon}{3}, \forall r, s \geq n_0$$

Now if $r, s \geq n_0$

$$\begin{aligned} \left\|Tx_s^{(s)} - T{x_r}^{(r)}\right\| &\leqslant \left\|Tx_s^{(s)} - T_{n_\varepsilon} x_s^{(s)}\right\| + \left\|T_{n_\varepsilon} x_s^{(s)} - T_{n_\varepsilon} x_r^{(r)}\right\| + \left\|T_{n_\varepsilon} x_r^{(r)} - T x_r^{(r)}\right\| \\ &< \|T - T_{n_\varepsilon}\| \left\|x_s^{(s)}\right\| + \frac{\varepsilon}{3} + \|T_{n_\varepsilon} - T\| \, \|{x_r}^{(r)}\| \\ &< \frac{\varepsilon}{3c} c + \frac{\varepsilon}{3} + \frac{\varepsilon}{3c} c = \varepsilon \end{aligned}$$

Thus $\left(T(x_n^{(n)})\right)$ is strongly Cauchy in Y and Y is a Banach space. Hence $\left(T(x_n^{(n)})\right)$ converges in Y. Thus for each bounded sequence (x_n) of elements of X, there is a subsequence $\left({x_n}^{(n)}\right)$ such that $\left(Tx_n^{(n)}\right)$ converges in Y. i.e. T is compact. □

Corollary 3. If X is a Banach space and (T_n) is a sequence of compact linear operators in X such that $T_n \to T$ uniformly, then T is compact

Remark 4. The result would not be valid if we replace uniform convergence by strong convergence i.e. T_n compact, $T_n \xrightarrow{s} T$ does not imply T is compact.

Counter Example:

Let $H = \ell^2(\mathbb{N})$ and for each $n \in \mathbb{N}$ and $x = (x_1, x_2, \ldots) \in \ell^2(\mathbb{N})$ define T_n by

$$T_n x = (x_1, x_2, \cdots, x_n, 0, 0, \ldots)$$

i.e. $x_i = 0 \quad \forall i > n$. Each T_n has $\mathcal{R}T_n$ of finite dimension n. Also, for all $x = (x_n) \in \ell^2(\mathbb{N})$.

$$\|Tx_n\|^2 = \sum_{i=1}^{n} \|x_i\|^2 \leqslant \sum_{i=1}^{\infty} \|x_i\|^2 = \|x\|^2.$$

So each T_n is bounded ($\|T_n\| < 1$). Therefore, each T_n is compact by proposition 9(i). Also, for each $x = (x_n) \in \ell^2(\mathbb{N})$, we have

$$\begin{aligned} \|T_n x - Ix\| &= \|\,(x_1, x_2, \cdots x_n, 0, 0, \ldots) - (x_1 x_2, \ldots x_n, x_{n+1}, \cdots)\,\| \\ &= \|\,(0, 0, \ldots 0, x_{n+1}, x_{n+2}, \cdots)\,\|. \\ &= \sum_{i=n+1}^{\infty} |x_i|^2 \to 0 \text{ as } n \to \infty \end{aligned}$$

for each $\sum_{i=n+1}^{\infty} |x_i|^2$ is the "tail" of the convergent series $\sum_{i=1}^{\infty} |x_i|^2 \cdot$ Thus $T_n \xrightarrow{s} I$. But since $\ell^2(\mathbb{N})$ is infinite-dimensional, the identity operator I is not compact. Thus the strong limit of (T_n) is not compact.

Example 4. *Consider $H = \ell^2(\mathbb{N})$ and let $T : H \to H$ be defined by $Te_n = \frac{1}{n} e_n \ \forall\ n \in \mathbb{N}$, given that (e_n) is a complete orthonormal set in H. The matrix of T is the diagonal matrix*

$$\begin{bmatrix} 1 & & & 0 & \\ & \frac{1}{2} & & & \\ & & \frac{1}{3} & & \\ 0 & & & \frac{1}{4} & \\ & & & & \ddots \end{bmatrix} \tag{6}$$

$(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ are in $P\sigma(T))$. Show that T is compact.

Solution

If $x = (x_n)_{n=1}^{\infty} \in \ell^2$, then

$$Tx = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4}, \ldots\right)$$

{For

$$\begin{aligned} Tx = T\left(\sum_{n=1}^{\infty} x_n e_n\right) &= \sum_{n=1}^{\infty} x_n T e_n = \sum_{n=1}^{\infty} x_n \frac{e_n}{n} = \sum_{n=1}^{\infty} \frac{x_n}{n} e_n = \\ &= \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \ldots\right) \\ &\text{Note} \sum_{i=1}^{\infty} |x_i|^2 < +\infty \Rightarrow \sum_{i=1}^{\infty} \frac{1}{i} \|x_i|^2 < \infty. \\ &\left(x_1, \frac{x_2}{2}, \cdots\right) \in \ell^2(\mathbb{N})\} \end{aligned}$$

Define for each $n \in \mathbb{N}$; operators T_n by $T_n x = \left(x_1, \frac{x_2}{2,}, \ldots, \frac{x_n}{n}, 0, 0,\right)$, for $x = (x_k) \in \ell^2(\mathbb{N})$. Each $T_n \in B\left(\ell^2(\mathbb{N})\right)$ and has finite dimensional range. Therefore, each T_n is compact.
Now $\|T_n - T\| = \sup\left\{\|(T_n - T)\,x\| : x \in \ell^2(\mathbb{N}) \text{ and } \|x\| = 1\right\}$ and

$$= \left\{\sup\left\{\|(x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, 0, 0, \ldots\right) - \left(x_1, \frac{x_2}{2}, \ldots, \frac{x_n}{n}, \frac{x_{n+1}}{n+1}\right)\right\}$$

$$= \sup \left\{ \left\| \left(0, 0, \cdots, \frac{x_{n+1}}{n+1}, \frac{x_{n+2}}{n+2}, \dots \right) \right\| : \|x\| = 1 \right\}$$

$$= \sup \left\{ \sum_{i=n+1}^{\infty} \frac{1}{i^2} |xi|^2 \right.$$

, where $\sum_{i=1}^{\infty} |x_i|^2 = 1\} \leq \sup \left\{ \frac{1}{(n+1)^2} \sum_{i=n+1}^{\infty} |x_i|^2 : \sum_{i=1}^{\infty} |x_i|^2 = 1 \right\}$

$$\leqslant \frac{1}{(n+1)^2} \sum_{i=1}^{\infty} |x_i|^2 = \frac{1}{(n+1)^2}.$$

If we let $n \to +\infty$, we get, $\|T_n - T\| \to 0$. i.e. $T_n \to T$ uniformly. Each T_n is compact. Therefore, T is compact (by proposition 10).

Proposition 11. *Let X be a Banach space and $\mathcal{D}_T$ be a linear subspace of X. Suppose $T = \mathcal{D}_T \to X$ is compact. Then its closure $\overline{T}$ is compact. (If T is closable and compact, then its closure $\overline{T}$ is compact).*

Proof. We can extend T by continuity to all of $\overline{\mathcal{D}}_T$ (If $x \in \overline{\mathcal{D}}_T$, (x_n) is any sequence of points of $\mathcal{D}$ and such that $x_n \xrightarrow{s} x$, then we define the extension $\overline{T}$ of T by $\overline{T}_x =$ strong limit Tx_n. Moreover, $\|\overline{T}\| = \|\mathrm{T}\|$. So $\overline{\mathcal{D}}_{\overline{T}} = \mathcal{D}_T$. Moreover, $\overline{T}$ is closed). Let (x_n) be any bounded sequence in $\overline{\mathcal{D}_T} = \overline{\mathcal{D}}_T$. Hence for each $n \in \mathbb{N}$, we can find a $y_n \in \mathcal{D}_T$ such that $\|y_n - x_n\| \leqslant \frac{1}{n}$. Clearly, (y_n) is bounded; since (x_n) is bounded. Indeed,

$$\begin{aligned} \|y_n\| &= \|(y_n - x_n) + x_n\| \\ &\leqslant \|y_n - x_n\| + \|x_n\| \\ &\leqslant \frac{1}{n} + \|x_n\| \end{aligned}$$

Therefore (y_n) is bounded. Since T is compact, and (y_n) is a bounded sequence in $\mathcal{D}_T$ so there exists a subsequence (y_{n_k}) of (y_n) such that (Ty_{n_k}) converges strongly in X to y, say.
Consider the corresponding subsequence (x_{nk}) of (x_0). Then

$$\begin{aligned} &\overline{T}x_{nk} - y = \overline{T}x_{n_k} - \overline{T}_{y_{nk}} + \overline{T}y_{nk} - y \\ &\text{Therefore } \left\|\overline{T}x_{n_k} - y\right\| \leq \|F\| \|x_{n_k} - y_{n_k}\| + \left\|\overline{T}y_{nk} - y\right\| \\ &\leqslant \|T\| \frac{1}{n_k.} + \|Ty_k - y\| \\ &\text{Therefore } \overline{T}x_{n_k} \xrightarrow{s} y \end{aligned}$$

This shows that for any bounded sequence $\{x_n\}$ there is a subsequence $\{x_{n_k}\}$ such that $\{\overline{T}x_{n_k}\}$ converges strongly in X. So $\overline{T}$ is compact. □

Definition 8. Let X and Y be normed linear spaces and $T : X \to Y$ a linear operator. We say that T is of **finite rank** m (where $m \in \mathbb{N}$) if the Hamel-dimension of the range $\mathcal{R}_T$ of T is m.

Proposition 12. *Let H and K be Hilbert spaces and $T : H \to K$ be a bounded linear operator. T is of finite rank m if and only if there exists linearly independent subsets $\{x_1, x_2, \dots, x_m\} \subset H$ and $\{y_1, y_2, \dots, y_m\} \subset K$ such that*

$$Tx = \sum_{j=1}^{m} \langle x_j, x_j \rangle \, y_j \quad \forall \ x \in H.$$

Then $T^* y = \sum_{j=1}^{m} \langle y_j, y_j \rangle \, x_j$ and $\|T\| \leqslant \sum_{j=1}^{m} \|x_j\| \, \|y_i\|$, T^* is of rank m if and only if T is of rank m. We can without loss of generality, take $\{x_1, x_2, \cdots, x_m\}$ or $\{y_1, y_2, \dots y_m\}$ to be an orthonormal system.

Proof. Take $\{y_1, \ldots, y_m\}$ to be an orthonormal system in $\mathcal{R}_T$. Then for any $x \in H$, we have

$$\begin{aligned} Tx &= \sum_{j=1}^{m} \langle Tx, y_j \rangle \, y_j \\ &= \sum_{j=1}^{m} \langle x, T^* y_j \rangle \, y_j \end{aligned}$$

Put $T^* y_j = x_j$ for each $j = 1, \ldots, m$ so

$$T_x = \sum_{j=1}^{m} \langle x, x_j \rangle \, y_j$$

We need to show that $\{x_1, \ldots, x_m\}$ is linearly independent. Assume the contrary then without loss of generality, we may suppose that x_1 is a linear combination of $x_2, \ldots, x_m$ say $x_2 = \sum_{j=2}^{m} a_j x_j$ for scalars $a_j \in \mathbb{C}$, therefore

$$\begin{aligned} Tx &= \sum_{j=2}^{m} \langle x, x_j \rangle \, y_j + \langle x, x_1 \rangle \, y_1 \\ &= \sum_{j=2}^{m} \langle x, x_j \rangle \, y_i + \left\langle x, \sum_{j=2}^{m} a_j x_j \right\rangle y_1 \\ &= \sum_{j=2}^{m} \langle x, x_j \rangle \, y_j + \left(\sum_{j=2}^{m} a_j \, \langle x, x_j \rangle \right) y_1 \\ &= \sum_{j=2}^{m} \langle x, x_j \rangle \, (y_j + \bar{a}_j y_1) \\ &= \text{ linear combination of } (m-1) \text{ vectors } \; \{y_2 + \bar{a}_2 y_1, y_3 + \bar{a}_3 y_2, \ldots, y_m + \bar{a}_m y_2\} \end{aligned}$$

which implies $dim \ \mathcal{R}_T$ is at most $(m-1)$, a contradiction since

$\mathcal{R}_T = m$

Hence $\{x_1, \ldots, x_m\}$ cannot be linearly dependent. Therefore, $\{x_2, \ldots, x_M\}$ is linearly independent. It is clear that

$$\mathcal{R}_T \subseteq [\{y_1, \cdots y_m\}]$$

Since $\{x_1, \ldots x_m]$ is linearly independent;

$$\begin{aligned} &x_1 \neq \bar{0} \text{ and } x_1 \notin [\{x_2, \ldots, x_m\}] \\ &\text{so } x_1 \in [\{x_2, \ldots, x_m\}]^{\perp} \end{aligned}$$

Hence we can find an $z_1 \in H$ such that $z_1 \perp x_1$ and $z_1 \perp x_2, \ldots, x_m$.
This can be done in general, that is for any $k \in \{1, 2, \ldots, m\}$ we can find a $z_k \in H$ such that $\langle z_k, x_k \rangle \neq 0$ and $\langle z_k, x_j \rangle = 0 \quad \forall \ j \neq k$. Putting $x = z_k$, we get

$$Tz_k = \sum_{j=1}^{n} \langle z_k, x_j \rangle \, y_j = \langle z_k, x_k \rangle y_k.$$

which implies that $y_k \in \mathcal{R}_T$ and this is true for $k = 1, 2, \ldots, m$. Thus $\mathcal{R}_T = [\{y_1, \ldots, y_m\}]$.
Also from

$$Tx = \sum_{j=1}^{m} \langle x, x_j \rangle \, y_j$$

we get

$$\|T_x\| \leq \sum_{j=1}^{m} |\langle x, x_j \rangle| \|y_j\| \leqslant \|x\| \sum_{j=1}^{m} \|x_j\| \, \|y_j\|$$

which shows that

$$\|T\| \leqslant \sum_{j=1}^{m} \|x_j\| \, \|y_j\|$$

For all $y \in K$

$$\langle T^* y, x\rangle = \langle y, Tx\rangle = \left\langle y, \sum_{j=1}^{m} \langle x, x_j > y_j\rangle \right\rangle = \sum_{j=1}^{m} \left\langle \overline{\{x, x_j\}} \right\rangle \langle y, y_j\rangle$$

$$= \sum_{j=1}^{m} \langle y, y_j\rangle \langle x_j, x\rangle = \left\langle \sum_{j=1}^{m} \langle y, y_j\rangle x_j, x \right\rangle$$

As this is true for all $x \in H$, we get

$$T^* y = \sum_{j=1}^{m} \langle y, y_j\rangle x_j$$

and rank $T^* = m$.
Conversely, if rank $T^* = m$, we may assume that $x_j's$ are orthonormal and show that T^{**} is of rank m i.e. T is of rank m. □

Proposition 13. *Let (X, ρ) be a metric space. If a subset A of X is totally bounded, then A is separable.*

Proof. Since A is totally bounded, for each $\varepsilon > 0$, there exists a finite $\varepsilon - net$ N_ε for A. (i.e. for each $x \in A$ there exists$y \in N_\varepsilon$ such that $\|x - y\| < \varepsilon$. Let $\varepsilon = \frac{1}{k}$ where $k \in \mathbb{N}$ and let k run through $\mathbb{N}$. Let $N = \bigcup_{k \in \mathbb{N}} N_{\frac{1}{k}}$. Clearly, N is at most countable and dense in A. For if $x \in A$, and $\varepsilon > 0$ choose a $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. Then there exists $y \in N_{\frac{1}{k}}$ such that $\|y - x\| < \frac{1}{k} < \varepsilon$. So $N(x; \varepsilon)$ has a point $y \in N_{\frac{1}{k}} \subseteq N$. Thus N is an atmost countable set which is dense in A. Thus A is separable. □

We have seen earlier in proposition 1 that:
Let (X, ρ) be a metric space and $A \subseteq X$. Then A is relatively compact (i.e. $\bar{A}$ is compact) if and only if every sequence (x_n) of points of A has a convergent subsequence (It's limit need not be in A). We now show:

Proposition 14. *Let (X, ρ) be a metric space and A a subset of X which is relatively compact. Then A is totally bounded.*

Proof. Assume the contrary i.e. suppose A was not totally bounded. Then there exists an $\varepsilon > 0$ for which A does not have a finite ε in X. In particular, A does not have a finite ε-net in A. (i.e. consisting of points of A). Choose $x_1 \in A$. Then there exists a points $x_2 \in A$ such that $\rho(x_1, x_2) \geq \varepsilon$ for if no such point x existed in A then $\rho(x, y) < \varepsilon \ \forall y \in A$ which means that $\{x_1\}$ is an (finite) $\varepsilon - net$ for A, a contradiction!

There exists a point $x_3 \in A$ such that $\rho(x_3, x_1) \geq \varepsilon$ and $\rho(x_3, x_2) \geq \varepsilon$, for if no such point x_3 existed, it would mean that $\{x_1, x_2\}$ is a finite $\varepsilon - net$ for A in (A), a contradiction!

Proceeding in this manner, we get a sequence (x_n) of distinct points of A such that $\rho(x_i, x_j) \geq \varepsilon$ for all $i \neq j$. This sequence (x_n) of points of A clearly has no convergent subsequence. Hence A cannot be relatively compact. Thus A is not totally bounded implies A not relatively compact. A is relatively compact implies A is totally bounded. □

Proposition 15. *Let (X, ρ) be a metric space and A be relatively compact. Then A is separable.*

Proof. A is relatively compact implies A is totally bounded which implies A is separable. □

Proposition 16. *Let X and Y be normed linear spaces and $T : X \to Y$ be compact. Then the range $\mathcal{R}_T$ of T is separable.*

Proof. Denote an open neighborhood of $\bar{0}$ of radius n by $N(\bar{0}; n)$. Then

$$x = \bigcup_{n \in N} N(\bar{0}; n)$$

Clearly, $x \supseteq \bigcup_{n \in N} N(\bar{0}; n)$. On the other hand, if $x \in X$, then $\|x\|$ is a non-negative real. Hence $\exists n_0 \in \mathbb{N}$ such that $\|x\| < n_0$ i.e.

$$x \in N(\bar{0}; n_0) \subseteq \bigcup_{n \in \mathbb{N}} N(\bar{0}; n) \text{ i.e. } X \subseteq \bigcup_{n \in \mathbb{N}} (\bar{0}, n).$$

Now $\mathcal{R}_T = T(x) = T\left(\bigcup_{n \in \mathbb{N}} N(\bar{0}; n)\right) = \bigcup_{n \in \mathbb{N}} T(N(\bar{0}; n)$
(Note:The last step would not be true if we had an intersection of a family of set.) Since each $N(\bar{0} : n)$ is bounded and T is compact, so $T(N(\bar{0}; n))$ is relatively compact and hence separable. Thus $\mathcal{R}_T$ being a countable union of separable sets is separable.

□

3 Compact Operators in Hilbert Spaces

We now investigate the action of a bounded linear operator on weakly convergent sequence. We shall work with Hilbert spaces as our main interest lies in these spaces.

Proposition 17. *Let H be a Hilbert space and $T \in B(H)$. Let (x_n) be a sequence of points of H such that $x_n \xrightarrow{\omega} x$. Then $Tx_n \xrightarrow{\omega} Tx$*

Proof. Since $T \in B(H)$, $T^* \in B(H)$. Also $x_n \overset{w}{\to} x$ implies $lim_{n\to\infty} \langle x_n, T^* y\rangle = \langle x, T^* y\rangle \quad \forall\, y \in H$. i.e. $\lim_{n\to\infty} \langle Tx_n, y\rangle = \langle Tx, y\rangle \quad \forall y \in H$, which implies $Tx_n \xrightarrow{\omega} Tx$ □

Remark 5. The result goes through if $T \in B(H, K)($ then $T^* \in B(K, H))$. We also know that strong convergence implies weak convergence but weak convergence does not imply strong convergence.

Example 5. *Consider $\ell^2(\mathbb{N})$ and take (e_n) to be the orthonormal basis.*

$$e_n = (0, 0, \ldots, 0, 1, \ldots)(n \in \mathbb{N})$$

Now $\|e_n - e_m\|^2 = 2 \quad \forall\; n \neq m$. So $\rho(e_n, e_m) = \sqrt{2} \quad \forall\; m \neq n$. So (e_n) cannot be strongly convergent. For every $x \in \ell^2$ we have
$x = \sum_{n\in\mathbb{N}} \langle x, e_n\rangle e_n$ and $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2$ (Parseval's equality)
So $\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 < \infty$ i.e. $\sum_{n\in\mathbb{N}} |\langle x, e_n\rangle|^2$ is convergent in $\mathbb{R}$ and hence

$$\lim_{n\to\infty} |\langle x, e_n\rangle| = 0 \text{ i.e. } \lim_{n\to\infty} \langle x, e_n\rangle = 0.$$

Thus

$$\lim_{n\to\infty} \langle e_n, x\rangle = \langle \bar{0}, x\rangle \quad \forall x \in H. \text{ i.e. } e_n \xrightarrow{\omega} e$$

Hence in example 1, for $T \in B(H)$, $Tx_n \xrightarrow{\omega} Tx$ does not imply strong convergence i.e. (Tx_n) is not strongly convergent. In distinction to this, we get a surprising result:

Proposition 18. *Let H and K be Hilbert spaces and (x_n) be a weakly convergent sequence in H. $T \in B(H, K)$ is compact if and only if T maps every weakly convergent sequence in H to a strongly convergent sequence.*

Proof. Let $x_n \xrightarrow{\omega} x$. So (x_n) is bounded (earlier result). Consider the set $M = \{Tx_n : n \in \mathbb{N}\}$. This is bounded and also relatively compact. If (Tx_n) is not strongly convergent, then there exist at least two subsequences (x_n'), (x_n'') of (x_n) such that $(Tx_n'), (Tx_n'')$ converge to distinct limits y', y'' respectively. Now $\forall x \in H$

$$\begin{aligned}\langle y', x\rangle &= \left\langle s.\lim_{n\to\infty} Tx_n', x\right\rangle \\ &= \lim_{n\to\infty} \langle Tx_n', x\rangle \\ &= \lim_{n\to\infty} \langle x_n', T^*x\rangle \\ &= \lim_{n\to\infty} \langle x_n'', T^*x\rangle\end{aligned}$$

{ Reason: Since $x_n \xrightarrow{\omega} x$ (note the weak limit is unique) so $(x_n'), (x_n'')$ being subsequences of (x_n) also converges weakly to x, i.e.
$\lim_{n\to\infty} \langle x_n', y\rangle = \langle x, y\rangle$, $\quad \lim_{n\to\infty} \langle x_n'', y\rangle = \langle x, y\rangle$ etc. }. Thus

$$\begin{aligned}&\langle y', x\rangle = \lim_{n\to\infty} \langle x_n'', Tx^*\rangle = \lim_{n\to\infty} \langle Tx'', x\rangle \\ &= \left\langle s.\lim_{n\to\infty} Tx_n'', x\right\rangle = \langle y'', x\rangle \\ &\text{That is } \langle y', x\rangle = \langle y'', x\rangle \quad \forall x \in H. \\ &\text{Thus } \langle y' - y'', x\rangle = 0 \quad \forall x \in H \\ &\text{Therefore, } y' - y'' \perp H \text{ i.e. } y' - y'' = \overline{0}. \\ &\text{i.e. } y' = y''! \text{ a contradiction.}\end{aligned}$$

Hence the assumption that (Tx_n) is not strongly convergent is unacceptable. Hence (Tx_n) must converge strongly.
Conversely, let (x_n) be a bounded sequence of elements of H. Then (x_n) has a subsequence (x_{n_k}) which convergent weakly to H [9]. By hypothesis, (Tx_{n_k}) converges strongly in K. Thus every bounded sequence (x_n) of elements in H has a subsequence (x_{n_k}) such that (Tx_{n_k}) converges strongly in K. Therefore, T is compact. (By proposition 2) □

Proposition 19. *Let H, K be Hilbert spaces. Let $T : H \to K$ be compact. If $x_n \xrightarrow{\omega} x$ then $Tx_n \xrightarrow{s} Tx$.*

Proof. Since $T \in B(H, K)$ we have already seen by proposition 17 that
$x_n \xrightarrow{\omega} x$ implies $Tx_n \xrightarrow{\omega} Tx$. Now since T is compact, (x_n) is weakly convergent implies (Tx_n) is strongly convergent, say to $y \in K$. Thus $Tx_n \xrightarrow{s} y$. But strong convergence implies weak convergence, so $Tx_n \xrightarrow{s} y$ implies $Tx_n \xrightarrow{w} y$. By uniqueness of weak limit (of a weakly convergent sequence) we have

$$\begin{aligned}&Tx_n \xrightarrow{\omega} y, Tx_n \xrightarrow{\omega} Tx \text{ implies } Tx = y \\ &\text{Therefore,} \quad Tx_n \xrightarrow{S} y = Tx \\ &\text{i.e. } Tx_n \xrightarrow{S} Tx\end{aligned}$$

□

Proposition 20. *Let H and K be Hilbert spaces and $T \in B(H, K)$. The following statements are equivalent.*

- *(i) T is compact*
- *(ii) T^*T is compact*
- *(iii) T^* is compact.*

Proof. $(i) \Leftrightarrow (ii)$
$T \in B(H, K)$ implies $T^* \in B(K, H)$. T is compact and T^*T is meaningful. T is compact, T^* is

bounded implies T^*T is compact.

$(ii) \Leftrightarrow (i)$
Let (x_n) be a weakly convergent sequence in H such that $x_n \xrightarrow{\omega} x$. Since T^*T is compact we have by proposition 19 $T^*Tx_n \xrightarrow{s} T^*T_x$ i.e. T^*T is strongly Cauchy in H. Since (x_n) is weakly convergent, (x_n) is bounded, hence there exists $c > 0$ such that $\|x_n\| \leq c \quad \forall\, n \in \mathbb{N}$.
Now for all $m, n \in \mathbb{N}$,

$$\begin{aligned}\|Tx_n - Tx_m\|^2 &= \langle T(x_n - x_m), T(x_n - x_m)\rangle \\ &= \langle T^*T(x_n - x_m), (x_n - x_m)\rangle \\ &= |\langle T^*T(x_n - x_m), (x_n - x_m)\rangle| \quad (\because T^*T \geq 0) \\ &\leq \|T^*T(x_n - x_m)\| \, \|x_n - x_m\|\end{aligned}$$

Since T^*T is strongly Cauchy in $H, \|T^*T(x_n - x_m)\| \to 0$ as both m, n $\longrightarrow \infty$. Since $\|x_k\| \leq c \;\forall\; k \in \mathbb{N}$, so $\|x_n - x_m\| \leqslant \|x_n\| + \|x_m\| \leqslant 2c$. Therefore, $\|Tx_n - Tx_m\| \to 0$ as both $m, n \to \infty$ i.e. (Tx_n) is strongly Cauchy in K. But K is strongly complete. Therefore, Tx_n converges strongly in K. So T maps weakly convergent sequences to strongly convergent sequences. Therefore, T is compact.

$(i) \Leftrightarrow (iii)$.
Since T is compact and $T^* \in B(K, H)$, so $TT^* (\in B(K))$ is compact i.e. $(T^*)\, T^*$ is compact. From (i) $\Leftrightarrow$ (ii), we conclude, T^* is compact. Now $T^* \in B(H)$ and $(T^*)^* = T$. So T^* is compact $\Rightarrow (T^*)^*$ is compact $\Rightarrow T$ is compact ∎

Proposition 21. *Let H be an infinite dimensional Hilbert space. If T is compact, then $0 \in \sigma(T)$.*

Proof. Since T is compact and H is infinite dimensional so T is not invertible (by proposition 7). {For if T^{-1} existed in $B(H)$ then $T^{-1}T$ would be compact i.e. I would be compact which is not possible when H is infinite dimensional}. i.e. $T - 0I$ is not invertible. Therefore, $0 \in \sigma(T)$. ∎

Proposition 22. *Let $T : H \to K$ be compact. Then $\eta_T^\perp$ is separable.*

Proof. Let $\{e_\alpha : x \in \Lambda\}$ be an orthonormal basis for $\eta_T^\perp$. For any sequence (e_{α_i}) from $\{e_\alpha : \alpha \in \Lambda]$ with $e_{\alpha_i} \neq e_{\alpha_j}$ if $i \neq j$, we know that $e_{\alpha i} \overset{\omega}{\to} \bar{0}$ i.e. $(e_{\alpha_i})_{i=1}^\infty$ converges weakly. Since T is compact, $(Te_{\alpha_i})_{i=1}^\infty$ converges strongly to $\bar{0}$.

Hence for each real $\varepsilon > 0$ there can be at most a finite number of $\alpha \in \Lambda$ such that

$$\|Te_{\alpha_i}\| \geqslant \varepsilon.$$

(For if $\{\alpha \in \Lambda : \|Te_{\alpha_i}\| \geq \varepsilon\}$ is of infinite cardinality, then we would select from this bounded sequence $\{e_{\alpha_i}]$ a subsequence $\left\{e'_{\alpha_i}\right\}$ which converges weakly to $\bar{0}$; therefore $Te'_{\alpha_i} \xrightarrow{s}= \bar{0}$ (for T is compact) and this is a contradiction! since $\|Te_{\alpha_i}\| \geq \varepsilon \quad \forall i \in \mathbb{N}$.
It now follows that the set

$$\{\alpha \in \Lambda : \|Te_\alpha\| > 0\}$$

is at most countable. To see this, put $\varepsilon = \frac{1}{k}$ and let k run through $\mathbb{N}$. Then

$$\{\alpha \in \Lambda : \|Te_\alpha\| > 0\} = \bigcup_{k \in \mathbb{N}} \{\alpha_i : \|Te_{\alpha_i}\| \geq \frac{1}{k}\}$$

Hence the orthormal basis spanning $\eta_T^\perp$ is at most countable. This shows that $\eta_T^\perp$ is separable.

We use: A normed linear space X is separable if and only if there exists an atmost countable family F of linearly independent elements of X such that $\overline{[F]} = X$. (For if $Q = \{\alpha \in \Lambda : \|Te_\alpha\| \geq \varepsilon\}$ is of infinite cardinality We would select a sequence (α_n) of elements of Q and for this sequence we would have (as already above) $T_{\alpha_n} \xrightarrow{s} \bar{0}$ which contradicts the condition $\|Te_\alpha\| \geq \varepsilon \;\forall\; \alpha \in Q$ ∎

Proposition 23. *Let $T \in B(H)$. Then T is compact if and only if there is a sequence (T_n) of elements of $B(H)$ which are of finite rank such that $T_n \xrightarrow{\|\cdot\|} T$ i.e. T_n converges to T uniformly.*

Proof. By proposition 10 we have seen that: If (T_n) is a sequence of compact linear operators in H and $T_n \to T$ in the norm of H, then T is compact. If each $T_n \in B(H)$ is of finite rank, then we have, seen that T_n is compact and then we apply proposition 22 to conclude the proof of the given theorem in one direction.

Conversely, let T be compact. Hence, by proposition 22, $\eta_T^{\perp}$ is separable. Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\eta_T^{\perp}$. Let P_n be the orthoprojector on H onto $M_n = \vee_{i=1}^{n} \{e_i\}$ and let P be the orthoprojector on H onto $\vee_{i=1}^{\infty} \{e_i\} = \eta_T^{\perp}$. Let $T_n = TP_n \forall\ n \in \mathbb{N}$, i.e. $T_n x = TP_n x\ \forall x \in H$. So domain of T_n is $M_n = \mathcal{R}_{P_n}$ and is of orthogonal dimension n and hence dimension of $\mathcal{R}_{T_n}$ is at most n. (Note that

$$T_n x = \sum_{i=1}^{n} \langle x, e_i \rangle Te_i \quad \forall x \in H$$

$$\begin{aligned}&\text{Therefore, } T_n x = TP_n x = T\left(P_n x\right)\\ &= T\left(\sum_{i=1}^{n} \langle x, e_i \rangle e_i\right) \quad \text{for} \quad \mathcal{R}_{P_n} = M_n = \vee_{i=1}^{n} \{e_i\}\\ &= \sum_{i=1}^{n} \langle x, e_i \rangle Te_i\end{aligned}$$

Since $M_n \subseteq M_{n+2} \quad \forall\ n \in \mathbb{N}$. So $\quad P_n \leq P_{n+1} \leqslant P \quad \forall n \in \mathbb{N}$ i.e. $\quad P_1 \leq P_2 \leq \ldots \leq P_n \leqslant \ldots \leq P$
Hence $P_n \xrightarrow{s} P$.
Consider the operators $T - T_n \in B(H)$. There exists a sequence (x_n) of element of H such that $\|x_n\| = 1$ and

$$\|(T - T_n)x_n\| \geq \frac{1}{2}\|T - T_n\| \tag{7}$$

(for $\|T - T_n\| = \sup\{\|(T - T_n)\, x\| : x \in H\}$ and $\|x\| = 1$).

For all $y \in H$, we have $\langle (P - P_n)\, x_n, y \rangle = \langle x_n, (P - P_n)\, y \rangle \to 0$ as $n \to \infty$ since $P_n \xrightarrow{s} P \Rightarrow P_n y \xrightarrow{s} P_y$ i.e. $(P_n - P)_y \xrightarrow{s} \bar{0}$.
Therefore $\langle (P - P_n)\, x_n, y \rangle \to 0 = \langle \bar{0}, y \rangle \quad \forall y \in H$
i.e. $(P - P_n)\, x_n \xrightarrow{\omega} \bar{0}$.
Since T is compact,

$$T\left(P - P_n\right) x_n \xrightarrow{s} \bar{0} \tag{8}$$

Now from (7)

$$\|T - T_n\| \leqslant 2\, \|(T - T_n)x_n\| = 2\, \|Tx_n - T_n x_n\| = 2\, \|Tx_n - TP_n x_n\|$$

Noting that (by projection theorem), we can write each x_n as $x_n = x_n' + x_n''$ where $x_n' \in \eta_T^{\perp}$ and $x_n'' \in \eta_T$, we have

$$Tx_n = Tx_n'' + Tx_n'' = Tx_n'$$

since $x_n'' \in \eta_T$, so $Tx_n'' = \bar{0}$. But $x_n' =$ component of x_n in $\eta_T^{\perp} =$ range P. So $x_n' = Px_n$. Therefore $Tx_n' = TPx_n$ i.e. $Tx_n = TPx_n$. Hence $\|T - T_n\| \leqslant 2\|TPx_n - (P_n x_n)\| = 2\|T(P - P_n)x_n\|$.
By (8) $T(P - P_n)x_n \xrightarrow{s} \bar{0}$ and thus we obtain $\lim_{n\to\infty} \|T - T_n\| = 0$
i.e. $T_n \longrightarrow T$ uniformly. ☐

Lemma 6. *Let T be a compact operator in H and (e_n) be any infinite orthonormal sequence in H. Then $\lim_{n\to\infty} \langle |T\alpha_n, e_n| \rangle = 0$.*

Proof. For $e_n \xrightarrow{\omega} \bar{0}$ and T being compact $Te_n \xrightarrow{s} \bar{0}$.
Now by Cauchy-Bunyakovsky-Shwarz inequality,

$$|\langle Te_1, e_n \rangle| < \|Te_n\|\, \|e_n\| = \|Te_n\| \quad \forall_n \in \mathbb{N}$$

But $Te_n \xrightarrow{s} \overline{0}$, therefore,

$$\|Te_n\| \to \|\overline{0}\| = 0$$

$$\text{thus } \lim_{n\to\infty} |\langle Te_n, e_n\rangle| = 0$$

□

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