Original Research Article

An extended finite element method for the elasticity interface problem

Abstract

In this paper, we propose an extended mixed finite element method for elasticity interface problems based on Mini finite element space. The stabilization term defined on edges of interface elements and the ghost penalty term are added, which ensures that the discrete inf-sup condition holds independent of how the interface intersects the triangulation. Finally, numerical experiments are carried out to verify the theoretical analysis.

Keywords: elasticity interface problem, extended mixed finite element method, optimal a priori error estimates, estimates of condition number

1. Introduction

Elasticity interface problems which describe how solid objects are deformed when external forces are applied on them have wide applications in continuum mechanics, multi-phase elasticity problems, solid mechanics, etc. Interface problems usually lead to differential equations with discontinuous or non-smooth solutions across interfaces.

Due to the discontinuity of the coefficients across the interface, if using standard finite element method to solve interface problems, one usually enforces mesh lines along the interface in order to get optimal a priori error estimates, which is the so-called interface-fitted finite element method. However, for many problems in which the interface is complicated or varies with time, the generation of interface-fitted meshes is very costly. Therefore, it is attractive to develop finite method based on interface-unfitted mesh whose generation is independent of the interface, which is called the interface-unfitted method. As far as we know, there are mainly two classes of interface-unfitted methods, the extended finite element method (XFEM) and the immersed finite element method (IFEM).

Z. Li firstly proposed the immersed finite element method in 1998 [19] to solve one dimensional elliptic interface problems. The main idea of IFEM is to take a simple Cartesian grid and modify the basis function to satisfy the interface jump conditions. In [20], Z. Li, T. Lin and X. Wu extended the immersed finite element method for two dimensional elliptic interface problems. For more details, please see [8], [12], [14], [17], [21]. However, The research of IFEM is not thorough and the construction of the discrete spaces is still a challenging work.

XFEM was originally proposed by T. Belytschko and T. Black [3] to solve elastic crack problems. It was designed by modifying the basis function of interface elements. Whereafter, A. Hansbo and P. Hansbo firstly presented the extended finite element method (Nitsche's-XFEM) based on Nitsche's method in [15]. The main idea of Nitsche's-XFEM is to combine a variant of Nitsche's method to enforce continuity on the interface with the idea of XFEM to modify the basis function of interface elements. Later on, based on Nitsche's-XFEM, there are a lot of research in the field of the computational mechanics and flow problems such as [7], [10], [13], [16], [18] and so on.

Among literatures mentioned in the above, most of the works are considered for elliptic interface problems and Stokes interface problems. As far as we know, there are two papers for elasticity interface problems. In

2004, A. Hansbo and P. Hansbo proposed a general approach that can handle both perfectly and imperfectly bonded interfaces for elasticity interface problems, but excluding the incompressible case. In order to deal with the incompressible case, R. Becker, E. Burman and P. Hansbo [2] applied the extended finite element spaces for both the displacement and the pressure in 2009. They obtained the optimal a priori estimates independent of the mesh size but not of the ratio of Lamé constants. What's more, they ignored the case where the interface cuts the mesh in a way that very small sub-elements were created, which would make system matrix of the method become ill-conditioned.

In this paper, we consider the mixed form for the elasticity interface problem based on XFEM $P_1^b - P_1$ finite element pair. By introducing some stabilization terms, we derive an inf-sup stability result for the discrete bilinear form uniform with respect to h and the quotient $\frac{\mu_1}{\mu_2}$. Based on this, we obtain the optimal a priori estimates in energy and L^2 norms. Meantime, in case of the instabilities because of "small cuts", we add the ghost penalty term near the interface and prove the condition number of stiffness matrix is $\mathcal{O}(h^{-2})$ independent of the location of the interface.

The outline of the paper is as follows. In Section 1, we introduce the extended finite element method. The analyses of the extended finite element method are presented in Section 2. In Section 3, we prove that the condition number of the stiffness matrix is independent of how the interface intersects the mesh. Finally, some numerical examples are carried out to verify our theoretical analyses in Section 4.

2. Preliminaries

2.1. The elasticity interface problem

Let Ω be a bounded domain in $\mathbb{R}^2$, with convex polygonal boundary $\partial\Omega$. A C^2-smooth interface defined by $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$ divides Ω into two open sets Ω_1 and Ω_2 such that $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \cap \Omega_2 = \emptyset$ (see Figure 1). Consider the following elasticity interface problem: find a displacement $\boldsymbol{u}$ and a pressure p such that

$$\begin{aligned} -\mathrm{div}(2\mu\epsilon(\boldsymbol{u})) + \mathrm{grad}p &= \boldsymbol{f} && \text{in } \Omega_1 \cup \Omega_2, \\ \mathrm{div}\boldsymbol{u} + \tfrac{1}{\lambda}p &= 0 && \text{in } \Omega_1 \cup \Omega_2, \\ [\boldsymbol{u}] &= 0 && \text{on } \Gamma, \\ [p\boldsymbol{n} - 2\mu\epsilon(\boldsymbol{u})\boldsymbol{n}] &= -\sigma\kappa\boldsymbol{n} && \text{on } \Gamma, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

Here, $\epsilon(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$ is the strain tensor. σ is the surface tension coefficient and κ is the curvature of the interface. $\boldsymbol{n}$ is the outward pointing normal to Ω_1 on Γ. μ, λ are piecewise Lamé constants

$$\mu = \begin{cases} \mu_1 & \text{in } \Omega_1, \\ \mu_2 & \text{in } \Omega_2, \end{cases} \qquad \lambda = \begin{cases} \lambda_1 & \text{in } \Omega_1, \\ \lambda_2 & \text{in } \Omega_2. \end{cases}$$

On the basis of Young's modulus E_i and Poisson's ratio ν_i, we have $\mu_i = \frac{E_i}{2(1+\nu_i)}$ and $\lambda_i = \frac{E_i\nu_i}{(1+\nu_i)(1-2\nu_i)}$, $i = 1, 2$. $\boldsymbol{f} \in (L^2(\Omega))^2$ is a given function, and $[v]|_\Gamma = v_1|_\Gamma - v_2|_\Gamma$ is the jump on the interface Γ, where $v_i = v|_{\Omega_i}, i = 1, 2$.

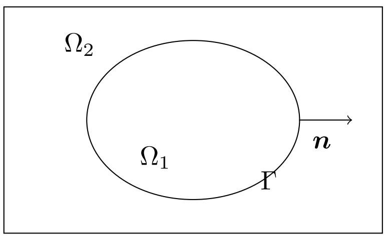

Figure 1: Domain Ω, its sub-domains Ω_1, Ω_2, and interface Γ.

For the weak formulation, we first introduce the space

$$L_0^2(\Omega) = \{p \in L^2(\Omega) : \int_\Omega p(x)\mathrm{d}x = 0\},$$

and the inner product between matrices $\sigma,\ \chi$,

$$(\sigma,\chi)_{L^2(\Omega)} = \int_\Omega \sigma : \chi \mathrm{d}x = \int_\Omega \sum_{i=1}^{2}\sum_{j=1}^{2} \sigma_{ij}\chi_{ij}\mathrm{d}x.$$

Then weak formulation of the problem can be read as follows: given $\boldsymbol{f} \in \boldsymbol{V}^{'}$, find $(\boldsymbol{u},p) \in \boldsymbol{V} \times Q = (H_0^1(\Omega))^2 \times L_0^2(\Omega)$ such that

$$B[(\boldsymbol{u},p),(\boldsymbol{v},q)] = L(\boldsymbol{v}), \quad \forall (\boldsymbol{v},q) \in \boldsymbol{V} \times Q, \tag{2.2}$$

where

$$\begin{aligned} B[(\boldsymbol{u},p),(\boldsymbol{v},q)] &= \int_{\Omega_1\cup\Omega_2} 2\mu\epsilon(\boldsymbol{u}) : \epsilon(\boldsymbol{v})\mathrm{d}x - \int_{\Omega_1\cup\Omega_2} p\mathrm{div}\boldsymbol{v}\mathrm{d}x \\ &\quad + \int_{\Omega_1\cup\Omega_2} q\mathrm{div}\boldsymbol{u}\mathrm{d}x + \int_{\Omega_1\cup\Omega_2} \lambda^{-1}pq\mathrm{d}x, \\ L(\boldsymbol{v}) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\mathrm{d}x + \int_\Gamma \sigma\kappa\boldsymbol{v}\cdot\boldsymbol{n}\mathrm{d}s. \end{aligned}$$

By the abstract theory of mixed problems ([5], [11]), the following theorem holds.

Theorem 2.1. *There exists a unique solution $(\boldsymbol{u},p) \in \boldsymbol{V} \times Q$ for the problem (2.2).*

2.2. The extended finite element method

Let $\mathcal{T}_h$ be the triangulation of Ω, generated independently of the location of the interface Γ. For any element $T \in \mathcal{T}_h$, denote the diameter of T by h_T and $h = \max_{T\in\mathcal{T}_h} h_T$. By $G_h = \{T \in \mathcal{T}_h : T \cap \Gamma \neq \emptyset\}$, we denote the set of elements that are intersected with the interface. For an element $T \in G_h$, let $\Gamma_T = \Gamma \cap T$ and $T_i = T \cap \Omega_i$. Define the sub-domains $\Omega_{i,h} = \{T \in \mathcal{T}_h : T \subset \Omega_i \text{ or meas}(T\cap\Gamma) > 0\}$, $\omega_{i,h} = \{T \in \Omega_{i,h} : T \subset \Omega_i \text{ and } T \cap \Gamma = \varnothing\}$, $i = 1,2$. Let $\mathcal{T}_{h,i} = \mathcal{T}_h|_{\Omega_{i,h}}$, $i = 1,2$. For the notation of edges, let $\mathcal{F}_i = \{e \subset \partial T : T \in G_h, e \not\subseteq \partial\Omega_{i,h}\}$, $i = 1,2$, which are called as the sets of transmission edges. For an edge $e \in \mathcal{F}_i$, let T^l and T^r be the two neighboring cells of the edge e. Denote $v^l = v|_{T^l}$, $v^r = v|_{T^r}$, where l, r represent the left and the right, respectively and set $[v]_e = v^l|_e - v^r|_e$. For an integer $k \geq 0$, $P_k(D)$ denotes the set of all scalar-valued on domain D with degree less than or equal to k. In the subsequent sections, the letter C or c, with or without subscript, denotes a generic constant that may not be the same at different occurrences for brevity and assume that Γ intersects with the boundary ∂T of an element T in G_h exactly twice and each (open) edge at most once..

2.3. Weak form of the discrete problem

Define

$$\begin{aligned} \boldsymbol{V}_{h,i} &= [\{v_h \in H^1(\Omega_{i,h}) : v_h|_T \in P_1(T),\ \forall T \in \mathcal{T}_{h,i},\ v_h|_{\partial\Omega} = 0,\ i = 1,2\} \oplus B]^2, \\ Q_{h,i} &= \{q_h \in H^1(\Omega_{i,h}) : q_h|_T \in P_1(T),\ \forall T \in \mathcal{T}_{h,i},\ i = 1,2\}, \end{aligned}$$

where

$$B = \{b \in C^0(\Omega_{i,h}) : b|_T \in P_3(T) \cap H_0^1(T),\ \forall T \in \mathcal{T}_{h,i},\ i = 1,2\}.$$

Thus, our extended Mini finite element spaces are

$$\begin{aligned} \boldsymbol{V}_h &= \boldsymbol{V}_{h,1} \times \boldsymbol{V}_{h,2}, \\ Q_h &= (Q_{h,1} \times Q_{h,2})/R = \{q_h \in Q_{h,1} \times Q_{h,2} : \int_\Omega q_h \mathrm{d}x = 0\}. \end{aligned}$$

To define the stabilization terms, we first decompose the interface zone of the triangulation $\mathcal{T}_{h,i}$ in $\mathcal{N}_{l,i}$ patches $\mathcal{P}_{l,i}$ with diameter $h_{\mathcal{P}_{l,i}} = \mathcal{O}(h)$ consisting of a moderate number of elements, in such a way that each interface element is involved in one patch $\mathcal{P}_{l,i}$. Denote $\Pi_{l,i} : L^2(\mathcal{P}_{l,i}) \mapsto P_1(\mathcal{P}_{l,i})$ be the L^2-projection onto $P_1(\mathcal{P}_{l,i})$. For more details, please see [6]. For any discontinuous function φ defined on Γ_T, we use the notations $\{\varphi\} = \omega_1\varphi_1 + \omega_2\varphi_2$ and $\{\varphi\}_* = \omega_2\varphi_1 + \omega_1\varphi_2$ with $\omega_i = \dfrac{|T_i|}{|T|}$, where $T \in G_h$, $|T| = \text{meas}(T), T_i = T \cap \Omega_i,\ i = 1, 2$. Clearly, $0 \le \omega_i \le 1$ and $\omega_1 + \omega_2 = 1$. Recalling the definition of $[\varphi]$, we have $[\varphi\psi] = \{\varphi\}[\psi] + [\varphi]\{\psi\}_*$.

Now, we present the discrete approximation of the elasticity interface problem: find $(\boldsymbol{u_h}, p_h) \in \boldsymbol{V}_h \times Q_h$, such that for any $(\boldsymbol{v_h}, q_h) \in \boldsymbol{V}_h \times Q_h$,

$$B_h[(\boldsymbol{u_h}, p_h), (\boldsymbol{v_h}, q_h)] + \gamma_1 J_p(p_h, q_h) + \gamma_2 J_u(\boldsymbol{u_h}, \boldsymbol{v_h}) = L_h(\boldsymbol{v_h}), \tag{2.3}$$

where

$$B_h[(\boldsymbol{u_h}, p_h), (\boldsymbol{v_h}, q_h)] = a_h(\boldsymbol{u_h}, \boldsymbol{v_h}) + b_h(\boldsymbol{v_h}, p_h) - b_h(\boldsymbol{u_h}, q_h) + c_h(p_h, q_h),$$

$$L_h(\boldsymbol{v_h}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v_h} \mathrm{d}x + \int_\Gamma \sigma\kappa\{\boldsymbol{v_h} \cdot \boldsymbol{n}\}_* \mathrm{d}s,$$

with

$$\begin{aligned}
a_h(\boldsymbol{u_h}, \boldsymbol{v_h}) &= \int_{\Omega_1\cup\Omega_2} 2\mu\epsilon(\boldsymbol{u_h}) : \epsilon(\boldsymbol{v_h})\mathrm{d}x - \int_\Gamma \{2\mu\epsilon(\boldsymbol{u_h})\boldsymbol{n}\}[\boldsymbol{v_h}]\mathrm{d}s \\
&\quad - \int_\Gamma \{2\mu\epsilon(\boldsymbol{v_h})\boldsymbol{n}\}[\boldsymbol{u_h}]\mathrm{d}s + \sum_{T\in G_h} \int_{\Gamma_T} \gamma_u h_T^{-1} \mu_{\max}[\boldsymbol{u_h}][\boldsymbol{v_h}]\mathrm{d}s, \\
b_h(\boldsymbol{v_h}, p_h) &= -\int_{\Omega_1\cup\Omega_2} p_h \mathrm{div}\boldsymbol{v_h}\mathrm{d}x + \int_\Gamma \{p_h\}[\boldsymbol{v_h} \cdot \boldsymbol{n}]\mathrm{d}s, \\
c_h(p_h, q_h) &= \int_{\Omega_1\cup\Omega_2} \lambda^{-1} p_h q_h \mathrm{d}x, \\
J_u(\boldsymbol{u_h}, \boldsymbol{v_h}) &= \sum_{i=1}^{2}\sum_{l=1}^{\mathcal{N}_{l,i}} \mu_i h_{\mathcal{P}_{l,i}}^{-2} \int_{\mathcal{P}_{l,i}} (\boldsymbol{u_{h,i}} - \Pi_{l,i}\boldsymbol{u_{h,i}})(\boldsymbol{v_{h,i}} - \Pi_{l,i}\boldsymbol{v_{h,i}})\mathrm{d}x, \\
J_p(p_h, q_h) &= \sum_{i=1}^{2} j_i(p_{h,i}, q_{h,i}) = \sum_{i=1}^{2} \mu_i^{-1} h^3 \sum_{e\in\mathcal{F}_i} \int_e [\nabla p_{h,i}][\nabla q_{h,i}]\mathrm{d}x.
\end{aligned}$$

$\gamma_1,\ \gamma_2 > 0$ are stabilization parameters independent of mesh size and Lamé constants.

Remark 2.1. *The fourth term in $a_h(\cdot, \cdot)$ is a standard penalty term in Nitsche's method [15]. The third term in $a_h(\cdot, \cdot)$ is introduced for the symmetry. The stabilization term $J_p(\cdot, \cdot)$ is used to gurantee the inf-sup condition of the discrete weak formulation. The ghost penalty term $J_u(\cdot, \cdot)$ is used to make sure the system matrix of the method is well-conditioned.*

3. Analysis of the scheme

Since the stabilization terms $J_u(\boldsymbol{u_h}, \boldsymbol{v_h})$, $J_p(p_h, q_h)$ are not the residual of the equations, the finite element formulation (2.3) is not consistent. We have the following weak consistent relation.

Lemma 3.1. *Assume $(\boldsymbol{u}, p) \in (H^2(\Omega_1\cup\Omega_2)\cap H_0^1(\Omega))^2 \times (H^1(\Omega_1\cup\Omega_2)\cap L_0^2(\Omega))$ be the solution of the problem (2.2) and $(\boldsymbol{u_h}, p_h) \in \boldsymbol{V}_h \times Q_h$ be the solution of the discrete problem (2.3). Then for any $(\boldsymbol{v_h}, q_h) \in \boldsymbol{V}_h \times Q_h$, there holds*

$$B_h[(\boldsymbol{u} - \boldsymbol{u_h}, p - p_h), (\boldsymbol{v_h}, q_h)] = \gamma_1 J_p(p_h, q_h) + \gamma_2 J_u(\boldsymbol{u_h}, \boldsymbol{v_h}). \tag{3.1}$$

Proof. Multiplying (1.1) and (1.2) by testing functions $\boldsymbol{v_h}$ and q_h, respectively, using integration by parts and noting that the interface conditions (1.3) and (1.4), we obtain

$$a_h(\boldsymbol{u}, \boldsymbol{v_h}) + b_h(\boldsymbol{v_h}, p) - b_h(\boldsymbol{u}, q_h) + c_h(p, q_h) = L_h(\boldsymbol{v_h}). \tag{3.2}$$

Subtracting (2.3) from (3.2), we get (3.1). □

In the following analysis, we need to introduce some mesh dependent norms, i.e.,

$$\begin{aligned}
&\|\boldsymbol{v}\|_{\frac{1}{2},h,\Gamma} = \Big(\sum_{T\in G_h} h_T^{-1}\|\boldsymbol{v}\|_{0,\Gamma_T}^2\Big)^{\frac{1}{2}}, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
&\|\boldsymbol{v}\|_{-\frac{1}{2},h,\Gamma} = \Big(\sum_{T\in G_h} h_T\|\boldsymbol{v}\|_{0,\Gamma_T}^2\Big)^{\frac{1}{2}}, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
&\|\boldsymbol{v}\| = \big(\|\mu^{\frac{1}{2}}\epsilon(\boldsymbol{v})\|_{0,\Omega_1\cup\Omega_2}^2 + \|\mu_{\max}^{\frac{1}{2}}[\boldsymbol{v}]\|_{\frac{1}{2},h,\Gamma}^2 + \|\mu_{\max}^{-\frac{1}{2}}\{2\mu\epsilon(\boldsymbol{v})\boldsymbol{n}\}\|_{-\frac{1}{2},h,\Gamma}^2\big)^{\frac{1}{2}}, \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
&\|\boldsymbol{v}\|_\star = \big(\|\mu^{\frac{1}{2}}\epsilon(\boldsymbol{v})\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + \|\mu_{\max}^{\frac{1}{2}}[\boldsymbol{v}]\|_{\frac{1}{2},h,\Gamma}^2\big)^{\frac{1}{2}}, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h,\\
&|||(\boldsymbol{v}, q)||| = \big(\|\boldsymbol{v}\|^2 + \|\mu^{-\frac{1}{2}}q\|_{0,\Omega_1\cup\Omega_2}^2 + \|\mu_{\max}^{-\frac{1}{2}}\{q\}\|_{-\frac{1}{2},h,\Gamma}^2\big)^{\frac{1}{2}}, \qquad \forall (\boldsymbol{v}, q) \in (\boldsymbol{V}_h, Q_h),\\
&|||(\boldsymbol{v}, q)|||_h = \big(\|\boldsymbol{v}\|_\star^2 + \|\mu^{-\frac{1}{2}}q\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + J_p(q,q) + J_u(\boldsymbol{v},\boldsymbol{v})\big)^{\frac{1}{2}}, \qquad \forall (\boldsymbol{v}, q) \in (\boldsymbol{V}_h, Q_h).
\end{aligned}$$

3.1. Continuity analysis

y the definition of $a_h(\boldsymbol{u_h}, \boldsymbol{v_h})$ and the Cauchy-Schwarz inequality, the following lemma holds.

Lemma 3.2. *For all $\boldsymbol{u_h} \in \boldsymbol{V}_h$, we have*

$$a_h(\boldsymbol{u_h}, \boldsymbol{v_h}) \le C_1\|\boldsymbol{u_h}\|\|\boldsymbol{v_h}\|, \quad \forall \boldsymbol{v_h} \in \boldsymbol{V}_h, \tag{3.3}$$

where C_1 is a positive constant independent of μ_1, μ_2 and the mesh size.

In the following analysis, we need a trace inequality on the interface and we just state the following lemma without proof (see [15]).

Lemma 3.3. *Map a triangle T onto the unit reference triangle $\tilde{T}$ by an affine map and denote by $\tilde{\Gamma}_{\tilde{T}}$ the corresponding image of Γ_T. Under the assumptions in Section 2, there exists a constant C, depending on Γ but independent of the mesh, such that*

$$\|\boldsymbol{w}\|_{0,\tilde{\Gamma}_{\tilde{T}}}^2 \le C\|\boldsymbol{w}\|_{0,\tilde{T}}\|\boldsymbol{w}\|_{1,\tilde{T}}, \quad \forall \boldsymbol{w} \in H^1(\tilde{T}). \tag{3.4}$$

Lemma 3.4. *For any $q_h \in Q_h$, we have*

$$\|\mu_{\max}^{-\frac{1}{2}}\{q_h\}\|_{-\frac{1}{2},h,\Gamma} \le C_2\|\mu_{\max}^{-\frac{1}{2}}q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}},$$

where C_2 is a positive constant independent of μ_1, μ_2 and the mesh size.

Proof. By the definition of $\|\cdot\|_{-\frac{1}{2},h,\Gamma}$, Lemma 3.3 and the inverse inequality, there holds

$$\begin{aligned}
\|\mu_{\max}^{-\frac{1}{2}}\{q_h\}\|_{-\frac{1}{2},h,\Gamma}^2 &\le C\sum_{T\in G_h} h_T\big(h_T^{-1}\|\mu_{\max}^{-\frac{1}{2}}q_h\|_{0,T}^2 + h_T|\mu_{\max}^{-\frac{1}{2}}q_h|_{1,T}^2\big)\\
&\le C\sum_{T\in G_h} h_T\cdot h_T^{-1}\|\mu_{\max}^{-\frac{1}{2}}q_h\|_{0,T}^2\\
&\le C_2\|\mu_{\max}^{-\frac{1}{2}}q_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2.
\end{aligned}$$

The proof is completed. □

Similarly, we can get the following result.

Lemma 3.5. *There exists a positive constant C_I independent of μ_1 and μ_2 such that*

$$\|\{2\mu\epsilon(\boldsymbol{u_h})\boldsymbol{n}\}\|^2_{-\frac{1}{2},h,\Gamma} \leq C_I\mu_{\max}\|\mu^{\frac{1}{2}}\epsilon(\boldsymbol{u_h})\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}, \quad \forall \boldsymbol{u_h} \in \boldsymbol{V}_h.$$

Combining the definitions of $|||(\cdot,\cdot)|||$ and $|||(\cdot,\cdot)|||_h$ with Lemmas 3.4-3.5, we can derive the following relation immediately.

Lemma 3.6. *For any $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$, there exists a constant $C > 0$, such that*

$$|||(\boldsymbol{v}_h, q_h)||| \leq C|||(\boldsymbol{v}_h, q_h)|||_h.$$

Together with Lemmas 3.2, 3.4-3.6 and the discrete Korn inequality $\big((1.15)$ in [4]$\big)$, we can derive the following result.

Lemma 3.7. *For any $(\boldsymbol{u_h}, p_h) \in \boldsymbol{V}_h \times Q_h$, $(\boldsymbol{v_h}, q_h) \in \boldsymbol{V}_h \times Q_h$, we have*

$$B_h[(\boldsymbol{u_h}, p_h), (\boldsymbol{v_h}, q_h)] + \gamma_1 J_p(p_h, q_h) + \gamma_2 J_u(\boldsymbol{u_h}, \boldsymbol{v_h}) \leq C_B|||(\boldsymbol{u_h}, p_h)|||_h|||(\boldsymbol{v_h}, q_h)|||_h,$$

where C_B is a positive constant independent of μ_1, μ_2 and the mesh size.

3.2. Stability analysis

In this part, we are devoted to the stability of the weak formulation. We first give the coercivity of $a_h(\cdot,\cdot)$.

Lemma 3.8. *There exists a positive constant C_a independent of μ_1 and μ_2 such that*

$$a_h(\boldsymbol{u_h}, \boldsymbol{u_h}) + \gamma_2 J_u(\boldsymbol{u_h}, \boldsymbol{u_h}) \geq C_a\|\boldsymbol{u_h}\|^2_{\star}, \qquad \forall \boldsymbol{u_h} \in \boldsymbol{V}_h.$$

Proof. According to the definition of $a_h(\boldsymbol{u_h}, \boldsymbol{u_h})$, one has

$$a_h(\boldsymbol{u_h}, \boldsymbol{u_h}) = 2\|\mu^{\frac{1}{2}}\epsilon(\boldsymbol{u_h})\|^2_{0,\Omega_1\cup\Omega_2} - 2\int_\Gamma \{2\mu\epsilon(\boldsymbol{u_h})\boldsymbol{n}\}[\boldsymbol{u_h}]\mathrm{d}s + \gamma_u\|\mu^{\frac{1}{2}}_{\max}[\boldsymbol{u_h}]\|^2_{\frac{1}{2},h,\Gamma}. \tag{3.5}$$

Using the Cauchy-Schwarz inequality, the Young's inequality with parameter ε, we derive

$$\begin{aligned} 2\int_\Gamma \{2\mu\epsilon(\boldsymbol{u_h})\boldsymbol{n}\}[\boldsymbol{u_h}]\mathrm{d}s &\leq 2\|\mu^{-\frac{1}{2}}_{\max}\{2\mu\epsilon(\boldsymbol{u_h})\boldsymbol{n}\}\|_{-\frac{1}{2},h,\Gamma}\|\mu^{\frac{1}{2}}_{\max}[\boldsymbol{u_h}]\|_{\frac{1}{2},h,\Gamma} \\ &\leq \varepsilon\|\mu^{-\frac{1}{2}}_{\max}\{2\mu\epsilon(\boldsymbol{u_h})\boldsymbol{n}\}\|^2_{-\frac{1}{2},h,\Gamma} + \frac{1}{\varepsilon}\|\mu^{\frac{1}{2}}_{\max}[\boldsymbol{u_h}]\|^2_{\frac{1}{2},h,\Gamma} \\ &\leq C_I\varepsilon\|\mu^{\frac{1}{2}}\epsilon(\boldsymbol{u_h})\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \frac{1}{\varepsilon}\|\mu^{\frac{1}{2}}_{\max}[\boldsymbol{u_h}]\|^2_{\frac{1}{2},h,\Gamma}. \end{aligned} \tag{3.6}$$

Combining (3.5) with (3.6), we obtain

$$a_h(\boldsymbol{u_h}, \boldsymbol{u_h}) \geq 2\|\mu^{\frac{1}{2}}\epsilon(\boldsymbol{u_h})\|^2_{0,\Omega_1\cup\Omega_2} - C_I\varepsilon\|\mu^{\frac{1}{2}}\epsilon(\boldsymbol{u_h})\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + (\gamma_u - \frac{1}{\varepsilon})\|\mu^{\frac{1}{2}}_{\max}[\boldsymbol{u_h}]\|^2_{\frac{1}{2},h,\Gamma}. \tag{3.7}$$

By Theorem 2.12 in [4], we know

$$\|\epsilon(\boldsymbol{u_h})\|^2_{0,\Omega_1\cup\Omega_2} + \|[\boldsymbol{u_h}]\|^2_{\frac{1}{2},h,\Gamma} \geq C\|\nabla\boldsymbol{u_h}\|^2_{0,\Omega_1\cup\Omega_2}, \quad \forall \boldsymbol{u_h} \in \boldsymbol{V}_h. \tag{3.8}$$

According to Lemma 4.2 in [6], there holds

$$\|\nabla\boldsymbol{u_h}\|^2_{0,\Omega_1\cup\Omega_2} + J_u(\boldsymbol{u_h}, \boldsymbol{u_h}) \geq C_J\|\nabla\boldsymbol{u_h}\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}, \quad \forall \boldsymbol{u_h} \in \boldsymbol{V}_h. \tag{3.9}$$

Applying (3.8)-(3.9) to (3.7), we deduce

$$\begin{aligned} &a_h(\boldsymbol{u_h}, \boldsymbol{u_h}) + \gamma_2 J_u(\boldsymbol{u_h}, \boldsymbol{u_h}) \\ \geq\, &C\|\mu^{\frac{1}{2}}\nabla\boldsymbol{u_h}\|^2_{0,\Omega_1\cup\Omega_2} - C_I\varepsilon\|\mu^{\frac{1}{2}}\epsilon(\boldsymbol{u_h})\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + (\gamma_u - \frac{1}{\varepsilon} - 1)\|\mu^{\frac{1}{2}}_{\max}[\boldsymbol{u_h}]\|^2_{\frac{1}{2},h,\Gamma} + \gamma_2 J_u(\boldsymbol{u_h}, \boldsymbol{u_h}) \\ \geq\, &(C_J - C_I\varepsilon)\|\mu^{\frac{1}{2}}\epsilon(\boldsymbol{u_h})\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + (\gamma_u - \frac{1}{\varepsilon} - 1)\|\mu^{\frac{1}{2}}_{\max}[\boldsymbol{u_h}]\|^2_{\frac{1}{2},h,\Gamma} + (\gamma_2 - 1)J_u(\boldsymbol{u_h}, \boldsymbol{u_h}). \end{aligned} \tag{3.10}$$

Let $\varepsilon = \frac{C_J}{2C_I}$, $\gamma_u = \frac{2C_I}{C_J} + 2$, $\gamma_2 = 2$. Choosing $C_a = \min\{\frac{C_J}{2}, 1\}$, we finally obtain the desired result. $\square$

According to Lemma 2.2 in [18], we can derive the following lemma directly.

Lemma 3.9. *Let $p_h = (p_{h,1}, p_{h,2}) \in Q_h$, there exists a constant $C > 0$ such that*

$$\|\mu_i^{-\frac{1}{2}} p_{h,i}\|^2_{0,\Omega_{i,h}} \le C\bigg(\|\mu_i^{-\frac{1}{2}} p_{h,i}\|^2_{0,\omega_{i,h}} + j_i(p_{h,i}, p_{h,i})\bigg), \quad i = 1, 2. \tag{3.11}$$

According to the LBB stability of the standard $\boldsymbol{P}_1^b$-P_1 on $\omega_{i,h}$, for any $p_{h,i} \in Q_{h,i}$, there exists $\boldsymbol{v}_{p_{h,i}} \in V_{h,i}$ with $supp(\boldsymbol{v}_{p_{h,i}}) \subset \bar{\omega}_{i,h}$ satisfies

$$b_h(\boldsymbol{v}_{p_{h,i}}, p_{h,i}) \ge C\|\mu_i^{-\frac{1}{2}} p_{h,i}\|^2_{0,\omega_{i,h}}, \tag{3.12}$$

and

$$\|\mu_i^{-\frac{1}{2}} p_{h,i}\|_{0,\omega_{i,h}} \ge C|\mu_i^{\frac{1}{2}} \boldsymbol{v}_{p_{h,i}}|_{1,\omega_{i,h}}, \ i = 1, 2. \tag{3.13}$$

By Lemma 3.9 and (3.12), there holds

$$C_p\|\mu_i^{-\frac{1}{2}} p_{h,i}\|^2_{0,\Omega_{i,h}} \le \frac{1}{C} b_h(\boldsymbol{v}_{p_{h,i}}, p_{h,i}) + J_i(p_{h,i}, p_{h,i}), \quad \forall p_{h,i} \in Q_{h,i}, \ i = 1, 2. \tag{3.14}$$

By the definition of $\|\cdot\|_\star$ and (3.13)-(3.14), we get the stability of $b_h(\cdot,\cdot)$ on $\boldsymbol{V}_h \times Q_h$.

Lemma 3.10. *For any $p_h = (p_{h,1}, p_{h,2}) \in Q_h$, there exists $\boldsymbol{v}_{p_h} = (\boldsymbol{v}_{p_{h,1}}, \boldsymbol{v}_{p_{h,2}}) \in \boldsymbol{V}_h$ satisfies $\boldsymbol{v}_{p_h}|_{G_h} = 0$ such that*

$$b_h(\boldsymbol{v}_{p_h}, p_h) + J_p(p_h, p_h) \ge C_b\|\mu^{-\frac{1}{2}} p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}, \tag{3.15}$$

and

$$\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \ge C\|\boldsymbol{v}_{p_h}\|_\star. \tag{3.16}$$

Proof. By (3.14), we can get (3.15) directly. Since $\boldsymbol{v}_{p_{h,1}}|_{G_h} = 0$, $\boldsymbol{v}_{p_{h,2}}|_{G_h} = 0$, it implies

$$\|\boldsymbol{v}_{p_h}\|_\star = \|\mu^{\frac{1}{2}} \epsilon(\boldsymbol{v}_{p_h})\|_{0,\omega_{1,h}\cup\omega_{2,h}}. \tag{3.17}$$

By (3.13), we know

$$\|\mu_1^{-\frac{1}{2}} p_{h,1}\|^2_{0,\omega_{1,h}} + \|\mu_2^{-\frac{1}{2}} p_{h,2}\|^2_{0,\omega_{2,h}} \ge C\|\boldsymbol{v}_{p_h}\|^2_\star. \tag{3.18}$$

Then we get (3.16). □

Now, we show the stability of the discrete formulation.

Theorem 3.1. *Let $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$, for sufficient small h, there exists a positive constant C_s, such that*

$$\sup_{(\boldsymbol{v_h}, q_h)\in \boldsymbol{V}_h\times Q_h} \frac{B_h[(\boldsymbol{u_h}, p_h), (\boldsymbol{v_h}, q_h)] + \gamma_1 J_p(p_h, q_h) + \gamma_2 J_u(\boldsymbol{u_h}, \boldsymbol{v_h})}{|||(\boldsymbol{v_h}, q_h)|||_h} \ge C_s|||(\boldsymbol{u_h}, p_h)|||_h. \tag{3.19}$$

Proof. Let $(\boldsymbol{v}_h, q_h) = (\boldsymbol{u}_h, p_h)$, by Lemma 3.8, one has

$$B_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)] = a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) \ge C_a\|\boldsymbol{u}_h\|^2_\star - \gamma_2 J_u(\boldsymbol{u_h}, \boldsymbol{u_h}). \tag{3.20}$$

Let $(\boldsymbol{v}_h, q_h) = (\boldsymbol{v}_{p_h}, 0)$ which satisfies Lemma 3.10, then we obtain

$$B_h[(\boldsymbol{u}_h, p_h), (\boldsymbol{v}_{p_h}, 0)] \ge a_h(\boldsymbol{u}_h, \boldsymbol{v}_{p_h}) + C_b\|p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} - J_p(p_h, p_h). \tag{3.21}$$

Using the Cauchy-Schwarz inequality and (3.16), we get

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_{p_h}) \le \|\boldsymbol{u}_h\|_\star\|\boldsymbol{v}_{p_h}\|_\star \le C\|\boldsymbol{u}_h\|_\star\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}. \tag{3.22}$$

By (3.21)-(3.22) and the arithmetic-geometric inequality, we derive

$$B_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{v}_{p_h},0)] \geq -\frac{C}{2\varepsilon}\|\boldsymbol{u}_h\|_\star^2 + \big(C_b - \frac{C\varepsilon}{2}\big)\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} - J_p(p_h,p_h). \tag{3.23}$$

Combining (3.20) with (3.23), we deduce

$$\begin{aligned}
&B_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h+\eta\boldsymbol{v}_{p_h},p_h)]\\
\geq &C_a\|\boldsymbol{u}_h\|_\star^2 - J_u(\boldsymbol{u}_h,\boldsymbol{u}_h) + \eta\Big(-\frac{C}{2\varepsilon}\|\boldsymbol{u}_h\|_\star^2 + \big(C_b - \frac{C\varepsilon}{2}\big)\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} - J_p(p_h,p_h)\Big)\\
\geq &\big(C_a - \frac{C\eta}{2\varepsilon}\big)\|\boldsymbol{u}_h\|_\star^2 - J_u(\boldsymbol{u}_h,\boldsymbol{u}_h) + \eta\big(C_b - \frac{C\varepsilon}{2}\big)\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} - \eta J_p(p_h,p_h).
\end{aligned}$$

Let $\varepsilon = \frac{C_b}{C},\ \eta = \frac{C_aC_b}{C^2},\ \gamma_1 = \eta+1,\ \gamma_2 = 2$. Choosing $C_s = \min\{\frac{C_a}{2}, \frac{C_aC_b^2}{2C^2}, 1\}$, we can get

$$B_h[(\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h+\eta\boldsymbol{v}_{p_h},p_h)] + \gamma_1 J_p(p_h,p_h) + \gamma_2 J_u(\boldsymbol{u_h},\boldsymbol{v_h}) \geq C_s|||(\boldsymbol{u}_h,p_h)|||_h^2. \tag{3.24}$$

It is easy to check

$$|||(\boldsymbol{u}_h,p_h)|||_h \geq C|||(\boldsymbol{u}_h+\eta\boldsymbol{v}_{p_h},p_h)|||_h. \tag{3.25}$$

In fact, since $\boldsymbol{v}_{p_h}|_{G_h} = 0$, we know

$$J_u(\boldsymbol{u_h}+\eta\boldsymbol{v}_{p_h},\boldsymbol{u_h}+\eta\boldsymbol{v}_{p_h}) \leq C(J_u(\boldsymbol{u}_h,\boldsymbol{u}_h) + \eta^2 J_u(\boldsymbol{v}_{p_h},\boldsymbol{v}_{p_h})),$$

and we only need to prove

$$J_u(\boldsymbol{v}_{p_h},\boldsymbol{v}_{p_h}) \leq C\|\mu^{-\frac{1}{2}}p_h\|^2_{0,\omega_{1,h}\cup\omega_{2,h}}, \tag{3.26}$$

$$|||\boldsymbol{u}_h+\eta\boldsymbol{v}_{p_h}|||_h \leq C|||(\boldsymbol{u}_h,p_h)|||_h. \tag{3.27}$$

By means of the approximation property of L^2 projection and (3.13), we obtain (3.26). Combining the definitions of $\|\cdot\|_\star$ and $|||(\cdot,\cdot)|||_h$ with inequality (3.16), using the triangle inequality, we can get (3.27) directly. This completes the proof. □

Remark 3.1. *Combining Lemma 3.7 with Theorem 3.1 and applying Babuška theorem, we can know that the solution of the discrete problem (2.3) is exist and unique.*

4. Approximation properties and optimal convergence

According to extension theorem (see [9] (Theorem 1, Sect. 5.4) or [1] (Theorem 5.24)), there are extension operators for $i = 1,2$, $E_i^s : H^s(\Omega_i) \to H^s(\Omega)$ such that $(E_i^s v)|_{\Omega_i} = v$ and

$$\|E_i^s v\|_{s,\Omega} \leq C\|v\|_{s,\Omega_i}, \quad \forall v \in H^s(\Omega_i), s = 0,1.$$

Let $I_h : \big(H^2(\Omega)\cap H_0^1(\Omega)\big)^2 \to \boldsymbol{V}_h$ be the nodal interpolation operation, $\pi_h : H^1(\Omega) \to Q_h$ be the local L^2 projection. Define

$$(I_h^*\boldsymbol{v},\pi_h^*q) = \big((I_{h,1}^*\boldsymbol{v_1}, I_{h,2}^*\boldsymbol{v_2}),(\pi_{h,1}^*q_1,\pi_{h,2}^*q_2)\big), \tag{4.1}$$

where $I_{h,i}^*\boldsymbol{v_i} = \big(I_hE_i^2\boldsymbol{v_i}\big)|_{\Omega_i}, \pi_{h,i}^*\boldsymbol{v_i} = \big(\pi_hE_i^1\boldsymbol{v_i}\big)|_{\Omega_i}$.

Theorem 4.1. *For $(\boldsymbol{u},p) \in \big(H^2(\Omega_1\cup\Omega_2)\cap H_0^1(\Omega)\big)^2 \times \big(H^1(\Omega_1\cup\Omega_2)\cap L_0^2(\Omega)\big)$, let (I_h^*,π_h^*) be the interpolation operator defined in (4.1). There exist a positive constant C independent of μ_1, μ_2 and how the interface Γ intersects the triangulation, such that*

$$|||(\boldsymbol{u}-I_h^*\boldsymbol{u}, p-\pi_h^*p)||| \leq Ch\big(\|\mu_{\max}^{\frac{1}{2}}\boldsymbol{u}\|_{2,\Omega_1\cup\Omega_2} + \|\mu^{-\frac{1}{2}}p\|_{1,\Omega_1\cup\Omega_2}\big). \tag{4.2}$$

Proof. Let $\boldsymbol{u_i} = \boldsymbol{u}|_{\Omega_i}, T_i = T \cap \Omega_i, i = 1, 2$, we have

$$\|\epsilon(\boldsymbol{u_i} - I_{h,i}^*\boldsymbol{u_i})\|_{0,T_i}^2 \le |E_i^2\boldsymbol{u_i} - I_h E_i^2\boldsymbol{u_i}|_{1,T}^2 \le Ch^2\|E_i^2\boldsymbol{u_i}\|_{2,T}^2. \tag{4.3}$$

Similarly, there holds

$$\|p_i - \pi_{h,i}^* p_i\|_{0,T_i}^2 \le Ch_T^2\|E_i^1 p_i\|_{1,T}^2. \tag{4.4}$$

Summing over all the triangles $T \in \mathcal{T}_{h,i}$ in (4.3) and (4.4), using the approximation property of I_h and the property of the extension operator, we get

$$\sum_{i=1}^{2}\sum_{T\in\mathcal{T}_{h,i}} \|\epsilon(\boldsymbol{u_i} - I_{h,i}^*\boldsymbol{u_i})\|_{0,T_i}^2 \le C\sum_{i=1}^{2} h^2\|E_i^2\boldsymbol{u_i}\|_{2,\Omega}^2 \le C\sum_{i=1}^{2} h^2\|\boldsymbol{u_i}\|_{2,\Omega_i}^2, \tag{4.5}$$

and

$$\sum_{i=1}^{2}\sum_{T\subset\Omega_i} \|p_i - \pi_{h,i}^* p_i\|_{0,T_i}^2 \le \sum_{i=1}^{2}\sum_{T\subset\Omega_i} Ch_T^2\|E_i^1 p_i\|_{1,T}^2 \le \sum_{i=1}^{2} Ch^2\|p_i\|_{1,\Omega_i}^2. \tag{4.6}$$

Next, we consider the jumps on the interface. By Lemma 3.3, the inverse inequality and the property of the extension operator, we derive

$$\begin{aligned}\|[\boldsymbol{u} - I_h^*\boldsymbol{u}]\|_{\frac{1}{2},h,\Gamma}^2 &\le \sum_{i=1}^{2}\sum_{T\in G_h} (h_T^{-2}\|E_i^2\boldsymbol{u_i} - I_h E_i^2\boldsymbol{u_i}\|_{0,T}^2 + |E_i^2\boldsymbol{u_i} - I_h E_i^2\boldsymbol{u_i}|_{1,T}^2) \\ &\le \sum_{i=1}^{2}\sum_{T\in G_h} Ch_T^2\|E_i^2\boldsymbol{u_i}\|_{2,T}^2 \le \sum_{i=1}^{2} Ch^2\|\boldsymbol{u_i}\|_{2,\Omega_i}^2,\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}\|\{p - \pi_h^* p\}\|_{-\frac{1}{2},h,\Gamma}^2 &\le C\sum_{i=1}^{2}\sum_{T\in G_h} (\|E_i^1 p_i - \pi_{h,i}E_i^1 p_i\|_{0,T}^2 + h_T^2|E_i^1 p_i - \pi_{h,i}E_i^1 p_i|_{1,T}^2) \\ &\le C\sum_{i=1}^{2}\sum_{T\in G_h} Ch_T^2\|E_i^1 p_i\|_{1,T}^2 \le C\sum_{i=1}^{2} Ch^2\|p_i\|_{1,\Omega_i}^2.\end{aligned} \tag{4.8}$$

According to Lemma 3.3, it holds

$$\|\nabla \boldsymbol{w} \cdot \boldsymbol{n}\|_{0,\Gamma_T}^2 \le C(h_T^{-1}|\boldsymbol{w}|_{1,T}^2 + h_T|\boldsymbol{w}|_{2,T}^2), \quad \forall \boldsymbol{w} \in H^2(T). \tag{4.9}$$

Therefore, combining (4.9) with the approximation property and stability of I_h, we deduce

$$\begin{aligned}\|\{\epsilon(\boldsymbol{u} - I_h^*\boldsymbol{u})\boldsymbol{n}\}\|_{-\frac{1}{2},h,\Gamma}^2 &\le C\sum_{i=1}^{2}\sum_{T\in G_h} h_T\|\epsilon(E_i^2\boldsymbol{u_i} - I_h E_i^2\boldsymbol{u_i})\boldsymbol{n}\|_{0,\Gamma_T}^2 \\ &\le C\sum_{i=1}^{2}\sum_{T\in G_h} (|E_i^2\boldsymbol{u_i} - I_h E_i^2\boldsymbol{u_i}|_{1,T}^2 + h_T^2|E_i^2\boldsymbol{u_i} - I_h E_i^2\boldsymbol{u_i}|_{2,T}^2) \\ &\le C\sum_{i=1}^{2}\sum_{T\in G_h} h_T^2\|E_i^2\boldsymbol{u_i}\|_{2,T}^2 \le C\sum_{i=1}^{2} h^2\|\boldsymbol{u_i}\|_{2,\Omega_i}^2.\end{aligned} \tag{4.10}$$

Together with (4.5), (4.7) and (4.10), we get

$$\|\!|\boldsymbol{u} - I_h^*\boldsymbol{u}|\!\| \le Ch\|\mu_{\max}^{\frac{1}{2}}\boldsymbol{u}\|_{2,\Omega_1\cup\Omega_2}, \tag{4.11}$$

The desired result follows from (4.6), (4.8) and (4.11). □

Theorem 4.2. *Assume* $(\boldsymbol{u},p)\in\left(H^2(\Omega_1\cup\Omega_2)\cap H_0^1(\Omega)\right)^2\times\left(H^1(\Omega_1\cup\Omega_2)\cap L_0^2(\Omega)\right)$ *be the solution of the elasticity interface problem (2.2), and* $(\boldsymbol{u_h},p_h)\in\boldsymbol{V_h}\times Q_h$ *be the solution of the discrete problem (2.3). It is true that*

$$|||(\boldsymbol{u}-\boldsymbol{u_h},p-p_h)|||\le Ch(\|\mu_{\max}^{\frac{1}{2}}\boldsymbol{u}\|_{2,\Omega_1\cup\Omega_2}+\|\mu^{-\frac{1}{2}}p\|_{1,\Omega_1\cup\Omega_2}), \tag{4.12}$$

where C *is a positive constant independent of* μ_1, μ_2 *and how the interface* Γ *intersects the triangulation.*

Proof. Note that

$$|||(\boldsymbol{u}-\boldsymbol{u_h},p-p_h)|||\le|||(\boldsymbol{u}-I_h^*\boldsymbol{u},p-\pi_h^*p)|||+|||(\boldsymbol{u_h}-I_h^*\boldsymbol{u},p_h-\pi_h^*p)|||_h, \tag{4.13}$$

by Theorem 4.1, we only need to deal with $|||(I_h^*\boldsymbol{u}-\boldsymbol{u_h},\pi_h^*p-p_h)|||_h$. Let $\boldsymbol{\xi}_h=\boldsymbol{u_h}-I_h^*\boldsymbol{u},\ \zeta_h=p_h-\pi_h^*p,\ \boldsymbol{\xi}=\boldsymbol{u}-I_h^*\boldsymbol{u},\ \zeta=p-\pi_h^*p$. By Theorem 3.1 and Lemma 3.1, there holds

$$\begin{aligned}|||(\boldsymbol{\xi}_h,\zeta_h)|||_h\le&C\sup_{(\boldsymbol{v_h},q_h)\in\boldsymbol{V_h}\times Q_h}\frac{B_h[(\boldsymbol{\xi}_h,\zeta_h),(\boldsymbol{v}_h,q_h)]+\gamma_1J_p(\zeta_h,q_h)+\gamma_2J_u(\boldsymbol{\xi}_h,\boldsymbol{v}_h)}{|||(\boldsymbol{v_h},q_h)|||_h}\\ \le&C\sup_{(\boldsymbol{v_h},q_h)\in\boldsymbol{V_h}\times Q_h}\frac{B_h[(\boldsymbol{\xi},\zeta),(\boldsymbol{v}_h,q_h)]-\gamma_1J_p(\pi_h^*p,q_h)-\gamma_2J_u(I_h^*\boldsymbol{u},\boldsymbol{v}_h)}{|||(\boldsymbol{v_h},q_h)|||_h}.\end{aligned} \tag{4.14}$$

Applying the Cauchy-Schwarz inequality and Lemma 3.6, we have

$$\begin{aligned}B_h[(\boldsymbol{u}-I_h^*\boldsymbol{u},p-\pi_h^*p),(\boldsymbol{v}_h,q_h)]\le&|||(\boldsymbol{u}-I_h^*\boldsymbol{u},p-\pi_h^*p)|||\,|||(\boldsymbol{v_h},q_h)|||\\ \le&|||(\boldsymbol{u}-I_h^*\boldsymbol{u},p-\pi_h^*p)|||\,|||(\boldsymbol{v_h},q_h)|||_h.\end{aligned} \tag{4.15}$$

Let $\boldsymbol{E}^2\boldsymbol{u}=(E_1^2\boldsymbol{u}_1,E_2^2\boldsymbol{u}_2)$, by the Cauchy-Schwarz inequality, the properties of L^2 projection, it holds

$$\begin{aligned}J_u(I_h^*\boldsymbol{u},\boldsymbol{v_h})=&J_u(I_h^*\boldsymbol{u}-\boldsymbol{E}^2\boldsymbol{u},\boldsymbol{v_h})+J_u(\boldsymbol{E}^2\boldsymbol{u},\boldsymbol{v_h})\\ \le&\sum_{i=1}^{2}\sum_{l=1}^{\mathcal{N}_{l,i}}\mu_ih_{\mathcal{P}_{l,i}}^{-2}\Big(\|(E_i^2\boldsymbol{u_i}-I_{h,i}^*\boldsymbol{u_i})-\Pi_{l,i}(E_i^2\boldsymbol{u_i}-I_{h,i}^*\boldsymbol{u_i})\|_{0,\mathcal{P}_{l,i}}\\ &+\|E_i^2\boldsymbol{u_i}-\Pi_{l,i}E_i^2\boldsymbol{u_i}\|_{0,\mathcal{P}_{l,i}}\Big)\|\boldsymbol{v_{h,i}}-\Pi_{l,i}\boldsymbol{v_{h,i}}\|_{0,\mathcal{P}_{l,i}}\\ \le&\sum_{i=1}^{2}\sum_{l=1}^{\mathcal{N}_{l,i}}\mu_ih_{\mathcal{P}_{l,i}}^{-2}ch^2\|E_i^2\boldsymbol{u_i}\|_{2,\mathcal{P}_{l,i}}\cdot ch\|\nabla\boldsymbol{v_{h,i}}\|_{0,\mathcal{P}_{l,i}}\\ \le&ch\|\mu_{\max}^{\frac{1}{2}}\boldsymbol{u}\|_{2,\Omega_1\cup\Omega_2}|||(\boldsymbol{v_h},q_h)|||_h.\end{aligned} \tag{4.16}$$

Using the Cauchy-Schwarz inequality, we get

$$J_p(\pi_h^*p,q_h)\le\sum_{i=1}^{2}\mu_i^{-1}h^3\sum_{e\in\mathcal{F}_i}\|[\nabla\pi_{h,i}^*p_i]\|_{0,e}\|[\nabla q_{h,i}]\|_{0,e}\le J_p(\pi_h^*p,\pi_h^*p)^{\frac{1}{2}}|||(\boldsymbol{v_h},q_h)|||_h. \tag{4.17}$$

By the inverse inequality and stability of π_h, it follows

$$\begin{aligned}J_p(\pi_h^*p,\pi_h^*p)=&\sum_{i=1}^{2}\sum_{e\in\mathcal{F}_i}\mu_i^{-1}h^3\|[\nabla\pi_{h,i}^*p_i]\|_{0,e}^2\le ch^2\sum_{i=1}^{2}\mu_i^{-1}\sum_{e\in\mathcal{F}_i}\|\nabla\pi_{h,i}^*p_i\|_{0,T_e^+\cup T_e^-}^2\\ \le&ch^2\sum_{i=1}^{2}\mu_i^{-1}\sum_{T\in\mathcal{T}_h}\|\pi_hE_i^1p_i\|_{1,T}^2\le ch^2\sum_{i=1}^{2}\mu_i^{-1}\sum_{T\in\mathcal{T}_h}\|E_i^1p_i\|_{1,T}^2\\ \le&ch^2\mu_i^{-1}\|p\|_{1,\Omega_1\cup\Omega_2}^2,\end{aligned} \tag{4.18}$$

where T_e^+,T_e^- are two elements sharing an edge e.

Together with (4.13)-(4.18), we get the desired result. □

5. Numerical experiments

In this section, we are going to present two numerical experiments to demonstrate our method. In the following tests, we choose the penalty parameters $\gamma_1 = \gamma_2 = 1,\ \gamma_u = 10$ in the model problem (2.1).

5.1. Example 1

In this example, we consider the straight interface $\Gamma : x - \frac{\pi}{8} = 0$ and take the exact solution

$$\boldsymbol{u}(x,y) = \begin{cases} \begin{pmatrix} \frac{1}{\mu_1}x(x-1)y(y-1)(x-\frac{\pi}{8})^2 \\ \frac{1}{\mu_1}x(x-1)y(y-1)(x-\frac{\pi}{8})^2 \end{pmatrix} & \text{in } \Omega_1, \\ \begin{pmatrix} \frac{1}{\mu_2}x(x-1)y(y-1)(x-\frac{\pi}{8})^2 \\ \frac{1}{\mu_1}x(x-1)y(y-1)(x-\frac{\pi}{8})^2 \end{pmatrix} & \text{in } \Omega_2, \end{cases}$$

in the domain $\Omega = (0,1)\times(0,1)$. The domain Ω is divided by the interface into two sub-domains $\Omega_1 = \{(x,y)\in\Omega : x > \frac{\pi}{8}\}$ and $\Omega_2 = \{(x,y)\in\Omega : x \le \frac{\pi}{8}\}$.

Table 1. The finite element errors with $\mu_1 = 40, \mu_2 = 4, \lambda_1 = 80, \lambda_2 = 8$.

n	$\lvert\boldsymbol{u}-\boldsymbol{u_h}\rvert_{H^1}$	order	$\lVert\boldsymbol{u}-\boldsymbol{u_h}\rVert_{L^2}$	order	$\lVert p-p_h\rVert_{L^2}$	order
8	2.6190e-03		8.8870e-05		6.1961e-03	
16	1.3311e-03	0.9764	2.1896e-05	2.0210	2.0403e-03	1.6026
32	6.6785e-04	0.9950	5.3448e-06	2.0345	6.9453e-04	1.5547
64	3.3421e-04	0.9988	1.3748e-06	1.9589	2.4402e-04	1.5090

Table 2. The finite element errors with $\mu_1 = 4, \mu_2 = 40, \lambda_1 = 8, \lambda_2 = 80$.

n	$\lvert\boldsymbol{u}-\boldsymbol{u_h}\rvert_{H^1}$	order	$\lVert\boldsymbol{u}-\boldsymbol{u_h}\rVert_{L^2}$	order	$\lVert p-p_h\rVert_{L^2}$	order
8	1.2791e-03		4.2218e-05		6.0436e-03	
16	6.3771e-04	1.0041	1.0651e-05	1.9869	2.0372e-03	1.5688
32	3.2067e-04	0.9918	2.7102e-06	1.9745	6.9444e-04	1.5527
64	1.6167e-04	0.9880	6.6027e-07	2.0373	2.4433e-04	1.5070

Table 3. The finite element errors with $\mu_1 = 400, \mu_2 = 4, \lambda_1 = 800, \lambda_2 = 8$.

n	$\lvert\boldsymbol{u}-\boldsymbol{u_h}\rvert_{H^1}$	order	$\lVert\boldsymbol{u}-\boldsymbol{u_h}\rVert_{L^2}$	order	$\lVert p-p_h\rVert_{L^2}$	order
8	2.6308e-03		9.3428e-05		6.6212e-03	
16	1.3318e-03	0.9821	2.2311e-05	2.0661	2.1151e-03	1.6464
32	6.6715e-04	0.9973	5.3365e-06	2.0638	6.9768e-04	1.6001
64	3.3385e-04	0.9988	1.3825e-06	1.9486	2.4484e-04	1.5107

Table 4. The finite element errors with $\mu_1 = 4, \mu_2 = 400, \lambda_1 = 8, \lambda_2 = 800$.

n	$\lvert\boldsymbol{u}-\boldsymbol{u_h}\rvert_{H^1}$	order	$\lVert\boldsymbol{u}-\boldsymbol{u_h}\rVert_{L^2}$	order	$\lVert p-p_h\rVert_{L^2}$	order
8	1.2059e-03		4.1114e-05		6.0709e-03	
16	6.2263e-04	0.9537	1.0454e-05	1.9756	2.0484e-03	1.5674
32	3.1401e-04	0.9876	2.7131e-06	1.9460	6.9982e-04	1.5494
64	1.5739e-04	0.9965	6.4646e-07	2.0693	2.4545e-04	1.5116

Table 5. The finite element errors with $\mu_1 = 4000, \mu_2 = 4, \lambda_1 = 8000, \lambda_2 = 8$.

n	$\vert\boldsymbol{u}-\boldsymbol{u_h}\vert_{H^1}$	order	$\Vert\boldsymbol{u}-\boldsymbol{u_h}\Vert_{L^2}$	order	$\Vert p-p_h\Vert_{L^2}$	order
8	2.6504e-03		9.8701e-05		7.5144e-03	
16	1.3341e-03	0.9903	2.2849e-05	2.1109	2.2377e-03	1.7476
32	6.6723e-04	0.9996	5.3403e-06	2.0971	7.0191e-04	1.6727
64	3.3384e-04	0.9990	1.3813e-06	1.9509	2.4431e-04	1.5226

Table 6. The finite element errors with $\mu_1 = 4, \mu_2 = 4000, \lambda_1 = 8, \lambda_2 = 8000$.

n	$\vert\boldsymbol{u}-\boldsymbol{u_h}\vert_{H^1}$	order	$\Vert\boldsymbol{u}-\boldsymbol{u_h}\Vert_{L^2}$	order	$\Vert p-p_h\Vert_{L^2}$	order
8	1.2055e-03		4.1110e-05		6.0754e-03	
16	6.2273e-04	0.9529	1.0467e-05	1.9736	2.0592e-03	1.5609
32	3.1511e-04	0.9827	2.8950e-06	1.8543	8.2469e-04	1.3202
64	1.5735e-04	1.0019	6.4564e-07	2.1647	2.5308e-04	1.7042

In Tables 1-6, we list the errors with respect to different mesh sizes $\left(h = \frac{1}{n},\ n = 8,\ 16,\ 32,\ 64\right)$ and different pairs of Lamé constants $(\mu_1, \mu_2, \lambda_1, \lambda_2)$=(40, 4, 80, 8), (4, 40, 8, 80), (400, 4, 800, 8), (4, 400, 8, 800), (4000, 4, 8000, 8), (4, 4000, 8, 8000). From Tables 1-6, we can observe that the convergence of $\boldsymbol{u}-\boldsymbol{u_h}$ in the energy norm and $p - p_h$ in the norm $\|\cdot\|_0$ are optimal. Moreover, the convergence of $\boldsymbol{u}-\boldsymbol{u_h}$ in the norm $\|\cdot\|_0$ is second order. All the numerical results mentioned in the above are optimal independent of the quotient $\frac{\mu_1}{\mu_2}$.

5.2. Example 2

In this part, we consider a circular interface $\Gamma : x^2 + y^2 - r_0^2 = 0$, with $r_0 = \frac{\pi}{8}$. The exact solution is given by

$$u(x,y) = \begin{cases} \begin{pmatrix} \frac{1}{\lambda_1}(x^2+y^2)^{\frac{\alpha_1}{2}} \\ \frac{1}{\lambda_1}(x^2+y^2)^{\frac{\alpha_2}{2}} \end{pmatrix} & \text{in } \Omega_1, \\ \begin{pmatrix} \frac{1}{\lambda_2}(x^2+y^2)^{\frac{\alpha_1}{2}} + (\frac{1}{\lambda_1} - \frac{1}{\lambda^+})^{r_0^{\alpha 1}} \\ \frac{1}{\lambda_2}(x^2+y^2)^{\frac{\alpha_2}{2}} + (\frac{1}{\lambda_1} - \frac{1}{\lambda^+})^{r_0^{\alpha 2}} \end{pmatrix} & \text{in } \Omega_2, \end{cases}$$

with $\alpha_1 = 5, \alpha_2 = 7$, in the domain $\Omega = (-1,1)\times(-1,1)$, where $\Omega_1 = \{(x,y) : x^2 + y^2 \geq r_0^2\}$ and $\Omega_2 = \{(x,y) : x^2 + y^2 < r_0^2\}$,

We vary the Lamé parameters λ_1, μ_1 and λ_2, μ_2, so that we can test problems with different discontinuities in Lamé parameters. From Tables 7-9, we can observe the same convergence holds as Example 1, which is consistent with our theoretical analysis.

Table 7. The finite element errors with $\mu_1 = 40, \mu_2 = 4, \lambda_1 = 80, \lambda_2 = 8$.

n	$\vert\boldsymbol{u}-\boldsymbol{u_h}\vert_{H^1}$	order	$\Vert\boldsymbol{u}-\boldsymbol{u_h}\Vert_{L^2}$	order	$\Vert p-p_h\Vert_{L^2}$	order
8	9.5374e-01		7.7226e-02		1.6847e+00	
16	4.9385e-01	0.9495	2.0001e-02	1.9490	6.5002e-01	1.3740
32	2.4657e-01	1.0020	4.7555e-03	2.0724	2.2391e-01	1.5376
64	1.2338e-01	0.9988	1.1841e-03	2.0058	8.1079e-02	1.4655

Table 8. The finite element errors with $\mu_1 = 400, \mu_2 = 4, \lambda_1 = 800, \lambda_2 = 8$.

n	$\|\boldsymbol{u}-\boldsymbol{u_h}\|_{H^1}$	order	$\|\boldsymbol{u}-\boldsymbol{u_h}\|_{L^2}$	order	$\|p-p_h\|_{L^2}$	order
8	9.5356e-01		7.7166e-02		1.7114e+00	
16	4.9030e-01	0.9597	1.8937e-02	2.0268	6.5731e-01	1.3805
32	2.4653e-01	0.9919	4.7613e-03	1.9918	2.2555e-01	1.5432
64	1.2338e-01	0.9987	1.1762e-03	2.0172	8.1107e-02	1.4755

Table 9. The finite element errors with $\mu_1 = 4000, \mu_2 = 4, \lambda_1 = 8000, \lambda_2 = 8$.

n	$\|\boldsymbol{u}-\boldsymbol{u_h}\|_{H^1}$	order	$\|\boldsymbol{u}-\boldsymbol{u_h}\|_{L^2}$	order	$\|p-p_h\|_{L^2}$	order
8	9.5401e-01		7.7154e-02		1.9592e+00	
16	4.9004e-01	0.9611	1.8955e-02	2.0251	6.3211e-01	1.6320
32	2.4653e-01	0.9911	4.7383e-03	2.0002	3.2082e-01	0.9784
64	1.2338e-01	0.9986	1.1677e-03	2.0207	8.3995e-02	1.9334

6. Conclusions

In this paper, we introduce the extended mixed finite element method for the elasticity interface problem. The stabilization terms defined on edges of interface elements ensure that the discrete inf-sup condition holds and the optimal a priori error estimates are obtained. By adding the ghost penalty term, we prove that the system matrix of the method is well-conditioned. Numerical experiments are also given to demonstrate our theoretical results.

As far as I know, there are mainly two ways to solve the "locking" of the elasticity problem: the nonconforming finite element method and the mixed finite element method. As we have tried to solve the problem with the mixed finite element method, we shall try to use the nonconforming finite element method to solve the elasticity interface problem in the future.

References

[1] R. A. Adams and J. J. F. Fournier. *Sobolev Spaces.* Academic Press, New York, 2003.

[2] R. Becker, E. Burman, and P. Hansbo. A Nitsche extended finite element method for incompressible elasticity with discontinuous modulus of elasticity. *Computer Methods in Applied Mechanics and Engineering*, 198:3352–3360, 2009.

[3] T. Belytschko and T. Black. Elastic crack growth in finite elements with minimal remeshing. *International Journal for Numerical Methods in Engineering*, 45:601–620, 1999.

[4] S. C. Brenner. Korn's inequalities for piecewise H^1 vector fields. *Mathematics of Computation*, 73:1067–1087, 2003.

[5] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Method.* Springer-Verlag, New York, 1991.

[6] E. Burman. Ghost penalty. *Comptes Rendus Mathematique*, 348:1217–1220, 2010.

[7] L. Cattaneo, L. Formaggia, G. F. Iori, A. Scotti, and P. Zunino. Stabilized extended finite elements for the approximation of saddle point problems with unfitted interfaces. *Calcolo*, 52:123–152, 2015.

[8] Y. Chen, S. Hou, and X. Zhang. A bilinear partially penalized immersed finite element method for elliptic interface problems with multi-domain and triple-junction points. *Results in Applied Mathematics*, 8:100100, 2020.

[9] L. C. Evans. *Partial Differential Equations.* American Mathematical Society, Providence, 1998.

[10] T.-P. Fries and T. Belytschko. The extended/generalized finite element method: An overview of the method and its applications. *International Journal for Numerical Methods in Engineering*, 84:253–304, 2010.

[11] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations.* Springer-Verlag, Berlin, Heidelberg, 1986.

[12] Y. Gong, B. Li, and Z. Li. Immersed-interface finite-element methods for elliptic interface problems with nonhomogeneous jump conditions. *SIAM Journal on Numerical Analysis*, 46:472–495, 2008.

[13] S. Gross and A. Reusken. *Numerical Methods for Two-phase Incompressible Flows.* Springer, Berlin, 2011.

[14] R. Guo, T. Lin, Y. Lin, and Q. Zhuang. Error analysis of symmetric linear/bilinear partially penalized immersed finite element methods for Helmholtz interface problems. *Journal of Computational and Applied Mathematics*, 390:113378, 2021.

[15] A. Hansbo and P. Hansbo. An unfitted finite element method, based on Nitsche's method, for elliptic interface problems. *Computer Methods in Applied Mechanics and Engineering*, 191:5537–5552, 2002.

[16] P. Hansbo, M. G. Larson, and S. Zahedi. A cut finite element method for a Stokes interface problem. *Applied Numerical Mathematics*, 85:90–114, 2014.

[17] G. Jo and D. Y. Kwak. A reduced Crouzeix-Raviart immersed finite element method for elasticity problems with interfaces. *Computational Methods in Applied Mathematics*, 20:501–516, 2019.

[18] M. Kirchhart, S. Gross, and A. Reusken. Analysis of an XFEM discretization for Stokes interface problems. *SIAM Journal on Scientific Computing*, 38:A1019–A1043, 2016.

[19] Z. Li. The immersed interface method using a finite element formulation. *Applied Numerical Mathematics*, 27:253–267, 1998.

[20] Z. Li, T. Lin, and X. Wu. New Cartesian grid methods for interface problems using the finite element formulation. *Numerische Mathematik*, 96:61–98, 2003.

[21] T. Lin, Y. Lin, and X. Zhang. Partially penalized immersed finite element methods for elliptic interface problems. *SIAM Journal on Numerical Analysis*, 53:1121–1144, 2015.