Equal and odd values of Generalized Euler Functions

Abstract： Euler function $\varphi(n)$ and generalized Euler function $\varphi_e(n)$ are two important functions in number theory. Using the idea of classified discussion and determination of prime types, we study the solutions of odd number of generalized Euler function equations $\varphi_e(n)=\varphi_e(n+1)$ and obtain all the solutions satisfying the corresponding conditions, where $e=2,3,4$.

Key Words： Euler function；Generalized Euler function；Parity；Diophantine equation

1 Introduction

Euler function $\varphi(n)$ is a relatively important in number theory, and it is also studied by the majority of researchers. Euler function $\varphi(n)$ is defined as the number of positive integers not greater than n and prime to n. If $n>1$，let canonical form of n be $n=p_1^{r_1}p_2^{r_2}...p_k^{r_k}$， where $p_1,p_2,...,p_k$ are different primes，$r_i\ge 1\ (1\le i\le k)$，then

$$\varphi(n)=n(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_k}).$$

Generalized Euler function $\varphi_e(n)$ is defined as

$$\varphi_e(n)=\sum_{\substack{i=1\\(i,n)=1}}^{\left[\frac{n}{e}\right]}1.$$

where $[x]$ is the greatest integer not greater than x. If $e=1$，the generalized Euler function is just Euler function.

Cai[1,8] studied the parity of $\varphi_e(n)$ when $e=2,3,4,6$, and gives the conditions that both $\varphi_e(n)$ and $\varphi_e(n+1)$ are odd numbers，Liang[3], Cao[2] studied the solutions to the equations involving Euler function，Zhang[4,5,6] investigated the solutions to two equations involving Euler function $\varphi(n)$ and generalized Euler function $\varphi_2(n)$, Jiang[7] investigated the solutions of generalized Euler function $\varphi_3(n)$.

In《Unsolved Problems in Number Theory》[13], proposing whether there are infinitely many pairs of consecutive integer pairs n and $n+1$ such that $\varphi(n)=\varphi(n+1)$? Jud McGranie found 1267 solutions to $\varphi(n)=\varphi(n+1)$ whit $n\le 10^{10}$, and the largest of which is $n=9985705$, $\varphi(n)=\varphi(n+1)=2^{11}3^{5}7\cdot 11$ We find the following conclusions on the basis of the fact that the documents [1] and [8], both $\varphi_e(n)$ and $\varphi_e(n+1)$ are odd numbers, and then obtain the solutions of the equation $\varphi_e(n)=\varphi_e(n+1)$.

Theorem 1.1 Both $\varphi_2(n)$ and $\varphi_2(n+1)$ are odd and equal if and only if $n=2$or 3.

Theorem 1.2 Both $\varphi_3(n)$ and $\varphi_3(n+1)$ are odd and equal if and only if $n=3$ or 4 or 5 or 15.

Theorem 1.3 Both $\varphi_4(n)$ and $\varphi_4(n+1)$ are odd and equal if and only if $n=4$ or 5 or 6 or 7.

2 Lemmas

Lemma 2.1[1] Except for $n=2,3,242$, both $\varphi_2(n)$ and $\varphi_2(n+1)$ are odd if and only if $n=2p^{\beta}$, where $\beta\ge 1, p\equiv 3 \pmod 4$, both $2p^{\beta}+1$ and p are primes.

Lemma 2.2[12] $\varphi_2(1)=$, $\varphi_2(2)=1$; when $n\ge 3$, $\varphi_2(n)=\frac{1}{2}\varphi(n)$.

Lemma 2.3[1] Except for $n=3,15,24$, both $\varphi_3(n)$ and $\varphi_3(n+1)$ are odd if and only if

（1）$n+1=2^{2^m}+1(m\ge 1)$ is prime; or

(2) $n=2q, q\equiv 5 \pmod 6$, both q and $\frac{2^q+1}{3}$ are primes, where $n=2q, q\equiv 5 \pmod 6$, or

（3）$n=3\cdot 2^{\beta}-1(\beta\ge 1)$ is prime.

Lemma 2.4[1] If $n>3$, $n=3^{\alpha}\prod_{i=1}^{k} p_i^{\alpha_i}, (p_i,3)=1, 1\le i\le k$, then

$$\varphi_3(n)=\begin{cases}\frac{1}{3}\varphi(n)+\frac{(-1)^{\Omega(n)}2^{\omega(n)-a-1}}{3}, a=0 \text{ or } 1, p_i\equiv 2 \pmod 3, 1\le i\le k,\\ \frac{1}{3}\varphi(n), \text{ otherwise},\end{cases}$$

where $\Omega(n)$ is the number of prime factors of n(counting repetitions) and $\omega(n)$ is the number of distinct prime factors of n.

Lemma 2.5[12] For any positive integer m, n ， we have

$$\varphi(mn)=\frac{(m,n)\varphi(m)\varphi(n)}{\varphi((m,n))},$$

where (m,n) represents the greatest common factor of m and n. $\varphi(mn)=\varphi(m)\varphi(n)$ when $(m,n)=1$.

Lemma 2.6[8] The value of n such that both $\varphi_4(n)$ and $\varphi_4(n+1)$ are odd are listed in Table 1.

Table 1 The value of n such that both $\varphi_4(n)$ and $\varphi_4(n+1)$ are odd

n	$n+1$	conditions
4	5	
7	8	
57121	57122	
p^2	$2q^2$	$p\equiv 7 \pmod 8, q\equiv 5 \pmod 8$ are primes
$2q^\beta-1$	$2q^\beta$	$2q^\beta-1\equiv 5 \pmod 8, q\equiv 3 \pmod 8$ are primes，and β is prime
$2q^\beta$	$2q^\beta+1$	$2q^\beta+1\equiv 7 \pmod 8, q\equiv 3 \pmod 8$ are primes，and β is prime
p^2	p^2+1	$p\equiv 5 \pmod 8, \frac{p^2+1}{2}\equiv 5 \pmod 8$ are primes
$5^\alpha-1$	5^α	$\frac{5^\alpha-1}{4}\equiv 3 \pmod 4$ is a prime
$4q^\beta$	$4q^\beta+1$	$4q^\beta+1, q\equiv 3 \pmod 4$ are primes， $\beta\ge 1$

Lemma 2.7[8] If $n>4$, $n=2^{a}\prod_{i=1}^{k}p_i^{\alpha_i},(p_i,2)=1,a\geq 0,1\leq i\leq k$, then

$$\varphi_4(n)=\begin{cases}\frac{1}{4}\varphi(n)+\frac{(-1)^{\Omega(n)}2^{\omega(n)-a}}{4},a=0\text{ or }1,p_i\equiv 3(\bmod 4),1\leq i\leq k,\\ \frac{1}{4}\varphi(n),\ \text{otherwise.}\end{cases}$$

3 Proof of Theorems

3.1 Proof of Theorem 1.1

We have $\varphi_2(2)=\varphi_2(3)=\varphi_2(4)=1$ by definition of the generalized Euler function $\varphi_2(n)$, and $\varphi_2(242)=55,\varphi_2(243)=81$ by Lemma 2.2.

By lemma 2.1, except for $n=2,3,242$, both $\varphi_2(n)$ and $\varphi_2(n+1)$ are odd if and only if $n=2p^{\beta}$, where $\beta\geq 1,p\equiv 3(\bmod 4)$, both $2p^{\beta}+1$ and p are primes. By lemma 2.2, When $n\geq 3,\varphi_2(n)=\frac{1}{2}\varphi(n)$, and $\varphi_2(n+1)=\frac{1}{2}\varphi(n+1)$. Then for the equation $\varphi_e(n)=\varphi_e(n+1)$, we just need to solve the equation

$$\varphi(n)=\varphi(n+1). \tag{1}$$

Put $n=2p^{\beta}$, $n+1=2p^{\beta}+1$ in (1), since $n+1=2p^{\beta}+1$ is prime, then $\varphi(n+1)=n$. We just need to solve the equation

$$\varphi(n)=n,$$

and it has only a solution $n=1$, but the solution is not satisfied with the form $n=2p^{\beta}$, so there is no solution.

Hence both $\varphi_2(n)$ and $\varphi_2(n+1)$ are odd and equal if and only if $n=2$ or 3.

3.2 Proof of Theorem 1.2

By the definition of $\varphi_3(n)$, We have

$$\varphi_3(3)=1,\varphi_3(4)=1,\varphi_3(15)=3,\varphi_3(16)=3,\varphi_3(24)=3,\varphi_3(25)=7,$$

hence $\varphi_3(3)=\varphi_3(4),\varphi_3(15)=\varphi_3(16)$. Except $n=3,15,24$, we discuss the solutions in 3 cases by lemma 2.3.

Case 1 When $n=2^{2^m}$, $n+1=2^{2^m}+1(m\geq1)$, and $n+1=2^{2^m}+1(m\geq1)$ is prime, by lemma 2.4, we have

$$\varphi_3(n)=\frac{1}{3}\varphi(n)+\frac{1}{3}.$$

Since $n+1=2^{2^m}+1$ is prime and $n+1\equiv2\pmod 3$, we have

$$\varphi_3(n+1)=\frac{1}{3}\varphi(n+1)-\frac{1}{3}.$$

If $\varphi_3(n)=\varphi_3(n+1)$, then

$$\frac{1}{3}\varphi(n)+\frac{1}{3}=\frac{1}{3}\varphi(n+1)-\frac{1}{3}.$$

Simplify it, we obtain $2^{2^m-1}+1=2^{2^m}-1$, thus we have $m=1$, $n=4$.

Case 2 When $n=2^q, n=2^q+1$, and both $q\equiv5\pmod 6$, $\frac{2^q+1}{3}$ are primes, by lemma 2.4, we have

$$\varphi_3(n)=\frac{1}{3}\varphi(n)-\frac{1}{3}.$$

Since $\frac{2^q+1}{3}$ is prime, $q\equiv5\pmod 6$ and $\varphi(9)=6$, we have

$$2^q+1\equiv2^5+1\equiv33\pmod 9,$$

thus $\frac{2^q+1}{3}\equiv11\equiv2\pmod 3$. $n+1=3\times\frac{2^q+1}{3}$, then by lemma 2.4, we obtain

$$\varphi_3(n+1)=\frac{\varphi(n+1)}{3}+\frac{1}{3}.$$

If $\varphi_3(n)=\varphi_3(n+1)$, then $\varphi(n)=\varphi(n+1)+2$, namely

$$2^q\cdot(1-\frac{1}{2})=2\times(\frac{2^q+1}{3}-1)+2,$$

simplified to $2^q=-4$, we have no solutions in this case.

Case 3 When $n=3\cdot2^\beta-1$, $n+1=3\cdot2^\beta$, and $n=3\cdot2^\beta-1(\beta\geq1)$ is prime, by lemma 2.4, we have

$$\varphi_3(n)=\frac{1}{3}\varphi(n)-\frac{1}{3},$$

meanwhile，

$$\varphi_3(n+1)=\frac{1}{3}\varphi(n+1)+\frac{(-1)^{1+\beta}2^{\alpha(n)-\alpha-1}}{3}=\frac{1}{3}\varphi(n+1)+\frac{(-1)^{1+\beta}}{3}.$$

If $\beta=2k,k>0$

$$\frac{1}{3}\varphi(n)-\frac{1}{3}=\frac{1}{3}\varphi(n+1)-\frac{1}{3},$$

simplified to $\varphi(n)=\varphi(n+1)$. Since $n=3\cdot 2^{\beta}-1(\beta\geq 1)$ is prime，then

$$3\cdot 2^{\beta}-2=3\cdot 2^{\beta}\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{3}),$$

We get $\beta=0$, this is contradicted with the condition $\beta\geq 1$. If $\beta=2k+1,k\geq 0$,

$$\frac{1}{3}\varphi(n)-\frac{1}{3}=\frac{1}{3}\varphi(n+1)+\frac{1}{3},$$

simplified to $\varphi(n)=\varphi(n+1)+2$，then

$$3\cdot 2^{\beta}-2=3\cdot 2^{\beta}\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{3})+2,$$

We have $\beta=1$，$n=5$.

Sum up，both $\varphi_3(n)$ and $\varphi_3(n+1)$ are odd and equal if and only if $n=3$ or 4 or 5 or 15.

3.3 Proof of Theorem 1.3

By lemma 2.7, we have $\varphi_4(4)=1,\varphi_4(5)=1$ ，$\varphi_4(7)=1,\varphi_4(8)=1$and

$$\varphi_4(57121)=14221,\ \varphi_4(57122)=6591,$$

hence $\varphi_4(4)=\varphi_4(5),\varphi_4(7)=\varphi_4(8)$. Then we discuss the solutions in 6 cases by lemma 2.6.

Case 1 When $n=p^2,n+1=2q^2$, and both $p\equiv 7(\bmod 8),q\equiv 5(\bmod 8)$ are primes. By lemma 2.7, we have $\varphi_4(n)=\frac{1}{4}\varphi(n)+\frac{1}{2}$. Since $q\equiv 1(\bmod 4)$，then $\varphi_4(n+1)=\frac{1}{4}\varphi(n+1),$

namely

$$\frac{1}{4}\varphi(n)+\frac{1}{2}=\frac{1}{4}\varphi(n+1).$$

Simplified to $\varphi(n)+2=\varphi(n+1),$ namely

$$p^2\cdot(1-\frac{1}{p})+2=2q^2\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{q}).$$

Then $q\cdot(q-1)-p\cdot(p-1)=2,$ by $p^2+1\equiv 2q^2$, we have $p=q^2+q+1.$ Then

$$p^2=(q^2+q+1)^2\geq(q^2+q)^2\geq 36q^2>2q^2,$$

which is contradicted with the condition $p^2+1\equiv 2q^2$, no solution.

Case 2 When $n=2q^\beta-1, n+1=2q^\beta$, and both $2q^\beta-1\equiv 5 \pmod 8$, $q\equiv 3 \pmod 8$ are primes, where β is a odd. By lemma 2.7, we have $\varphi_4(n+1)=\frac{1}{4}\varphi(n+1)+\frac{1}{2}.$

Since $2q^\beta-1\equiv 1 \pmod 4$, we have $\varphi_4(n)=\frac{1}{4}\varphi(n),$ namely

$$\frac{1}{4}\varphi(n)=\frac{1}{4}\varphi(n+1)+\frac{1}{2}.$$

Simplified to $\varphi(n)=\varphi(n+1)+2,$ namely

$$(2q^\beta-1)-1=2q^\beta\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{q})+2.$$

Then $(q+1)\cdot q^{\beta-1}=4,$ since both q and $q+1$ are positive integers, and $q\equiv 3 \pmod 8$, so $q+1\geq 4$, then $q=3, \beta=1$, $n=5.$

Case 3 When $n=2q^\beta, n+1=2q^\beta+1$, and both $2q^\beta+1\equiv 7 \pmod 8$, $q\equiv 3 \pmod 8$ are primes, where β is a odd. By lemma 2.7, we have $\varphi_4(n)=\frac{1}{4}\varphi(n)+\frac{1}{2}$ and

$$\varphi_4(n+1)=\frac{1}{4}\varphi(n+1)-\frac{1}{2},$$

then

$$\frac{1}{4}\varphi(n)+\frac{1}{2}=\frac{1}{4}\varphi(n+1)-\frac{1}{2}.$$

Simplified to $\varphi(n)+4=\varphi(n+1)$, namely

$$2q^{\beta}\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{q})+4=2q^{\beta}.$$

Then $(q+1)\cdot q^{\beta-1}=4$, since q and $q+1$ both are positive integers , and $q\equiv 3(\mathrm{mod}\,8)$, so $q+1\geq 4$, then $q=3,\beta=1,\ n=6$.

Case 4 When $n=p^2, n+1=p^2+1$, and both $p\equiv 5(\mathrm{mod}\,8)$, $\frac{p^2+1}{2}\equiv 5(\mathrm{mod}\,8)$ are primes. By lemma 2.7, we have $\varphi_4(n)=\frac{1}{4}\varphi(n)$ and

$$\varphi_4(n+1)=\frac{1}{4}\varphi(n+1).$$

When $\varphi_4(n)=\varphi_4(n+1)$, we have

$$\frac{1}{4}\varphi(n)=\frac{1}{4}\varphi(n+1).$$

Simplified to

$$\mathrm{p}^2\cdot(1-\frac{1}{p})=\frac{p^2+1}{2}-1,$$

then $p=1$. Which contradicts $p\equiv 5(\mathrm{mod}\,8)$.

Case 5 When $n=5^{\alpha}-1, n+1=5^{\alpha}$, and $\frac{5^{\alpha}-1}{4}\equiv 3(\mathrm{mod}\,4)$ is a prime, then $n=4\cdot\frac{5^{\alpha}-1}{4}=2^2\cdot\frac{5^{\alpha}-1}{4}$. By lemma 2.7, we have $\varphi_4(n)=\frac{1}{4}\varphi(n)$ and

$$\varphi_4(n+1)=\frac{1}{4}\varphi(n+1),$$

namely $\frac{1}{4}\varphi(n)=\frac{1}{4}\varphi(n+1)$, simplified to $\varphi(n)=\varphi(n+1)$, i.e., $2\cdot(\frac{5^{\alpha}-1}{4}-1)=5^{\alpha}\cdot\frac{4}{5}$,

Then $5^{\alpha}=-\frac{25}{3}$, which is impossible.

Case 6 When $n=4q^{\beta}, n+1=4q^{\beta}+1$, and both $4q^{\beta}+1, q\equiv 3(\mathrm{mod}\,4)$ are primes, where $\beta\geq 1$.

By lemma 2.7, we have $\varphi_4(n)=\frac{1}{4}\varphi(n)$ and $\varphi_4(n+1)=\frac{1}{4}\varphi(n+1)$, namely

$$\frac{1}{4}\varphi(n)=\frac{1}{4}\varphi(n+1).$$

Simplified to $\varphi(n)=\varphi(n+1)$, namely

$$4q^{\beta}\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{q})=4q^{\beta}.$$

Then $q=-1$. Which contradicts the condition that $q\equiv 3(\bmod 4)$ is a prime.

Sum up, both $\varphi_4(n)$ and $\varphi_4(n+1)$ are odd and equal if and only if $n=4$ or 5 or 6 or 7.

4 Expectation

Euler function $\varphi(n)$ and generalized Euler function $\varphi_e(n)$ are two important functions in number theory. which this article has studied is the odd solutions of generalized Euler function equation $\varphi_e(n)=\varphi_e(n+1)$, where $e=2,3,4$. Similarly, we can use a similar method to study the odd solutions of $\varphi_6(n)=\varphi_6(n+1)$in combination with the relevant conclusions of the literature [8]. In the future, we can study all the solutions of the equations $\varphi_e(n)=\varphi_e(n+1)$ and $\varphi_e(n)=\varphi_e(n+k)$ for positive k further.

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