UNDER PEER REVIEW

Existence of positive solutions for the Kirchhoff type equations involving general critical growth in $\mathbb{R}^N$

Abstract: In this paper, we consider the following nonlinear Kirchhoff type problem

$$-\left(a+\lambda\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u+V(x)u=f(u),\quad x\in\mathbb{R}^N,$$

where $N\geq 3$, a is a positive constant, $\lambda\geq 0$ is a parameter. Under some sufficient assumptions on $V(x)$ and $f(u)$, the existence of positive solution to the above problem is proved by variational methods and Mountain Pass Theorem. Specially, with the aid of a cut-off function and a monotonic trick, we obtain the boundedness of Palais-smale sequences. Our results improve the previous results in the literature.
Keywords: Kirchhoff type equation; Positive solutions; Cut-off function; Variational methods

Mathematics Subject Classification. 35J20; 35J60

1. Introduction

Consider the following Kirchhoff type problem

$$\begin{cases}-(a+\lambda\int_{\mathbb{R}^N}|\nabla u|^2dx)\Delta u+V(x)u=f(u), & x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N),\quad u>0,\end{cases}\tag{1.1}$$

where $N\geq 3$, a is a positive constant, $\lambda\geq 0$ is a parameter, $F(t)=\int_0^t f(s)ds$. $V(x)$ is a positive continuous potential. This type of equation is an extension of the classic d'Alembert wave equations for free vibration of elastic strings, because it takes into account the effects of the strings' length changes during vibration. For the purpose of stating our statement, $V(x)$ and $f(u)$ are assumed to satisfy the following basic assumptions:

(V0) $V(x)\leq\liminf_{|x|\to\infty}V(x):=V_\infty<\infty$ *for any* $x\in\mathbb{R}^N$.

(F1) $f(x)\in C(\mathbb{R}^+,\mathbb{R}^+)$, $\mathbb{R}^+=[0,+\infty)$ *and there exists* $C>0$ *such that* $|f(t)|\leq C(|t|+|t|^{p-1})$ *for all* $t\in\mathbb{R}^+$ *and some* $p\in(2,2^*)$, *where* $2^*=\frac{2N}{N-2}$ *for* $N\geq 3$.

As problem (1.1) involves the term $\int_{\mathbb{R}^N}|\nabla u|^2dx$, it is no longer a local problem, which causes some analytical difficulties. Moreover, in recent years, in purely mathematical research and practical applications, non-local operators have appeared in the description of various phenomena, such as physics and chemistry [1], obstacle problems [2], optimization and finance [3], etc.

If we consider problem (1.1) on a bounded smooth region Ω and $\lambda \equiv b$ is a positive constant, then problem (1.1) becomes to the following problem

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u + V(x)u = f(u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Problem (1.2) originally derived from the following problem

$$u_{tt} - \left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = f(x,u), \tag{1.3}$$

which is a practical equation proposed by Kirchhoff [4] in the process of studying the classical D'Alembert wave equation of the free vibration of the retractable rope. For more details of problem (1.3) in physical aspects, please see [5, 6] and the references therein. The early classical research of Kirchhoff equations is dedicated by Bernstein [6] and Pohožaev [7]. However, the Kirchhoff-type equation was greatly brought into focus only after Lions [8] investigated problem (1.2) involving an abstract framework. For more researches on Kirchhoff-type equations, please refer to the literature [9, 10] and the references therein. Under various conditions on potential $V(x)$ and nonlinearity $f(x)$, the existence, non-existence and multiplicity of problem (1.2) have been studied in the literature by variational methods. On unbounded domains, many existence and multiplicity results are also obtained for problem (1.2). For example, in [11], Xie discussed problem (1.2) with an asymptotically 4-linear nonlinearity f and the existence of a least energy nodal solution for the problem was obtained by variational methods. With the aid of a monotonic technique and a new version of global compactness lemma, Li and Ye [12] proved that problem (1.2) has a positive ground state solution when $f(u) = |u|^{p-1}u$ with $p \in (2,5)$. Chen and Tang [13] obtained the existence of infinitely many high energy solutions for problem (1.2) on $\mathbb{R}^3$ by using Symmetric Mountain Pass Theorem. Especially, they introduced some new techniques to overcome the competitive effect of non-local terms. For more recent results related to Kirchhoff equations, please see e.g. [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24].

Although the Kirchhoff-type Dirichlet problem have been extensively studied, as far as we know, there is very little literature on Kirchhoff-type problems (such as (1.1)), which involved parameters. Very recently, Xu [25] studied the following nonlinear Kirchhoff type equations

$$\begin{cases} -(a+\lambda\int_{\mathbb{R}^n} |\nabla u|^2dx)\Delta u + V(x)u = |u|^{p-1}u, & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \quad u>0, \end{cases} \tag{1.4}$$

where $N \geq 3$, a is a positive constant, λ is a parameter, $1 < p < \max\{3, \frac{N+2}{N-2}\}$ and the potential $V(x)$ satisfies (V_0). We point out that they obtained the existence of at least one positive solution and nonexistence of nontrivial solutions for problem (1.4) by using variational methods and a cut-off technique. Problem (1.4) is the special form of the problem (1.1), in other words, problem (1.1) is more general than problem (1.4). Hence, our results can be seemed as the complementary work of [25].

In [26], Li et al. considered the following nonlinear Kirchhoff type equation

$$\left(a+\lambda\int_{\mathbb{R}^N} |\nabla u|^2dx + \lambda b\int_{\mathbb{R}^N} u^2dx\right)[-\Delta u + bu] = f(u), \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $N \geq 3$, λ is a parameter, a, b are positive constants. When the nonlinearity f is subcritical, the existence of at least one positive radial solution was proved in [26] for $\lambda \geq 0$ is small, by using variational methods and a cut-off functional.

Motivated by the above mentioned literatures, we consider more general nonlinear Kirchhoff type elliptic problem (1.1) in this paper. The main purpose of this paper is to obtain the existence and multiplicity of positive solutions for problem (1.1) when f is subcritical. To prossess this, we must overcome the following three main difficulties: (I) Due to the emergence of the term $\int_{\mathbb{R}^N} |\nabla u|^2 dx$, problem (1.1) is a non-local problem, which means that equation (1.1) is not a point-wise identity. This causes some mathematical difficulties and makes the study of (1.1) more interesting; (II) The second main difficulty comes from subcritical growth. We should show the boundedness of Palais-Smale ((PS) for short) sequences without the Ambrosetti-Rabinowitz ((AR) for short) condition. Indeed, it is not easy to verify the existence of bounded (PS) sequences. In order to overcome this difficulty, we shall adopt the technique used in [26]; (III) We need to get the convergent subsequence of the bounded (PS) sequence. Since V(x) is non-constant, the method used in [25] cannot be applied to our results. We will make some assumptions on $\nabla V(x)$ and use some tricks to solve this problem.

In this paper, due to focusing on the positive solution of (1.1), we assume that $f(s) = 0$ when $s < 0$. Moreover, we shall make the following assumptions.

(V1) $(\nabla V(x), x) \in L^2(\mathbb{R}^N)$, $(\nabla V(x), x) \leq \frac{a\theta}{2x^2}$ *for* $x \in \mathbb{R}^N$, *where* $(\cdot,\cdot)$ *denotes the usual inner product,* $\theta \in (0, 1)$.
(F2) $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$.
(F3) $\lim_{t\to \infty} \frac{f(t)}{t} = \infty$.

Our main results are given in the following.

Theorem 1.1. *Assume that* $N \geq 3$, *a is a positive constant,* $\lambda \geq 0$ *is a parameter. If the conditions* (V0)-(V1) *and* (F1)-(F3) *hold. Then there exists* $\lambda_0 > 0$ *such that for all* $\lambda \in [0, \lambda_0)$, *problem* (1.1) *has at least one positive solution.*

When $\lambda = 0$, $V(x) \equiv b$, the problem (1.1) reduces to

$$-a\Delta u + bu = f(u), \quad in \ \ \mathbb{R}^N. \tag{1.6}$$

Then we have the following corollary.

Corollary 1.2. *Assume* $N \geq 3$ *and b is a positive constant. If the conditions* (F1)-(F3) *hold, then problem* (1.6) *has at least one positive solution.*

Theorem 1.3. *Under the conditions of Theorem 1.1 and* $N > 3$, *there exists* $\lambda_1 > 0$ *such that problem* (1.1) *has no nontrivial solution when* $\lambda \geq \lambda_1$.

Remark 1.4. *In this paper, we considered the existence and multiplicity of positive solutions for the more general Kirchhoff type problems than the work in the literature. We must point out that the authors in [25] and [26] both considered the general nonlinearity f which only involves subcritical growth. Compared to [26], the nonlinearity* $f(u)$ *in our results is more general. Besides, since* $V(x)$ *is not a constant, the methods used in [25] is not suitable for problem* (1.1). *So we use a "monotonic" trick and make assumption on* $\nabla V(x)$ *to obtain our results. In Section 2, the condition* (V1) *guarantees that for enough large* $T > 0$, *there exists* $\lambda_0 > 0$ *such that* $\int_{\mathbb{R}^N} |\nabla u_n|^2 \leq T^2$ *for any* $0 < \lambda < \lambda_0$ *in Lemma 2.9. It is pointed out that our results extend the mentioned results to a certain extent.*

The rest of this paper is organized as follows: in Section 2, some framework are demonstrated. In Section 3, the proofs of the main results are given. In section 4, the conclusion is given. In the following, C_i denotes different positive constants in different spaces.

2. Preliminaries

In this section, we first introduce some marks. Let $H := H^1(\mathbb{R}^N)$ be the usual Sobolev space equipped with the inner product and norm

$$\langle u, v\rangle_H = \int_{\mathbb{R}^N} (\nabla u \nabla v + uv)dx, \quad \|u\|_H = \langle u, u\rangle^{\frac{1}{2}}.$$

We define the working space by

$$E = \left\{ u \in H : \int_{\mathbb{R}^N} V(x)u^2 dx < +\infty \right\}$$

endowed with the inner product and norm

$$\langle u, v\rangle = \int_{\mathbb{R}^N} (a\nabla u \nabla v + V(x)uv)dx, \quad \|u\| = \langle u, u\rangle^{\frac{1}{2}}.$$

The norm of the usual Lebesgue space $L^p(\mathbb{R}^N)$ is denoted by $\|\cdot\|_p$. Since $V(x)$ satisfies the condition (V_0), $\|\cdot\|$ is equivalent to the standard norm $\|u\|_H$ on $H^1(\mathbb{R}^N)$. Since the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for $p \in (2, 6)$, there exists γ_p such that

$$\|u\|_p \leq \gamma_p \|u\|, \quad \forall u \in E.$$

From [27], we can know that the continuous embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $p \in (2, 6)$. In this paper, we consider the existence of positive solution of problem (1.1), so we suppose that $f(s) = 0$ for $s < 0$. Obviously, assume that the conditions (F1)-(F3) hold, then, weak solutions to problem (1.1) are the critical points of the functional $I_\lambda(u)$ defined in $H^1(\mathbb{R}^N)$ by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} (a|\nabla u|^2 + V(x)u^2)dx + \frac{\lambda}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^N} F(u)dx. \tag{2.1}$$

By the condition (F1), it is easy to examine that $I_\lambda \in C^1(E, \mathbb{R})$ for all $\lambda > 0$, and

$$\langle I'_\lambda(u), v\rangle = \left(a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 dx\right) \int_{\mathbb{R}^N} \nabla u \nabla v dx + \int_{\mathbb{R}^N} V(x)uv dx - \int_{\mathbb{R}^N} f(u)v dx, \quad u, v \in E. \tag{2.2}$$

Firstly, we shall show that I_λ has the mountain path geometry when λ is small.

Lemma 2.1. *For $\lambda \geq 0$ small, the functional I_λ satisfies the following conditions.*

(i) There exists $\alpha > 0$, $\rho > 0$ such that $I_\lambda(u) > \alpha$ for $\|u\| = \rho$.
(ii) There exists an $e \in B^c_\rho(0)$ such that $I_\lambda(e) < 0$, where $B_\rho(0)$ is a ball centred at origin with radius ρ.
Proof. (i) From the Sobolev embedding $H^1(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ for $2 < p < \frac{2N}{N-2}$ and the condition (F1), we obtain

$$I_\lambda(u) \geq I_0(u) \geq \frac{1}{2}\|u\|^2 - \frac{C}{2}|u|_2^2 - \frac{C}{p}|u|_p^p,$$

where $C >$ is a constant. Let $\{e_j\}$ is an orthogonal basis of E and define $X_j = \mathbb{R}e_j$,

$$Y_m = \bigoplus_{j=1}^{m} X_j, \quad Z_m = \bigoplus_{j=m+1}^{\infty} X_j, \quad m \in \mathbb{Z}.$$

Set

$$\eta_m(s) = \sup_{u \in Z_m, \|u\|=1} \|u\|_s, \quad \forall\, m \in \mathbb{N}, \quad 2 < s < 2^*.$$

Due to the result of [27], since the embedding $E \hookrightarrow L^p(\mathbb{R}^N)$ is compact for $p \in (2, 6)$, there holds

$$\eta_m(s) \to 0 \quad as \quad m \to \infty. \tag{2.3}$$

Then we have

$$I_\lambda(u) \geq \frac{1}{2}\|u\|^2 - \frac{C}{2}\eta_m^2(2)\|u\|^2 - \frac{C}{p}\eta_m^p(p)\|u\|^p$$

By (2.3), there exists an integer $M \geq 1$ such that

$$\eta_m^2(2) \leq \frac{1}{2C} \quad \text{and} \quad \eta_m^p(p) \leq \frac{p}{4C}, \quad \forall\, m \geq M. \tag{2.4}$$

Hence, we can take some $\rho \in (0,1)$ with $\|u\| = \rho$ such that

$$I_\lambda(u) \geq \frac{1}{4}\|u\|^2(1 - \|u\|^{p-2}) = \alpha > 0.$$

(ii) Take $e \in H^1(\mathbb{R}^N)$ such that $I_0(e) < 0$, Obviously, $e \in B_\rho^c(0)$. Then for $0 \leq \lambda \leq \frac{-8I_0(e)}{3(\int_{\mathbb{R}^N} |\nabla e|^2 dx)^2}$,

$$I_\lambda(e) = I_\lambda(e) + \frac{\lambda}{4}\left(\int_{\mathbb{R}^N} |\nabla e|^2 dx\right)^2 \leq \frac{1}{3}I_0(e) < 0.$$

□

Lemma 2.2. *When $N = 4$ with λ large or when $N \geq 5$ with $\lambda \geq 0$, I_λ is bounded from below in $H^1(\mathbb{R}^N)$.*

Proof. By (F1) and (2.1), we have

$$I_\lambda(u) \geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx + \frac{\lambda}{4}\|\nabla u\|_2^4 - \frac{C}{2}\|u\|_2^2 - \frac{C}{p}\|u\|_p^p. \tag{2.5}$$

From Holder's inequality, we have

$$|u|_p \leq |u|_2^r |u|_{\frac{2N}{N-2}}^{1-r}, \quad u \in L^2(\mathbb{R}^N) \cap L^{\frac{2N}{N-2}}(\mathbb{R}^N), \tag{2.6}$$

where $\frac{r}{2} + \frac{(N-2)(1-r)}{2N} = \frac{1}{p}$. From Sobolev imbedding Theorem, there exists a constant $d > 0$ such that

$$\|u\|_{\frac{2N}{N-2}} \leq d\|\nabla u\|_2, \quad u \in H^1(\mathbb{R}^N). \tag{2.7}$$

From (2.6) and (2.7), we have

$$\int_{\mathbb{R}^N} |u|^p dx \leq d\|u\|_2^{\frac{2N+(2-N)p}{2}} \|\nabla u\|_2^{\frac{N(p-2)}{2}}, \quad u \in H^1(\mathbb{R}^N). \tag{2.8}$$

Since $p \in (2, 2^*)$, we have $0 < \frac{2N+(2-N)p}{2} < 2$, then from Young's inequality and (2.8), for any $\varepsilon > 0$, there exists a constant $d(\varepsilon) > 0$ such that

$$\int_{\mathbb{R}^N} |u|^p dx \leq \varepsilon \int_{\mathbb{R}^N} |u|^2 dx + d(\varepsilon)\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^{\frac{N}{N-2}}. \tag{2.9}$$

By (2.3), we can choose ε such that $(\frac{pC+2C\varepsilon}{2p})\eta_k^2(2) < \frac{1}{4}$, then from (2.5) and (2.9), we have

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 dx + \frac{\lambda}{4}\|\nabla u\|_2^4 - \left(\frac{C}{2} + \frac{C\varepsilon}{p}\right)\|u\|_2^2 - \frac{Cd(\varepsilon)}{p}\|\nabla u\|_2^{\frac{2N}{N-2}} \\ &\geq \frac{1}{2}\|u\|^2 + \frac{\lambda}{4}\|\nabla u\|_2^4 - \left(\frac{C}{2} + \frac{C\varepsilon}{p}\right)\eta_k^2(2)\|u\|^2 - \frac{Cd(\varepsilon)}{p}\|\nabla u\|_2^{\frac{2N}{N-2}} \\ &\geq \frac{1}{4}\|u\|^2 + \frac{\lambda}{4}\|\nabla u\|_2^4 - \frac{Cd(\varepsilon)}{p}\|\nabla u\|_2^{\frac{2N}{N-2}}. \end{aligned} \tag{2.10}$$

When $N = 4$, if $\frac{\lambda}{4} \geq \frac{Cd(\varepsilon)}{p}$, then $I_\lambda(u) \geq 0$ for all $u \in E$. When $N \geq 5$, we have $\frac{2N}{N-2} \leq 4$, then when $\|\nabla u\|_2 \geq (\frac{4Cd(\varepsilon)}{p\lambda})^{\frac{N-2}{2N-8}}$, we can get $I_\lambda(u) \geq 0$. Hence, it is obvious that I_λ is bounded from below when $N \geq 5$ for all $\lambda > 0$.

□

It is crucial to obtain the boundedness of (PS) sequences for the associated functional I_λ. However, the standard arguments is not available for proving the boundedness of the (PS) sequences. In order to overcome the difficulty, following the idea of [28, 29], we use a cut-off function $\varphi \in (\mathbb{R}^+, [0,1])$ satisfying

$$\begin{cases} \varphi(t) = 1, & t \in [0,1] \\ 0 \le \varphi(t) \le 1, & t \in (1,2) \\ \varphi(t) = 0, & t \in [2,\infty) \\ 0 \le \varphi'(t) \le 2. \end{cases}$$

For any $T > 0$, we modify the original functional I_λ to a new functional $I^T_{\lambda,\mu}$ defined by

$$I^T_{\lambda,\mu}(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\lambda h_T(u)\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \mu \int_{\mathbb{R}^N} F(u)dx, \quad u \in E. \tag{2.11}$$

It can be verified that $I^T_{\lambda,\mu}$ is of class C^1 and for any $u, v \in E$,

$$\begin{aligned} \langle (I^T_{\lambda,\mu})'(u), v\rangle &= \left[a + \lambda h_T(u)\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{2T^2} h'_T(u)\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2\right]\int_{\mathbb{R}^N} \nabla u \nabla v dx \\ &\quad + \int_{\mathbb{R}^N} V(x)uvdx - \mu\int_{\mathbb{R}^N} f(u)vdx, \end{aligned} \tag{2.12}$$

where

$$h_T(u) = \varphi\left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx}{T^2}\right).$$

Since $h_T(u)\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 \le 4T^4$, the functional $I^T_{\lambda,\mu}$ has a mountain path critical level for any fixed $T > 0$. Actually, for $T > 0$ sufficiently large and $\lambda > 0$ sufficiently small, we can find a bounded (PS) sequence $\{u_n\}$ of $I^T_{\lambda,\mu}$ such that $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \le T^2$ for all n large, which is also a (PS) sequence of I_λ. We recall the following result. The "monotonic trick" introduced in [30, 31] is the core of the following Theorem.

Theorem 2.3.[30, 31] *Let $(X, \|.\|)$ be a Banach space and $J \in \mathbb{R}^+$ be an interval. Consider the family of C^1-functional on X*

$$\varphi_{\lambda,\mu}(u) = A(u) - \mu B(u), \quad \forall \lambda \in J,$$

with B nonnegative and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to \infty$ and such that $\varphi_{\lambda,\mu}(0) = 0$.

For any $\mu \in J$, we set

$$\Gamma_\mu = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \varphi_{\lambda,\mu}(\gamma(1)) < 0\}.$$

If for every $\mu \in J$, the set Γ_μ is nonempty and

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} \varphi_{\lambda,\mu}(\gamma(t)) > 0,$$

then for almost every $\mu \in J$, there is a sequence $\{u_n\} \subset X$ such that

(i) $\{u_n\}$ is bounded;

(ii) $\varphi_{\lambda,\mu}(u_n) \to c_{\lambda,\mu}$;

(iii) $\varphi'_{\lambda,\mu}(u_n) \to 0$ in the dual X^* of X.

In our case, $X = E$,

$$A(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\lambda h_T(u)\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2, \quad B(u) = \int_{\mathbb{R}^N} F(u)dx.$$

In the following, we will prove that $I^T_{\lambda,\mu}$ satisfies the conditions of Theorem 2.3.

Lemma 2.4. *$\Gamma_\mu \neq \emptyset$ for all $\mu \in J = [\delta, 1]$, where $\delta \in (0,1)$ is a positive constant.*

Proof. For any $\mu \in J$, a radial function $\omega \in E$ can be chosen with $\omega > 0$ and $\int_{\mathbb{R}^N} |\nabla\omega|^2 dx = 1$. By (F2) and (F3), we have that, for any $C_1 > 0$ with $(\mu C_1 - \frac{V_\infty}{2}) \int_{\mathbb{R}^N} |\omega|^2 dx > a$, there exists C_2 such that

$$F(s) \geq C_1|s|^2 - C_2, \quad \forall\ s \in \mathbb{R}^+. \tag{2.13}$$

For any $t \geq 2\sqrt{T}$, from (2.13), we have

$$\begin{aligned} I^T_{\lambda,\mu}(t\omega) &= \frac{1}{2}at^2 \int_{\mathbb{R}^N} |\nabla\omega|^2 dx + \frac{1}{2}\int_{\mathbb{R}^N} V(x)t^2\omega^2 dx \\ &\quad + \frac{1}{4}\lambda\varphi\left(\frac{t^2\int_{\mathbb{R}^N}|\nabla\omega|^2dx}{T^2}\right)\left(\int_{\mathbb{R}^N}|\nabla t\omega|^2dx\right)^2 - \mu\int_{\mathbb{R}^N}F(t\omega)dx \\ &\leq \frac{1}{2}at^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_\infty t^2\omega^2 dx + \frac{1}{4}\lambda\varphi\left(\frac{t^2}{T^2}\right)\left(\int_{\mathbb{R}^N}|\nabla t\omega|^2dx\right)^2 - \mu\int_{\mathbb{R}^N}F(t\omega)dx \\ &\leq \frac{1}{2}at^2 + \frac{1}{2}V_\infty t^2\int_{\mathbb{R}^N}\omega^2dx - \mu C_1 t^2\int_{\mathbb{R}^N}|\omega|^2dx + \mu C_2. \end{aligned} \tag{2.14}$$

Hence, (2.14) implies that $I^T_{\lambda,\mu}(t\omega) < 0$ for t sufficiently large. The proof is completed. □

Lemma 2.5. *For any $\mu \in J$, there exists $\theta > 0$ such that $c_{\lambda,\mu} = \inf_{\gamma\in\Gamma_\mu}\max_{t\in[0,1]} I^T_{\lambda,\mu}(\gamma(t)) \geq \theta > 0$.*

Proof. For any $u \in E$ and $\mu \in J$, by (F1) and (F2), for any $\epsilon > 0$, there exist $C(\epsilon) > 0$ such that

$$\begin{aligned} I^T_{\lambda,\mu}(u) &\geq \frac{1}{2}\|u\|^2 + \frac{1}{4}\lambda h_T(u)\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2 - \epsilon\|u\|_2^2 - C(\epsilon)\|u\|_p^p \\ &\geq \frac{1}{2}\|u\|^2 - \epsilon\|u\|_2^2 - C(\epsilon)\|u\|_p^p. \end{aligned}$$

Similar to the proof of Lemma 2.1 (i), there exists $\rho \in (0,1)$ such that $I_{\lambda,\mu}(u) \geq \theta > 0$ for any $\mu \in J$ and $u \in E$ with $0 < \|u\| \leq \rho$. By the continuity of γ and the definition of Γ_γ, we have $\|\gamma(1)\| > \rho$, then there exists $t_\gamma \in (0,1)$ such that $\|\gamma(t_\gamma)\| = \rho$. Hence, for any $\mu \in J$,

$$c_{\lambda,\mu} = \inf_{\gamma\in\Gamma_\mu}\max_{t\in[0,1]} I^T_{\lambda,\mu}(\gamma(t)) \geq \theta > 0.$$

The proof is completed. □

Lemma 2.6. *For any $\mu \in J$ and $4\lambda T^2 < 1$, each bounded (PS) sequence of the functional $I^T_{\lambda,\mu}$ admits a convergent subsequence.*

Proof. It is easy to see that $A(u) \to +\infty$ as $\|u\| \to +\infty$. From (F2), we have $B(u) \to +\infty$ as $\|u\| \to +\infty$. From Theorem 2.3, Lemma 2.4 and Lemma 2.5, we know that there exists a bounded sequence $\{u_n\} \subset E$. Without loss of generality, for any $\mu \in J$, let $\{u_n\}$ be a bounded (PS) sequence of $I^T_{\lambda,\mu}$, that is, $\{u_n\}$ and $I^T_{\lambda,\mu}$ are bounded and $(I^T_{\lambda,\mu})'(u) \to 0$ in E^*. Up to a subsequence, assume that there exists $u \in E$ such that

$$\begin{aligned} u_n &\rightharpoonup u \quad \text{in } E, \\ u_n &\to u \quad \text{in } L^p\left(\mathbb{R}^N\right), \\ u_n &\to u \quad \text{a.e. in } \mathbb{R}^N. \end{aligned} \tag{2.15}$$

By (F1) and (F2), for any $\varepsilon \in (0, \frac{1}{2})$, there exists C_ε such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon |t|^{p-1}, \quad t \in \mathbb{R}. \tag{2.16}$$

Hence, by (2.16), we have

$$\begin{aligned}
\int_{\mathbb{R}^N} (f(u_n) - f(u))(u_n - u)dx &\leq \int_{\mathbb{R}^N} |f(u_n) - f(u)||u_n - u|dx \\
&\leq \int_{\mathbb{R}^N} (|f(u_n)| + |f(u)|)|u_n - u|dx \\
&\leq \varepsilon \int_{\mathbb{R}^N} (|u_n| + |u|)|u_n - u|dx + C_\varepsilon \int_{\mathbb{R}^N} (|u_n|^{p-1} + |u|^{p-1})|u_n - u|dx \\
&\leq \varepsilon(\|u_n\|_2 + \|u\|_2)\|u_n - u\|_2^2 + C_\varepsilon(\|u_n\|_p^{p-1} + \|u\|_p^{p-1})\|u_n - u\|_p \\
&\leq (\|u_n\|^2 + \|u\|^2)\|u_n - u\|^2 + C_\varepsilon(\|u_n\|_p^{p-1} + \|u\|_p^{p-1})\|u_n - u\|_p.
\end{aligned} \tag{2.17}$$

It follows from (2.15) and (2.17) that

$$\int_{\mathbb{R}^N} (f(u_n) - f(u))(u_n - u)dx \to 0, \quad \text{as} \quad n \to \infty. \tag{2.18}$$

Then, by (2.11) and let $X_\lambda^T(u) = \frac{\lambda}{2T^2}\varphi'\left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2}{T^2}\right)(\int_{\mathbb{R}^N} |\nabla u|^2)^2$, we have

$$\begin{aligned}
&\langle (I_{\lambda,\mu}^T)'(u_n) - (I_{\lambda,\mu}^T)'(u), u_n - u\rangle \\
= \ & a\int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 dx + \int_{\mathbb{R}^N} V(x)|u_n - u|^2 dx + \lambda h_T(u_n)\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 dx \\
&+X_\lambda^T(u_n)\int_{\mathbb{R}^N} |\nabla(u_n - u)|^2 dx + [X_\lambda^T(u_n) - X_\lambda^T(u)]\int_{\mathbb{R}^N} \nabla u\nabla(u_n - u))dx \\
&-\lambda h_T(u)\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\int_{\mathbb{R}^N} \nabla u\nabla(u_n - u)dx + \lambda h_T(u_n)\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla u\nabla(u_n - u)dx \\
&-\mu\int_{\mathbb{R}^N} (f(u_n) - f(u))(u_n - u)dx \\
\geq \ & \|u_n - u\|^2 - \left(\lambda h_T(u)\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda h_T(u_n)\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla u\nabla(u_n - u)dx \\
&-[X_\lambda^T(u) - X_\lambda^T(u_n)]\int_{\mathbb{R}^N} \nabla u\nabla(u_n - u))dx - \mu\int_{\mathbb{R}^N} (f(u_n) - f(u))(u_n - u)dx.
\end{aligned} \tag{2.19}$$

Hence, from (2.19), we have

$$\begin{aligned}
\|u_n - u\|^2 \leq \ & \left(\lambda h_T(u)\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda h_T(u_n)\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N} \nabla u\nabla(u_n - u)dx \\
&+[X_\lambda^T(u) - X_\lambda^T(u_n)]\int_{\mathbb{R}^N} \nabla u\nabla(u_n - u))dx + \mu\int_{\mathbb{R}^N} (f(u_n) - f(u))(u_n - u)dx \\
&+\langle (I_{\lambda,\mu}^T)'(u_n) - (I_{\lambda,\mu}^T)'(u), u_n - u\rangle.
\end{aligned} \tag{2.20}$$

In the following, in order to obtain our result, we should discuss $\int_{\mathbb{R}^N} \nabla u\nabla(u_n - u)dx$. Define a functional $H_u : E \to \mathbb{R}$ by

$$H_u(v) = \int_{\mathbb{R}^N} \nabla u\nabla v dx, \quad \forall\ v \in E.$$

It is clearly that H_u is a linear functional on E. Since

$$|H_u(v)| \leq \int_{\mathbb{R}^N} |\nabla u\nabla v|dx \leq \|u\|\|v\|,$$

we have that H_u is bounded on E, i.e., $H_u \in E^*$. Therefore, $\lim_{n\to\infty} H_u(u_n) = H_u(u)$ if $u_n \rightharpoonup u$ in E, hence, we have

$$\int_{\mathbb{R}^N} \nabla u \nabla (u_n - u) dx \to 0 \quad \text{as} \quad n \to \infty. \tag{2.21}$$

Since $\{u_n\}$, X_λ^T and h_T are bounded, from (2.20), we have,

$$\left(\lambda h_T(u) \int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda h_T(u_n) \int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right) \int_{\mathbb{R}^N} \nabla u \nabla (u_n - u) dx \to 0, \quad \text{as } n \to \infty, \tag{2.22}$$

and

$$[X_\lambda^T(u) - X_\lambda^T(u_n)] \int_{\mathbb{R}^N} \nabla u \nabla (u_n - u)) dx \to 0, \quad \text{as } n \to \infty. \tag{2.23}$$

It is obvious that $\langle (I_{\lambda,\mu}^T)'(u_n) - (I_{\lambda,\mu}^T)'(u), u_n - u\rangle \to 0$ since $(I_{\lambda,\mu}^T)'(u) \to 0$ and $u_n \rightharpoonup u$ in E. Consequently, from (2.18), (2.20), (2.22) and (2.23), we have

$$\|u_n - u\|^2 \to 0.$$

Hence, for any $\mu \in J$ and $4\lambda T^2 < 1$, each bounded (PS) sequence of the functional $I_{\lambda,\mu}^T$ has a convergent subsequence. The proof is completed. □

Lemma 2.7. *Let $4\lambda T^2 < 1$, then for almost every $\mu \in J$, there exists $u^\mu \in E\backslash\{0\}$ such that $(I_{\lambda,\mu}^T)'(u^\mu) = 0$, and $(I_{\lambda,\mu}^T)(u^\mu) = c_{\lambda,\mu}$.*

Proof. By the definition of $A(u)$ and $B(u)$, we can see that $A(u) \to \infty$ as $n \to \infty$ and $B(u)$ is nonnegative. According to Lemma 2.1 and Theorem 2.3, for almost $\mu \in J$, there exists a bounded sequence $\{u_n^\mu\}$ such that

$$(I_{\lambda,\mu}^T)'(u_n^\mu) = 0, \quad \text{and} \quad I_{\lambda,\mu}^T(u_n^\mu) = c_{\lambda,\mu_n}.$$

By Lemma 2.6, we can conclude that there exists $u^\mu \in E$ such that $u_n^\mu \to u^\mu$. Therefore, $(I_{\lambda,\mu}^T)'(u^\mu) = 0$, and $(I_{\lambda,\mu}^T)(u^\mu) = c_{\lambda,\mu}$. From $f(s) = 0$ for $s < 0$ and the Lemma 2.5, we have $u^\mu \in E\backslash\{0\}$. □

From Lemma 2.7, there exists $\{u_n\} \in J$ with $\mu_n \to 1^-$ and a sequence $\{u_n\} \subset E$ such that

$$(I_{\lambda,\mu_n}^T)'(u_n) = 0, \quad \text{and} \quad I_{\lambda,\mu_n}^T(u_n) = c_{\lambda,\mu_n}.$$

Next, we will introduce the following Pohožaev identity, which is crucial to obtain that $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx < T^2$. So we give the following Lemma.

Lemma 2.8. *Let $4\lambda T^2 < 1$ and $N \geq 3$, if $u \in H$ is a weak solution of*

$$-\left[a + \lambda h_T(u) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{2T^2} \varphi'\left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2}{T^2}\right) \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2\right] \Delta u + V(x) u = \mu f(u), \quad x \in \mathbb{R}^N,$$

then, u satisfies the following Pohožaev identity

$$\begin{aligned} & N\mu \int_{\mathbb{R}^N} F(u) dx - \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x), x) u^2 dx \\ = & \frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx \left[a + \lambda h_T(u) \int_{\mathbb{R}^N} |\nabla u|^2 dx \right. \\ & \left. + \frac{\lambda}{2T^2} \varphi'\left(\frac{\int_{\mathbb{R}^N} |\nabla u|^2}{T^2}\right) \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2\right] + \frac{N}{2} \int_{\mathbb{R}^N} V(x) u^2 dx. \end{aligned} \tag{2.24}$$

Proof. Similar to Lemma 2.2 in [32], we can prove the above conclusion. Here, we omit the details of proof. □

The following Lemma implies that $\int_{\mathbb{R}^N}|\nabla u_n|^2dx \le T^2$, which is the core of this paper.

Lemma 2.9. *Let $\{u_n\}$ be a (PS) sequence of I^T_{λ,μ_n} at level c_{λ,μ_n}, then for $T>0$ sufficiently large, there exists $\lambda_0>0$ with $4\lambda_0T^2<1$ such that for any $\lambda\in[0,\lambda_0)$, $\int_{\mathbb{R}^N}|\nabla u_n|^2dx \le T^2$ for all $n\in\mathbb{Z}$.*

Proof. We will discuss by contradiction. Assume that there exists no subsequence $\{u_n\}$ such that $\int_{\mathbb{R}^N}|\nabla u_n|^2dx$ is bounded by T. Then we can suppose $\int_{\mathbb{R}^N}|\nabla u_n|^2dx > T^2$.

Firstly, since $(I^T_{\lambda,\mu_n})'(u_n)=0$, by (2.24), $\{u_n\}$ satisfies the following Pohožaev identity:

$$\begin{aligned}
&N\mu_n\int_{\mathbb{R}^N}F(u_n)dx-\frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x),x)u_n^2dx\\
=\ &\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2dx\left[a+\lambda h_T(u_n)\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right.\\
&\left.+\frac{\lambda}{2T^2}\varphi'\left(\frac{\int_{\mathbb{R}^N}|\nabla u_n|^2}{T^2}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)^2\right]+\frac{N}{2}\int_{\mathbb{R}^N}V(x)u_n^2dx. \qquad (2.25)
\end{aligned}$$

Since $I^T_{\lambda,\mu_n}(u_n)=c_{\lambda,\mu_n}$, we have

$$c_{\lambda,\mu_n}N=\frac{N}{2}\int_{\mathbb{R}^N}(a|\nabla u_n|^2+V(x)u_n^2)dx+\frac{\lambda N}{4}h_T(u_n)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)^2-\mu_nN\int_{\mathbb{R}^N}F(u_n)dx \qquad (2.26)$$

Then, according to (2.25), (2.26), (V1) and Hardy inequality, we can get that

$$\begin{aligned}
a\int_{\mathbb{R}^N}|\nabla u_n|^2dx\ &\le\ \left[a+\lambda h_T(u_n)\int_{\mathbb{R}^N}|\nabla u_n|^2dx+\frac{\lambda}{2T^2}\varphi'\left(\frac{\int_{\mathbb{R}^N}|\nabla u_n|^2dx}{T^2}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)^2\right]\int_{\mathbb{R}^N}|\nabla u_n|^2dx\\
&=\ c_{\lambda,\mu_n}N+\frac{N}{4}\lambda h_T(u_n)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)^2+\frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x),x)u_n^2dx\\
&\quad+\frac{N\lambda}{4T^2}\varphi'\left(\frac{\int_{\mathbb{R}^N}|\nabla u_n|^2dx}{T^2}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)^3\\
&\le\ c_{\lambda,\mu_n}N+\frac{N}{4}\lambda h_T(u_n)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)^2+\theta a\int_{\mathbb{R}^N}|\nabla u_n|^2dx\\
&\quad+\frac{N\lambda}{4T^2}\varphi'\left(\frac{\int_{\mathbb{R}^N}|\nabla u_n|^2dx}{T^2}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)^3. \qquad (2.27)
\end{aligned}$$

From (2.27), we have

$$(1-\theta)a\int_{\mathbb{R}^N}|\nabla u_n|^2dx\le Nc_{\lambda,\mu_n}+\frac{N}{4}\lambda h_T(u_n)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2dx\right)^2+\frac{N\lambda}{4T^2}\varphi'\left(\frac{\int_{\mathbb{R}^N}|\nabla u_n|^2dx}{T^2}\right)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2\right)^3. \qquad (2.28)$$

By Lemma 2.5 and (2.13), we have

$$\begin{aligned}
c_{\lambda,\mu_n}&\le\max_t I^T_{\lambda,\mu_n}(t\omega)\\
&=\max_t\left\{\frac{1}{2}t^2+\frac{1}{2}t^2\int_{\mathbb{R}^N}V(x)\omega^2dx-\mu_n\int_{\mathbb{R}^N}F(t\omega)dx\right\}+\max_t\left\{\frac{1}{4}\lambda h_T(t\omega)\left(\int_{\mathbb{R}^N}|\nabla t\omega|^2\right)^2dx\right\}\\
&\le\max_t\left\{\frac{1}{2}t^2+\frac{1}{2}V_\infty t^2\int_{\mathbb{R}^N}\omega^2dx-\mu_nC_1t^2\int_{\mathbb{R}^N}\omega^2dx+\mu_nC_2\right\}+\max_t\left\{\frac{1}{4}\lambda\varphi\left(\frac{t^2}{T^2}\right)t^4\right\}\\
&=C_3+A_1(T).
\end{aligned} \qquad (2.29)$$

If $t > \sqrt{2}T$, then $\varphi(\frac{t^2}{T^2}) = 0$. Hence, we obtain

$$A_1(T) \leq \lambda T^4. \tag{2.30}$$

We also get that

$$\frac{N}{4}\lambda h_T(u_n)\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^2 \leq \lambda N T^4, \tag{2.31}$$

and

$$\frac{N\lambda}{4T^2}\varphi'\left(\frac{\int_{\mathbb{R}^N} |\nabla u_n|^2 dx}{T^2}\right)\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^3 \leq 4\lambda N T^4. \tag{2.32}$$

From (2.28)-(2.32), we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \leq \frac{1}{(1-\theta)a}(NC_3 + 6\lambda N T^4). \tag{2.33}$$

Since we suppose $\int_{\mathbb{R}^N} |\nabla u_n|^2 dx > T^2$, $n \in \mathbb{N}$, then from (2.33), we have

$$T^2 < \frac{1}{(1-\theta)a}(NC_3 + 6\lambda N T^4),$$

which is not true for T large and $4\lambda T^4 < 1$. So we can choose $\lambda_0 = \frac{1}{4T^4}$. The proof is completed. □

3. Proof of the main results

Proof of Theorem 1.1. We define T and λ_0 as in Lemma 2.9. And let $\{u_n\}$ be a sequence for I^T_{λ,μ_n}. Then by Lemma 2.9, we have

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \leq T^2,$$

and

$$I^T_{\lambda,\mu_n}(u_n) = \frac{1}{2}\|u_n\|^2 + \frac{1}{4}\lambda\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \mu_n \int_{\mathbb{R}^N} F(u_n)dx.$$

Hence, we have

$$\langle I'_\lambda(u_n), v\rangle = \langle (I^T_{\lambda,\mu_n})'(u_n), v\rangle - (1-\mu_n)\int_{\mathbb{R}^N} F(u_n)dx.$$

Consequently, when $\mu_n \to 1$, $\{u_n\}$ is also a bounded (PS) sequence of I_λ. According to Lemma 2.6, $\{u_n\}$ has a convergent subsequence, we may assume $u_n \to u_0$. Thus $I'_\lambda(u_0) = 0$. According to Lemma 2.5, we have that $I_\lambda(u_0) = \lim_{n\to\infty} I_\lambda(u_n) = \lim_{n\to\infty} I^T_{\lambda,\mu_n}(u_n) \geq \theta > 0$. And the condition (F1) implies that u_0 is a positive solution. The proof is completed. □

Proof of Corollary 1.2. By Theorem 1.1, it is obvious that Corollary 1.2 holds. □

Proof of Theorem 1.3. Assume that $N \geq 4$ and $u \in H^1_0(\Omega)$ is a nontrivial solution of the problem (1.1). Multiply (1.1) by u and integrate by parts, we obtain

$$a\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} V(x)u^2 + \lambda\|\nabla u\|_2^4 = \int_{\mathbb{R}^N} F(u)dx.$$

Similarly, by (F1) and (2.16), there exists ε_1 and C_{ε_1} such that

$$\int_{\mathbb{R}^N} F(u)dx \leq \varepsilon_1\|u\|_2^2 + C_{\varepsilon_1}\|\nabla u\|^2_{\frac{N}{N-2}}.$$

Now, we can choose $\varepsilon_1 = \frac{1}{2}V_0$. Then, we have that

$$a\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} V(x)u^2dx + \lambda\|u\|_2^4 \leq \frac{V_0}{2}\|u\|_2^2 + C_{\varepsilon_1}\|\nabla u\|^2_{\frac{N}{N-2}}.$$

Since $N \geq 4$, then, $2 < \frac{2N}{N-2} \leq 4$. We can easily use Young's inequality to obtain that

$$C_{\varepsilon_1}\|\nabla u\|^2_{\frac{N}{N-2}} \leq a\|\nabla u\|_2^2 + C_4\|\nabla u\|_2^4.$$

Consequently, we get

$$\int_{\mathbb{R}^N} V(x)u^2dx + \lambda\|u\|_2^4 \leq \frac{V_0}{2}\|u\|_2^2 + C_4\|\nabla u\|_2^4.$$

If we choose $\lambda \geq C_4$, we have

$$\|\nabla u\|_2 = \|u\|_2 = 0.$$

Hence, when $N > 3$ and for enough large λ, problem (2.3) has no nontrivial solution. The proof is completed. □

4. Conclusion

In this paper, we first proved that the energy functional I_λ has the mountain structure. Secondly, we defined a cut-off functional φ and established a modified functional $I^T_{\lambda,\mu}$. We showed that $I^T_{\lambda,\mu}$ has a bounded palais-smale sequence $\{u_n\}$. Then, we proved that there exists λ_0 such that for any $0 < \lambda < \lambda_0$, $\int_{\mathbb{R}^N} |\nabla u_n|^2dx \leq T^2$, which implies the critical value of $I^T_{\lambda,\mu}$ is also the critical value of I_λ. Finally, we deduce our results by the variational method. Obviously, our results are more general. We hope our results can be widely used in the Kirchhoff system as discussed in [27].

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