Tempered distribution version of the Tumarkin result

A b s t r a c t. We give a tempered distribution analogue of the Tumarkin result that concerns approximation of some functions by sequence of rational functions with given poles..

1. Background and the Tumarkin result

For the needs of our subsequent work we will define the Blashcke product in the upper half plane Π^+ , $\Pi^+ = \{z \in C | Imz > 0\}$. Assume

$$\sum_{n=1}^{\infty} \frac{y_n}{1+|z_n|^2} < \infty, z_n = x_n + iy_n \in \Pi^+ \quad (1.1)$$

Then the Blaschke product with zeros z_n is

$$B(z) = \left(\frac{z-i}{z+i}\right)^m \prod_{n=1}^{\infty} \frac{(|z_n^2+1|)}{z_n^2+1} \frac{z-z_n}{z-\overline{z_n}}, z \in \Pi^+ \qquad (1.2)$$

Let

$$z_{k_1}, z_{k_2}, \dots, z_{k_{N_k}}, k = 1,2, \dots, Imz \neq 1, N_k \leq \infty \qquad (1.3)$$

be given complex numbers. Some of the numbers in (1.3) might be equal and also some of them might be equal to ∞(in that case $Imz = 0$)

$Let\ R_k$be the rational function of the form

$$R_k(z) = \frac{c_0 z^p + c_1 z^{p-1} + \dots + c_p}{(z-z_{k_1})(z-z_{k_1})\dots(z-z_{k_{n_p}})}, z \in \Pi^+, \qquad (1.4)$$

whose poles are some of the number in (1.3) and $c_0, c_1, \dots, c_p$ are arbitrary numbers (if some $z_{k_i=\infty,}$ then in (1.4) we put 1 instead $z - z_{k_i}$).

All z_{k_i}, for which $Imz_{k_i} > 0$, will be denoted by z'_{k_i} , and all those z_{k_i}, for which $Imz_{k_i} < 0$ will be denoted by z''_{k_i}.

Let $S'_k = \sum_i \frac{Imz_{k_i}}{1+\left|z'_{k_i}\right|^2}$ и $S''_k = \sum_i \frac{(-Imz''_{k_i})}{1+\left|z''_{k_i}\right|^2}$.

With (1.5) we denote the following conditions

$$\overline{\lim_{k\to\infty}} S'_k < \infty, \qquad \overline{\lim_{k\to\infty}} S''_k = \infty. \qquad (1.5)$$

Let B_k ebe the Blaschke product whose zeros are the numbers, $z_{k_1}, z_{k_2}, \dots, z_{k_{N_k}}$, from the numbers (1.3), k = 1, 2, 3,.... Assume (1.5). Then $\mu(z) = \lim_{k\to\infty} log|B_k(z)|$ is subharmonic on Π^+ and differs from -∞. Let $u(z)$ be the harmonic majorant of $\mu(z)$ in Π^+. Since $\mu(z) \le 0$ we have that $u(z) \le 0$. Let $\phi(z) = e^{u(z)+v(z)}$, where $v(z)$) is the harmonic conjugate of $u(z)$.

Tumarkin has proved the following results

Theorem 1. [4] Assume that (1.5) holds and that ϕ is as above. For a continuous function F on R there exists a sequence $\{R_k\}$ of rational functions of the form (1.4) which converges uniformly on R to F if and only if F coincide almost everywhere on R with the boundary value of meromorphic function F on Π^+ of the form

$$\mathrm{F(z)} = \frac{\psi(\mathrm{z})}{\mathrm{B(z)}\phi(\mathrm{z})}, \mathrm{z} \in \Pi^+, \qquad (1.6)$$

where ψ is any bounded analytic function on Π^+.

Let σ be a nondecreasing function of bounded variation on R. By Lp(dσ; R), p > 0 is denoted the set of all complex valued functions F, for which the Lebesgue-Stieltjes integral exists i.e. $\int_R |F(x)|^p d\sigma(x) < \infty$.

With (1.7) we denote the following condition:

$$\int_R \frac{\log\sigma'(x)}{1+x^2}dx < -\infty. \qquad (1.7)$$

Theorem 2. Assume (1.5) and (1.7). For a function $F \in L^p(d\sigma, R), p > 0$ there exists a sequence $\{R_k\}$ of rational functions of the form (1.4) such that $\lim_{k\to\infty}\int_R |F(x) - R_k(x)|^p d\sigma(x) = 0$ if and only if F coincide almost everywhere on R with the boundary value of a meromorphic function F on Π^+ of the form (1.6), where B and φ are as in theorem 1, and ψ is analytic function on Π^+ of the class N^+.

Note. N^+ is the class of all analytic functions on N^+ which satisfy the following condition

$$\lim_{y\to 0^+}\int_R \frac{\log^+|f(x+iy)|}{1+x^2}dx = \int_R \frac{\log^+|f(x)|}{1+x^2}dx$$

$S = S(R^n)$ denotes the space of all infinitely differentiable complex valued function φ on R^n satisfying $\sup_{t\in R^n}\left|t^\beta D^\alpha \varphi(t)\right| < \infty$
for all n-tuple α and β of nonnegative integers. Convergence in S is defined in the following way: a sequence $\{\varphi_\lambda\}$ of functions $\varphi_\lambda \in S$ in S as $\lambda \to \lambda_0$ if and only if

$$\lim_{\lambda\to\lambda_0}\sup_{t\in R^n}\left|t^\beta D_t^\alpha[\varphi_\lambda(t) - \varphi(t)]\right| = 0$$

for all n-tuple α and β of nonnegative integers.

Again, S' is the space of all continuous, liear functionals on S, called the space of tempered distributions.

The space S' is called the space of tempered distributions. We use the convention $\langle T,\varphi\rangle = T(\varphi)$ for the value of the functional Tacting on the function φ.

Let $\varphi\epsilon S$ and $f(x) \in L^1_{loc}(R^n)$. Then the functional T_f on S defined with

$$\langle \mathrm{T}_f, \varphi \rangle = \int_{R^n} f(t)\varphi(t)dt, \varphi \epsilon S,$$

is an element in S' and it is called the regular distribution generated by the function f.

2.Main result

Let f is a locally integrable function on R. With T_f we denote the corresponding regular distribution defined by $\langle \mathrm{T}_f, \varphi \rangle = \int_R \ f(x)\varphi(x)dx,\ \varphi \epsilon S.$

Theorem 3. Let $z_{k_1}, z_{k_2}, \dots, z_{k_{N_k}}, k = 1,2, \dots, Imz \neq 1, N_k \leq \infty,$ are given complex numbers which satisfy (1.5) and F be of the form (1.6) (as in Theorem 2). Let T_{F^*}, $F^* \in L^p$(R), be the distribution in S' generated by the boundary value $F^*(x)$ of $F(z)$ on Π^+.

Then there exists a sequence, $\{R_k\}$, of rational functions of the form (1.4) and, respectively, a sequence of distributions $\{T_{R_k}\}, T_{R_k} \in S'$ generated by R_k, satisfying

(i) $T_{R_k} \to T_{F^*},\ k \to \infty\ in\ S'$

(ii) $\overline{\lim_{n\to\infty}} \int_{-\infty}^{\infty} |R_k(x)|^p |\varphi(x)|\, dx < \infty$, for all $\varphi \epsilon S$.

Proof. (i) General idea is to prove the convergence in S' in weak sense and to use Banach Steinhaus theorem to obtain the strong convergence (note that the space S is of second category).

We start with applying Theorem 2, and obtain a sequence $\{R_k\}$, of rational functions of the form (1.4) for which the following holds

$$\lim_{k\to\infty} \int_R \ |F^*(x) - R_k(x)|^p\, dx = 0. \quad (2.1)$$

Triangle inequality implies

$$\|R_k\|_{L^p(\sigma)} = (\textstyle\int_R \ |R_k(x)|^p d\sigma(x)\ )^{\frac{1}{p}} \ = (\int_R \ |R_k(x) - F^*(x) + F^*(x)|^p d\sigma(x)\ )^{\frac{1}{p}}$$

$$\leq (\textstyle\int_R \ |R_k(x) - F^*(x)|^p d\sigma(x)\ )^{\frac{1}{p}} + (\int_R \ |F^*(x)|^p d\sigma(x)\ )^{\frac{1}{p}} < C,$$

Hence $R_k(x) \in L^p(R)$.

Now choose arbitrary $\varphi \epsilon S$ and fix it. We denote with q the Hölder conjugate of p, i.e.

$$\frac{1}{p} + \frac{1}{q} = 1,$$

$$\left|\int_R f(x)\varphi(x)dx\right| \leq \int_R |f(x)\varphi(x)dx| \leq (\int_R |f(x)|^p dx)^{\frac{1}{p}} (\int_R |\varphi(x)|^q dx)^{\frac{1}{q}}$$

for arbitrary function $f \in L^p, \varphi \in S$.

We use the previous and obtain

$$\left|\langle T_{R_k}, \varphi\rangle - \langle T_{F^*}, \varphi\rangle\right| = \left|\int_{-\infty}^{\infty} R_k(x) \quad \varphi(x)dx - \int_{-\infty}^{\infty} F^*(x)\varphi(x)dx\right|$$

$$= \left|\int_{-\infty}^{\infty}[R_k(x) - F^*(x)]\varphi(x)dx\right| \leq \int_{-\infty}^{\infty} |R_k(x) - F^*(x)||\varphi(x)|dx$$

$$\leq \left(\int_R |R_k(x) - F^*(x)|^p dx\right)^{\frac{1}{p}} \left(\int_R |\varphi(x)|^q dx\right)^{\frac{1}{q}}$$

$\leq M\left(\int_R |R_k(x) - F^*(x)|^p dx\right)^{\frac{1}{p}} \to 0,$ (2.1) when $k \to \infty$. In the previous calculations we used $M = \sup\{|\varphi(x)| \,|\, x \in R\}$ which is obviously finite since the inclusion $S \subset L^q$ is continuous and dense for arbitrary $1 \leq q < \infty$. The discussion on the start of the proof implies the claim or

$T_{R_k} \to T_{F^*},\; k \to \infty \; in \; S'$ in the strong topology.

(ii) Let $\varphi \epsilon S$ be arbitrary and fixed. Then Minkowski inequality implies

$$\left(\int_R |R_k(x)|^p \varphi(x)dx\right)^{\frac{1}{p}} \leq M^{\frac{1}{p}}\left(\int_R |R_k(x) - F^*(x) + F^*(x)|^p dx\right)^{\frac{1}{p}}$$

$$\leq M^{\frac{1}{p}}\left(\int_R |R_k(x)-F^*(x)|^p dx\right)+M^{\frac{1}{p}}\left(\int_R |F^*(x)|^p dx\right)^{\frac{1}{p}} = I_1 + I_2.$$

The integral I_1 tends to 0 when $k \to \infty$ which implies that $\int_R |R_k(x)|^p \varphi(x)dx \leq M_\varphi^{\frac{1}{p}} \|F^*\|_p + C$, for arbitrary $k \in N$. The latter implies (ii).

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