Original Research Article

Global existence for compressible Euler equations with damping in partial space-period domains

Abstract

In this paper, we are concerned with the global existence of solutions to isentropic compressible Euler equations with damping in partial space-period domains. Based on the uniform energy estimates, we obtain the global existence for any spatial dimension if the initial data is sufficiently close to an equilibrium. Simultaneously, we show that the vorticity and its derivatives decay exponentially to zero in two and three dimensions.

Keywords: Compressible Euler equations, damping, partial space-period domains, energy estimates, global existence

Mathematical Subject Classification 2010: 35L70, 35L65, 35L67, 76N15

1 Introduction

In this paper, we consider the isentropic compressible Euler equations with damping for $(t,x) \in [0,+\infty)\times\Omega$:

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u\otimes u + p\,\mathrm{I}_d) = -\rho u, \end{cases} \tag{1.1}$$

where $\Omega = \mathbb{T}^n \times \mathbb{R}^m, \mathbb{T}^n = [0,2\pi]^n$ is the n-dimensional torus, $m+n=d, n\geq 1, d\geq 2$, ρ is the density, $u=(u_1,u_2,...,u_d)$ is the velocity, $p=p(\rho)$ is the pressure with state equation $p(\rho)=A\rho^\gamma$ with constants $A>0$ and $\gamma>1$. From now on, for convenience, we assume that $A=\frac{1}{\gamma}$.

The initial data of the system (1.1) are given as follows

$$(\rho(0,x),u(0,x)) = (1+\varepsilon\rho_0(x),\varepsilon u_0(x)), \quad x\in\Omega, \tag{1.2}$$

where $1+\varepsilon\rho_0(x)>0$ for all $x\in\Omega$, $\varepsilon>0$ is sufficiently small. There are many literatures on the global existence of the Cauchy problem (1.1)-(1.2). In one-dimensional case, the global existence was proved by T. Nishida in [11] and the long-time behavior of (ρ,u) was established in [5]-[7]. For more results about the L^∞ entropy weak solution and the BV solution, one can see [2], [12] and the references therein. When it comes to the general multi-dimensional case, the global existence and long-time behavior of the solution were established in [13]-[17]. In particular, Lu used the semigroup method and proved the exponential stability of constant steady state on torus in [9]. Fang and Xu obtained the global C^1 solution in the framework of Besov space in [3]. It should be pointed out that the smallness of the parameter ε plays a key role in the above results. For instance, the authors in [14] proved that the C^1 solutions will blow up for large amplitude initial data in three dimension. For the non-isentropic compressible Euler equations with constant damping, the global solutions were also obtained, we refer it to [19], [20] and the references therein. Moreover, the work on the general hyperbolic systems including (1.1) was studied in [4], [8], [18] and so on.

In the present paper, we will consider the problem in partial space-period domains. Motivated by the energy method in [14], we establish the uniform energy estimates and obtain the global existence for any spatial dimension. In addition, we can also obtain that the vorticity and its derivatives decay exponentially to zero for $d = 2, 3$.

For convenience, we shall use the following convention throughout this paper:

- $f \lesssim g$ means there exists a generic constant C such that $f \leq Cg$;
- The differential operator ∂ denotes the time-spatial derivatives, i.e. $\partial = (\partial_t, \partial_{x_1}, ..., \partial_{x_d})$;
- The Latin letters a, b, c denote multiple indices, for example, $a = (a_0, a_1, ..., a_d)$, $|a| = a_0 + a_1 + ... + a_d$ is its length;
- For two multiple indices a, b, $b \leq a$ means $b_i \leq a_i$ for all $i = 0, 1, ..., d$;
- Define the operator $\partial^a = \partial_t^{a_0} \partial_{x_1}^{a_1} ... \partial_{x_d}^{a_d}$ and $\partial^{\leq a} = \sum\limits_{0 \leq b \leq a} \partial^b$, where a_i is some non-negative integer for all $i = 0, 1, ..., d$;
- $\|f(t, \cdot)\|_2$ stands for $\|f(t, x, y)\|_{L_x^2(\mathbb{T}^n)L_y^2(\mathbb{R}^m)}$ and $\|f(t, \cdot)\|_\infty$ stands for $\|f(t, x, y)\|_{L_x^\infty(\mathbb{T}^n)L_y^\infty(\mathbb{R}^m)}$.

At first, we reformulate the Cauchy problem (1.1)-(1.2). Let

$$\sigma(\rho) = \frac{2}{\gamma - 1}(c(\rho) - 1), \tag{1.3}$$

where $c(\rho) = \sqrt{p'(\rho)} = \rho^{\frac{\gamma-1}{2}}$ is the sound speed. Then the problem (1.1)-(1.2) can be rewritten as the following symmetric form

$$\begin{cases} \partial_t \sigma + u \cdot \nabla \sigma + \text{div} u + \frac{\gamma-1}{2} \sigma \text{div} u = 0, \\ \partial_t u + u \cdot \nabla u + u + \nabla \sigma + \frac{\gamma-1}{2} \sigma \nabla \sigma = 0, \\ \sigma(0, x) = \frac{2}{\gamma-1}[(1 + \varepsilon \rho_0(x))^{\frac{\gamma-1}{2}} - 1], \\ u(0, x) = \varepsilon u_0(x). \end{cases} \tag{1.4}$$

By the standard method in [10], the problem (1.4) has a local solution $(\sigma, u) \in C^1([0, t] \times \mathbb{R}^d)$ for some $t > 0$. To obtain the global existence, the main task is to derive the uniform prior estimates by making full use of the damping term. To this end, for any $k \in \mathbb{N}^+$, we define the energy functionals

$$E[f](t) = \|f(t, \cdot)\|_2, \quad E_k[f](t) = \sum_{1 \leq |a| \leq k} E[\partial^a f](t) \tag{1.5}$$

and the dissipation functionals

$$D[f](t) = \left(\int_0^t \int_{\mathbb{T}^n \times \mathbb{R}^m} f^2(\tau, x, y) dx dy d\tau \right)^{\frac{1}{2}}, \quad D_k[f](t) = \sum_{1 \leq |a| \leq k} D[\partial^a f](t). \tag{1.6}$$

Denote

$$E[f_1, f_2](t) = E[f_1](t) + E[f_2](t), \quad E_k[f_1, f_2](t) = E_k[f_1](t) + E_k[f_2](t) \tag{1.7}$$

and the similar definitions for $D[f_1, f_2](t), D_k[f_1, f_2](t)$.

Our main results are

Theorem 1.1 *Assume that* $(\rho_0(x), u_0(x)) \in H^{2N+1}(\mathbb{R}^d)$, *where* $N = [\frac{m}{2}] + [\frac{n}{2}] + 2$, $1 + \varepsilon\rho_0(x) > 0$ *for all* $x \in \Omega$, $\varepsilon > 0$ *is sufficiently small. Then there exists a global solution* $(\rho, u) \in C([0, +\infty), H^{2N+1}(\mathbb{T}^n \times \mathbb{R}^m)) \cap C^1([0, +\infty), H^{2N}(\mathbb{T}^n \times \mathbb{R}^m))$ *for the problem* (1.1)-(1.2) *and the solution satisfies*

$$\|(\rho - 1)(t, \cdot)\|_{H^{2N+1}(\mathbb{R}^d)} + \|u(t, \cdot)\|_{H^{2N+1}(\mathbb{R}^d)} \lesssim \varepsilon, \quad \forall\, t > 0. \tag{1.8}$$

Theorem 1.2 *Under the conditions in theorem 1.1, for any* C^1 *solution* (ρ, u), *we have that the vorticity* $\omega = \nabla \times u$ *and its derivatives decay exponentially to zero when* $d = 2, 3$.

Remark 1.1 *The condition that* $1+\varepsilon\rho_0(x) > 0$ *for all* $x \in \Omega$ *is very important to exclude the appearance of vacuum for the classical solutions to* (1.1)-(1.2). *In fact, for any solution* $(\rho, u) \in C^1([0, t] \times \Omega)$ *to* (1.1)-(1.2), *if we set*

$$\frac{dx}{dt} = u(t, x), \quad x(0) = x_0,$$

then by the method of characteristic and the first equation of (1.1), *we can get*

$$\rho(t, x(t)) = (1 + \varepsilon\rho_0(x_0)) \exp\left(-\int_0^t \mathrm{div} u(s, x(s))ds\right) > 0.$$

Remark 1.2 *The unknown function transformation* (1.3) *is valid without vacuum. In this sense, for any* $t > 0$, $(\rho, u) \in C^1([0, t] \times \Omega)$ *solves the problem* (1.1)-(1.2) *with* $\rho > 0$, *then* $(\sigma, u) \in C^1([0, t] \times \Omega)$ *solves the problem* (1.4) *with* $\frac{\gamma-1}{2}\sigma + 1 > 0$; *Conversely, if* $(\sigma, u) \in C^1([0, t] \times \Omega)$ *solves the problem* (1.4) *with* $\frac{\gamma-1}{2}\sigma + 1 > 0$, *let* $\rho = c^{-1}(\frac{\gamma-1}{2}\sigma + 1)$, *then* $(\rho, u) \in C^1([0, t] \times \Omega)$ *solves the problem* (1.1)-(1.2) *with* $\rho > 0$.

Remark 1.3 *Our result has no restriction on the spatial dimensions. In addition, based on the method in this paper, the conclusion can be extended to the compressible Euler equations with constant damping in the half space* R^d_+.

2 Proof of Theorem 1.1

Throughout this section, we will always assume that

$$\sup_{0 \le \tau \le t} E[\sigma, u](\tau) + \sup_{0 \le \tau \le t} E_{2N}[\sigma, u](\tau) \lesssim \varepsilon, \tag{2.1}$$

where $N = [\frac{m}{2}] + [\frac{n}{2}] + 2$. First, we introduce a Sobolev embedding theorem, one can see [1] for details.

Lemma 2.1 *Assume that* $f(t, x, y) \in H^N(\Omega), (x, y) \in \mathbb{T}^n \times \mathbb{R}^m$, *then we have*

$$\|f(t, \cdot)\|_\infty \lesssim \|\partial_x^{\le [\frac{n}{2}]+1} \partial_y^{\le [\frac{m}{2}]+1} f(t, x, y)\|_2 \lesssim E[f](t) + E_N[f](t). \tag{2.2}$$

Thus it follows from lemma 2.1 and the assumption (2.1) that

$$\sup_{0\le\tau\le t}\|\partial^a\sigma(\tau,\cdot)\|_\infty+\sup_{0\le\tau\le t}\|\partial^a u(\tau,\cdot)\|_\infty\lesssim\varepsilon,\quad 0\le|a|\le N. \tag{2.3}$$

In the following, to prove theorem 1.1, we split the process into several lemmas.

Lemma 2.2 *Under the assumption* (2.1)*, for all* $t>0$*, we have the following connection between the dissipation functionals* $D_k[\sigma](t)$ *and* $D_k[u](t)$ *that*

$$D_k[\sigma](t)\lesssim D_k[u](t)+D[u](t),\quad 1\le k\le 2N+1. \tag{2.4}$$

Proof. Rewrite the equations in (1.4) as follows

$$\begin{cases}\partial_t\sigma=-\mathrm{div}u-u\cdot\nabla\sigma-\frac{\gamma-1}{2}\sigma\mathrm{div}u,\\ \nabla\sigma=-\partial_t u-u-u\cdot\nabla u-\frac{\gamma-1}{2}\sigma\nabla\sigma.\end{cases} \tag{2.5}$$

By taking the L^2 norm of (2.5), squaring, integrating them over $[0,t]$ and adding up the two resulting equalities, we arrive at

$$\begin{aligned}D_1^2[\sigma](t)\ \lesssim\ & D_1^2[u](t)+D^2[u](t)\\ &+\int_0^t\int_{\mathbb{T}^n\times\mathbb{R}^m}\Big(|u\cdot\nabla\sigma|^2+|u\cdot\nabla u|^2+|\sigma\mathrm{div}u|^2+|\sigma\nabla\sigma|^2\Big)dxdyd\tau.\end{aligned} \tag{2.6}$$

This together with (2.3) yields

$$D_1^2[\sigma](t)\lesssim D_1^2[u](t)+D^2[u](t)+\varepsilon^2D_1^2[\sigma,u](t).$$

By the smallness of ε and Young's inequality, we conclude that

$$D_1[\sigma](t)\lesssim D_1[u](t)+D[u](t), \tag{2.7}$$

which implies that (2.4) holds for $k=1$. Next we prove (2.4) for $k>1$. Applying the operator $\partial^a(1\le|a|\le 2N)$ to (2.5) together with the Leibniz's rule derives

$$\begin{cases}\partial_t(\partial^a\sigma)=-\mathrm{div}(\partial^a u)-Q_1^a,\quad Q_1^a=\sum\limits_{b+c=a}C_{bc}^aQ_1^{bc},\\ \nabla(\partial^a\sigma)=-\partial_t(\partial^a u)-\partial^a u-Q_2^a,\quad Q_2^a=\sum\limits_{b+c=a}C_{bc}^aQ_2^{bc},\end{cases} \tag{2.8}$$

where

$$Q_1^{bc}=\partial^b u\cdot\nabla(\partial^c\sigma)+\frac{\gamma-1}{2}\partial^b\sigma\mathrm{div}(\partial^c u),\quad Q_2^{bc}=\partial^b u\cdot\nabla(\partial^c u)+\frac{\gamma-1}{2}\partial^b\sigma\nabla(\partial^c\sigma), \tag{2.9}$$

and C_{bc}^a are some constants. Employing the similar process as (2.6) together with (2.7), we can get

$$D_2^2[\sigma](t)\lesssim D_2^2[u](t)+D^2[u](t)+\sum_{|b|+|c|\le 1}D^2[Q_1^{bc},Q_2^{bc}](t). \tag{2.10}$$

By Hölder inequality and the estimate (2.3), one has

$$\sum_{|b|+|c|\le 1}D^2[Q_1^{bc},Q_2^{bc}](t)\lesssim\varepsilon^2D_2^2[\sigma,u](t).$$

This together with (2.10) yields (2.4) holds for $k = 2$. After an iterative process, we have

$$D_k^2[\sigma](t) \lesssim D_k^2[u](t) + D^2[u](t) + \sum_{|b|+|c|\leq k-1} D^2[Q_1^{bc}, Q_2^{bc}](t), \quad 1 \leq k \leq 2N+1. \tag{2.11}$$

By Hölder inequality and taking the L^∞ norm in Q_1^{bc}, Q_2^{bc} for those terms with $|b| \leq N$ or $|c| \leq N-1$ along with (2.3), one has

$$\sum_{|b|+|c|\leq k-1} D^2[Q_1^{bc}, Q_2^{bc}](t) \lesssim \varepsilon^2 D_k^2[\sigma, u](t), \quad 1 \leq k \leq 2N+1. \tag{2.12}$$

Substituting (2.12) into (2.11) and using the smallness of ε derive

$$D_k[\sigma](t) \lesssim D_k[u](t) + D[u](t), \quad 1 \leq k \leq 2N+1,$$

where we have used the Young's inequality. Thus the proof of lemma 2.2 is finished. □

Now, we begin to estimate the L^2 norm of the solution (σ, u).

Lemma 2.3 *Under the assumption* (2.1)*, for all $t > 0$, it holds that*

$$E^2[\sigma, u](t) + D^2[u](t) \lesssim E^2[\sigma, u](0) + \varepsilon D_1^2[u](t). \tag{2.13}$$

Proof. Multiplying the first equation of (1.4) by σ and the second one by u, and adding the resulting equations together derive

$$\frac{1}{2}\partial_t(\sigma^2 + |u|^2) + |u|^2 + \mathrm{div}(\sigma u) + \frac{\gamma-1}{2}\mathrm{div}(\sigma^2 u) = -\sigma u \cdot \nabla\sigma - \langle u \cdot \nabla u, u\rangle + \frac{\gamma-1}{2}\sigma\nabla\sigma \cdot u, \tag{2.14}$$

where $\langle \cdot, \cdot \rangle$ represents the standard inner product in $\mathbb{R}^d$. Integrating (2.14) by parts over $[0,t] \times \Omega$ gives

$$\begin{aligned} E^2[\sigma, u](t) + D^2[u](t) \quad &\lesssim \quad E^2[\sigma, u](0) + \int_0^t \int_{\mathbb{T}^n\times\mathbb{R}^m} |\sigma||u||\nabla\sigma| dxdyd\tau \\ &\quad + \int_0^t \int_{\mathbb{T}^n\times\mathbb{R}^m} |\nabla u||u|^2 dxdyd\tau. \end{aligned} \tag{2.15}$$

Applying Cauchy-Schwarz inequality to (2.15) and using (2.3), one can obtain

$$E^2[\sigma, u](t) + D^2[u](t) \lesssim E^2[\sigma, u](0) + \varepsilon D^2[u](t) + \varepsilon D_1^2[\sigma](t),$$

which together with lemma 2.2 yields

$$E^2[\sigma, u](t) + D^2[u](t) \lesssim E^2[\sigma, u](0) + \varepsilon D^2[u](t) + \varepsilon D_1^2[u](t). \tag{2.16}$$

Combining (2.16) and the smallness of ε derives the desired result (2.13). □

Next, we establish the estimate of the derivatives for the solution (ρ, u).

Lemma 2.4 *Under the assumption* (2.1)*, for all $t > 0$, $1 \leq k \leq 2N+1$, it holds that*

$$E_k^2[\sigma, u](t) + D_k^2[u](t) \lesssim E_k^2[\sigma, u](0) + \varepsilon D^2[u](t). \tag{2.17}$$

Proof. Multiplying the first equation of (2.8) by $\partial^a\sigma$ and the second one by $\partial^a u$, and adding the resulting equations together give

$$\begin{aligned}
&\frac{1}{2}\partial_t\big(|\partial^a\sigma|^2+|\partial^a u|^2\big)+|\partial^a u|^2\\
+\quad&\mathrm{div}\Big[\partial^a\sigma\partial^a u+\frac{1}{2}u\big(|\partial^a\sigma|^2+|\partial^a u|^2\big)+\frac{\gamma-1}{2}\sigma\partial^a\sigma\partial^a u\Big]\\
=\quad&\frac{1}{2}\mathrm{div}u\big(|\partial^a\sigma|^2+|\partial^a u|^2\big)+\frac{\gamma-1}{2}\partial^a\sigma\partial^a u\cdot\nabla\sigma\\
&-\sum_{\substack{b+c=a\\|c|<|a|}}\big(\partial^a\sigma Q_1^{bc}+\partial^a u\cdot Q_2^{bc}\big),
\end{aligned}\tag{2.18}$$

where $Q_i^{bc}, i=1,2$ are defined in (2.9) and the multi-index a satisfies $1\le|a|\le k\le 2N+1$. Integrating (2.18) by parts over $[0,t]\times\Omega$ and adding the resulting equalities with $|a|$ from 1 to k, we can obtain

$$\begin{aligned}
E_k^2[\sigma,u](t)+D_k^2[u](t)\ \lesssim\ & E_k^2[\sigma,u](0)\\
&+\sum_{|a|\le k}\Big\{\int_0^t\int_{\mathbb{T}^n\times\mathbb{R}^m}|\mathrm{div}u|\big(|\partial^a\sigma|^2+|\partial^a u|^2\big)dxdyd\tau\\
&+\int_0^t\int_{\mathbb{T}^n\times\mathbb{R}^m}|\partial^a\sigma||\partial^a u||\nabla\sigma|dxdyd\tau\\
&+\sum_{\substack{b+c=a\\|c|<|a|}}\int_0^t\int_{\mathbb{T}^n\times\mathbb{R}^m}\big(|\partial^a\sigma||Q_1^{bc}|+|\partial^a u||Q_2^{bc}|\big)dxdyd\tau\Big\}.
\end{aligned}\tag{2.19}$$

Applying (2.3) to the terms div$u$ and $\nabla\sigma$ in the second and third terms of (2.19) on the right-hand side, we arrive at

$$\sum_{|a|\le k}\int_0^t\int_{\mathbb{T}^n\times\mathbb{R}^m}\Big\{|\mathrm{div}u|\big(|\partial^a\sigma|^2+|\partial^a u|^2\big)+|\partial^a\sigma||\partial^a u||\nabla\sigma|\Big\}dxdyd\tau\lesssim\varepsilon D_k^2[\sigma,u](t),\tag{2.20}$$

where the Cauchy-Schwarz inequality has been used. Taking the L^∞ norm of in Q_1^{bc}, Q_2^{bc} for $|b|\le N$ or $|c|\le N-1$ and using (2.3) again, one has

$$\sum_{|a|\le k}\sum_{\substack{b+c=a\\|c|<|a|}}\int_0^t\int_{\mathbb{T}^n\times\mathbb{R}^m}\big(|\partial^a\sigma||Q_1^{bc}|+|\partial^a u||Q_2^{bc}|\big)dxdyd\tau\lesssim\varepsilon D_k^2[\sigma,u](t).\tag{2.21}$$

Substituting (2.20), (2.21) into (2.19), we get

$$E_k^2[\sigma,u](t)+D_k^2[u](t)\lesssim E_k^2[\sigma,u](0)+\varepsilon D_k^2[\sigma,u](t).$$

This together with lemma 2.2 yields

$$E_k^2[\sigma,u](t)+D_k^2[u](t)\lesssim E_k^2[\sigma,u](0)+\varepsilon(D_k^2[u](t)+D^2[u](t))\tag{2.22}$$

which implies (2.17) by the smallness of ε. Thus we complete the proof of lemma 2.4. □

Based on lemma 2.3-2.4, we are ready to prove theorem 1.1.

Proof of theorem 1.1 By lemma 2.3-2.4 and the smallness of ε, we get

$$E^2[\sigma,u](t)+E^2_{2N+1}[\sigma,u](t)+D^2[u](t)+D^2_{2N+1}[u](t)\lesssim E^2[\sigma,u](0)+E^2_{2N+1}[\sigma,u](0).$$

Since $(\rho_0(x),u_0(x))\in H^{2N+1}(\mathbb{R}^d)$, then we have

$$E^2[\sigma,u](t)+E^2_{2N+1}[\sigma,u](t)\lesssim\varepsilon^2. \tag{2.23}$$

Combining (2.23) with the local existence result and the continuation argument, we obtain the global C^1 solution (ρ,u) to (1.1). □

3 Proof of Theorem 1.2

When $d=3$, by the second equation of (1.1), we get the equation of the vorticity that

$$\partial_t\omega+\omega+u\cdot\nabla\omega+\omega\,\mathrm{div}u=\omega\cdot\nabla u, \tag{3.1}$$

where $\omega=\mathrm{curl}u=(\partial_2u_3-\partial_3u_2,\partial_3u_1-\partial_1u_3,\partial_1u_2-\partial_2u_1)^T$. Applying the operator $\partial^a(0\le|a|\le3)$ to (3.1) gives

$$\begin{aligned}&\partial_t(\partial^a\omega)+\partial^a\omega+u\cdot\nabla\partial^a\omega\\=&-\sum_{0<b\le a}K_{a,b}\partial^bu\cdot\nabla\partial^{a-b}\omega\\&-\sum_{0\le c\le a}K_{a,c}(\partial^c\omega\,\partial^{a-c}\mathrm{div}u-\partial^c\omega\cdot\nabla\partial^{a-c}u),\end{aligned} \tag{3.2}$$

where $K_{a,b},K_{a,c}$ are some constants. Multiplying (3.2) by the factor $\partial^a\omega$ and integrating the resulting equality over Ω derive

$$\begin{aligned}&\frac{d}{dt}\|\partial^a\omega(t,\cdot)\|_2^2+\|\partial^a\omega(t,\cdot)\|_2^2\\\lesssim&\int_{\mathbb{T}^n\times\mathbb{R}^m}|\mathrm{div}u||\partial^a\omega|^2dxdy\\&+\sum_{0<b\le a}\int_{\mathbb{T}^n\times\mathbb{R}^m}|\partial^bu||\nabla\partial^{a-b}\omega||\partial^a\omega|dxdy\\&+\sum_{0\le c\le a}\int_{\mathbb{T}^n\times\mathbb{R}^m}(|\partial^c\omega||\partial^{a-c}\mathrm{div}u||\partial^a\omega|+|\partial^c\omega||\partial^{a-c}u||\partial^a\omega|)dxdy.\end{aligned} \tag{3.3}$$

Then summing up (3.3) with $|a|$ from 0 to 3 yields

$$\begin{aligned}&\frac{d}{dt}\|\partial^{a\le3}\omega(t,\cdot)\|_2^2+\|\partial^{a\le3}\omega(t,\cdot)\|_2^2\\\lesssim&\|\partial^{a\le3}u(t,\cdot)\|_\infty\|\partial^{a\le3}\omega(t,\cdot)\|_2^2\\&+\|\omega(t,\cdot)\|_\infty\|\partial^{a\le3}\omega(t,\cdot)\|_2\|\partial^{a\le4}u(t,\cdot)\|_2.\end{aligned} \tag{3.4}$$

By lemma 2.1, we have $\|\omega(t,\cdot)\|_\infty\lesssim\|\partial^{a\le3}\omega(t,\cdot)\|_2$. Since $\|\partial^{a\le3}u(t,\cdot)\|_\infty\lesssim\varepsilon$, $\|u(t,\cdot)\|_{H^4}\lesssim\varepsilon$, then (3.4) becomes

$$\frac{d}{dt}\|\partial^{a\le3}\omega(t,\cdot)\|_2^2+\|\partial^{a\le3}\omega(t,\cdot)\|_2^2\lesssim\varepsilon\|\partial^{a\le3}\omega(t,\cdot)\|_2^2. \tag{3.5}$$

This together with the smallness of ε yields

$$\frac{d}{dt}\|\partial^{a\leq 3}\omega(t,\cdot)\|_2^2 + \|\partial^{a\leq 3}\omega(t,\cdot)\|_2^2 \leq 0. \tag{3.6}$$

Thus it follow from (3.6) that

$$\|\partial^{a\leq 3}\omega(t,\cdot)\|_2 \leq \varepsilon e^{-\frac{t}{2}} \tag{3.7}$$

and we obtain the exponential decay of the vorticity and its derivatives when $d = 3$.

Remark 3.1 *For $d = 2$, the equation of the vorticity becomes*

$$\partial_t\omega + \omega + u\cdot\nabla\omega + \omega\, divu = 0, \tag{3.8}$$

where $\omega = curlu = \partial_1 u_2 - \partial_2 u_1$. Compared with the equation of the vorticity for $d = 3$, (3.8) *is much simpler. One only needs to remove the term deriving from $\omega\cdot\nabla u$ in the proof for $d = 3$ above and can also obtain the exponential decay of the vorticity and its derivatives when $d = 2$. Thus we finish the proof of theorem 1.2.*

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