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Original Research Article

Steady flow of blood plasma through a non-deformed artery.

ABSTRACT

A mathematical model is developed here with the aim to study the laminar flow of blood plasma through a non-deformed arterial segment. The Navier- Stokes and continuity equations were solved analytically to obtain the axial velocity of blood plasma through the artery. Furthermore, the axial velocity was plotted against varied values of the radius of a sampled artery, results and conclusions were made between the relationships.

Keywords: Newtonian fluid, blood pressure, laminar flow.

INTRODUCTION

The human blood circulatory system provides essential substances such as nutrients and oxygen to the cells and transports metabolic waste products away from the same cells. Human blood is composed of blood cells suspended in blood plasma. The blood plasma which constitutes 55% of blood fluid, is mostly water (92% by volume), and contains dissipated proteins, glucose, mineral ions, hormones and blood cells themselves (Blessy Thomas and K.S Sunam, 2016). The blood cells are mainly red blood cells (also called RBCs or erythrocytes) and white blood cells, including leukocytes and platelets. The red blood cells are small semisolid particles, increase the viscosity of blood and will affect the behaviour of fluid. It has been noted that plasma behaves as a Newtonian fluid whereas the whole blood displays non-Newtonian character.

There are three major types of blood vessels: the arteries through which blood is carried away from the heart at higher physiologic pressures, the capillaries, which enable the actual exchange of water and chemicals between the blood and the tissues, and the veins, which carry blood from the capillaries and back toward the heart at lower physiologic pressures. Because of their

different roles, their structures and wall constituents are also different. The walls of blood vessels have a circumferentially layered structure. The most important layers are intima, media, and adventitia. The internal intima composed of the endothelium cell. The media, which is a layered one, is responsible for most of the vessel mechanical properties. The outer layer is adventitia. The artery possesses the thickest wall amongst the three major blood vessels which enables them to withstand the high pressure of arterial blood. It has a more elastic media which varies according to the size of the artery, with a thin collagenous adventitia compared to both the veins and capillaries.

Laminar flow of fluid is characterized by fluid particles following smooth paths in layers, with each layer moving smoothly past the adjacent layers with little or no mixing. Arterial blood flow is considered as a laminar flow. The study of the laminar flow of blood in arteries plays an important role in the diagnosis and clinical treatment as well as in the fundamental understanding of many cardiovascular diseases.

The Casson model for blood flow through a cylindrical tube states that

$$\sqrt{\tau} = \sqrt{\tau_y} + \sqrt{\eta\dot{\gamma}} \tag{1}$$

Equation 1 (David A. Rubenstein, 2012, p.145) is the relationship between shear stress (τ) and shear rate ($\dot{\gamma}$), where τ_y is a constant yield stress and η is an experimentally fit constant which approximates the fluid's viscosity. The shear stress is defined as

$$\tau = -\frac{r}{2}\frac{dp}{dx} \tag{2}$$
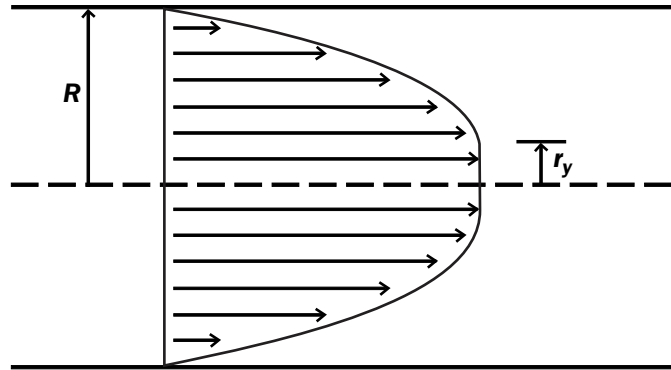
The Casson model can be rewritten as

$$\sqrt{-\frac{r}{2}\frac{dp}{dx}} = \sqrt{\tau_y} + \sqrt{\eta\dot{\gamma}} \tag{3}$$

Solving (3) for the shear rate, we have

$$-\frac{du}{dr} = \dot{\gamma} = \frac{1}{\eta}\left(\sqrt{-\frac{r}{2}\frac{dp}{dx}} - \sqrt{\tau_y}\right)^2 \tag{4}$$

Integrating (4) from r_y to R (tube radius), over the appropriate cross-sectional area, we have the velocity profile as

$$u(r) = \begin{cases} -\frac{1}{4\eta}\frac{dp}{dx}\left(R^2 - r^2 - \frac{8}{3}r_y^{0.5}(R^{1.5} - r^{1.5}) + 2r_y(R - r)\right), r_y \leq r \leq R \\ \qquad -\frac{1}{4\eta}\frac{dp}{dx}(\sqrt{R} - \sqrt{r_y})^3\left(\sqrt{R} + \frac{1}{3}\sqrt{r_y}\right), \ r \leq r_y \end{cases}$$

Fig 1. The velocity profile of blood flowing through a cylindrical vessel using the Casson model. The profile is blunter than a purely Newtonian fluid's velocity profile because the fluid that is between the centreline and r_y flows as a solid. At r_y, the shear stress (τ) exceeds the yield stress (τ_y) and the viscous forces take effect.

The Casson model is a very important tool in the representation of blood flow through the arterial system. Since blood plasma behaves as a Newtonian fluid (Verma and Parihar, 2010, Biswas and Chakraborty, 2010), we make an approximation to the Casson model by allowing the yield stress, $\tau_y = 0$ and treat the flow of blood plasma through a non-deformed artery as a Hagen-Poiseuille flow.

MATHEMATICAL FORMULATION

Consider the one dimensional steady flow of blood plasma through a cross-section of artery of radius r_0, centre o and length L.
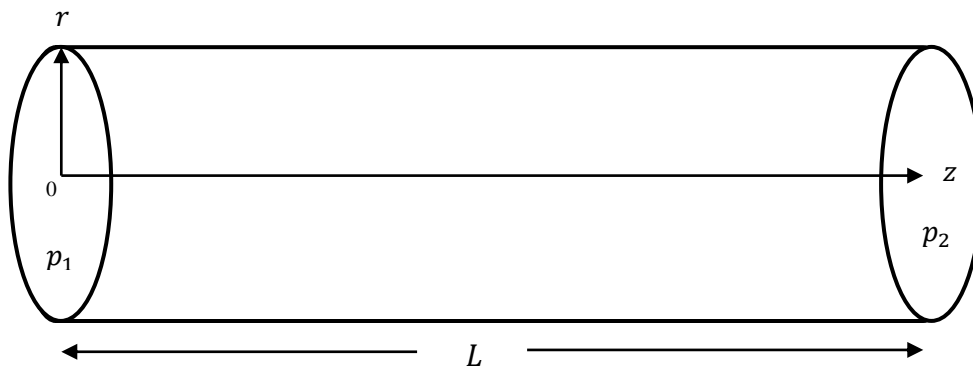
It is well established that the motion of fluids is governed by the Navier-Stokes equations which essentially can be seen as Newton's second law of motion for fluids.

For a compressible Newtonian fluid, this yields

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u}\right) = -\nabla p + \mu\nabla^2\vec{u} + \rho F \qquad (5)$$

Where $\vec{u}$, is the fluid velocity, ρ is the fluid density and μ is the fluid dynamic viscosity. The terms on the left correspond to the inertial forces, the first term on the right is the pressure forces, the second term is the viscous force and the last term is the applied external forces on the fluid.

These equations are always solved together with the continuity equation (conservation of mass)

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\vec{u}) = 0 \qquad (6)$$

Since blood plasma is an incompressible fluid,

$$\frac{\partial \rho}{\partial t} = 0 \text{ and upon dividing by } \rho, \text{ equation (6) becomes:}$$

$$\nabla.\vec{u} = 0 \qquad (7)$$

Equation (5) becomes

$$\rho\frac{D\vec{u}}{Dt} = -\nabla p + \mu\nabla^2\vec{u}$$

Where $\frac{D\vec{u}}{Dt} = \left(\frac{\partial \vec{u}}{\partial t} + (\vec{u}.\nabla)\vec{u}\right)$ and $\rho F = 0$ (since external force $F = 0$).

Dividing through by ρ gives

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\vec{u}$$

But $\frac{\mu}{\rho} = v$ (kinematic viscosity)

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + v\nabla^2\vec{u} \qquad (8)$$

This equation (8) is called the momentum equation.

The boundary conditions for the solutions of equations (7) and (8) are

$$\vec{u}(r) = u \quad \text{and} \quad \vec{u}(r_0) = 0.$$

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Since we are considering the flow of blood plasma through the artery (which has a cylindrical structure), we transform equations (7) and (8) from rectangular coordinates (x, y, z) to cylindrical coordinates (r, θ, z). The cylindrical coordinates (r, θ, z) is a special case of orthogonal curvilinear coordinates (u_1, u_2, u_3).

Where

$$u_1 = r, u_2 = \theta, u_3 = z.$$

Recall the continuity equation for blood plasma
$$\nabla \cdot \vec{u} = 0$$

Now, in the rectangular coordinates (x, y, z) the position vector $\vec{r}$ of a plasmic particle

$$\vec{r} = xi + yj + zk$$

The relationship between the rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) is

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z.$$

Now, in the cylindrical coordinates the position vector is
$$\vec{r} = r\cos\theta i + r\sin\theta j + zk$$

Consider the unit vectors e_r, e_θ and e_z in the cylindrical coordinates

$$e_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|},$$

Where

$$h_r = \left|\frac{\partial \vec{r}}{\partial r}\right|$$

$$= |\cos\theta i + \sin\theta j|$$
$$= \sqrt{\cos^2\theta + \sin^2\theta} = 1$$

Therefore

$$h_r = 1.$$

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$$e_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|}$$

Where

$$h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right|$$

$$= |-rsin\theta i + rcos\theta j|$$

$$= \sqrt{r^2 sin^2\theta + r^2 cos^2\theta} = r$$

Therefore

$$h_\theta = r.$$

$$e_z = \frac{\frac{\partial \vec{r}}{\partial z}}{\left|\frac{\partial \vec{r}}{\partial z}\right|}$$

Where

$$h_z = \left|\frac{\partial \vec{r}}{\partial z}\right|$$

$$= |k| = 1$$

Therefore

$$h_z = 1.$$

The terms h_r, h_θ and h_z are called scale factors.

Now, consider a scalar field v and let dv be the change from point A to B. If the position vector of A is $\vec{r}$, then that of B is $\vec{r} + d\vec{r}$.

Then

$$dv = \frac{\partial v}{\partial r} \, dr + \frac{\partial v}{\partial \theta} \, d\theta + \frac{\partial v}{\partial z} \, dz$$

Let

$$\nabla v = (\nabla v)_r e_r + (\nabla v)_\theta e_\theta + (\nabla v)_z e_z$$

Where $(\nabla v)_{r,\theta,z}$ is the components of ∇v in the r, θ and z directions

Also

$$d\vec{r} = \frac{\partial \vec{r}}{\partial r} \, dr + \frac{\partial \vec{r}}{\partial \theta} \, d\theta + \frac{\partial \vec{r}}{\partial z} \, dz$$

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$$= h_r dr e_r + h_\theta d\theta e_\theta + h_z dz e_z$$

Again,

$$dv = (\nabla v).d\vec{r}$$

$$= [(\nabla v)_r e_r + (\nabla v)_\theta e_\theta + (\nabla v)_z e_z].[h_r dr e_r + h_\theta d\theta e_\theta + h_z dz e_z]$$

$$= (\nabla v)_r h_r dr + (\nabla v)_\theta h_\theta d\theta + (\nabla v)_z h_z dz$$

But

$$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta + \frac{\partial v}{\partial z} dz$$

Equating coefficients, we have

$$\frac{\partial v}{\partial r} = (\nabla v)_r h_r$$

$$\therefore (\nabla v)_r = \frac{1}{h_r}\frac{\partial v}{\partial r}$$

Similarly,

$$(\nabla v)_\theta = \frac{1}{h_\theta}\frac{\partial v}{\partial \theta} \quad \text{and}$$

$$(\nabla v)_z = \frac{1}{h_z}\frac{\partial v}{\partial z}$$

Therefore

$$\nabla v = \frac{1}{h_r}\frac{\partial v}{\partial r}e_r + \frac{1}{h_\theta}\frac{\partial v}{\partial \theta}e_\theta + \frac{1}{h_z}\frac{\partial v}{\partial z}e_z$$

Hence

$$\nabla = \frac{e_r}{h_r}\frac{\partial}{\partial r} + \frac{e_\theta}{h_\theta}\frac{\partial}{\partial \theta} + \frac{e_z}{h_z}\frac{\partial}{\partial z}$$

Let $\vec{u} = u_r e_r + u_\theta e_\theta + u_z e_z$, so equation (7) becomes

$$\nabla.\vec{u} = \left[\frac{1}{h_r}\frac{\partial}{\partial r}e_r + \frac{1}{h_\theta}\frac{\partial}{\partial \theta}e_\theta + \frac{1}{h_z}\frac{\partial}{\partial z}e_z\right].[u_r e_r + u_\theta e_\theta + u_z e_z]$$

$$= \frac{1}{h_r}\frac{\partial u_r}{\partial r} + \frac{1}{h_\theta}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{h_z}\frac{\partial u_z}{\partial z}$$

$$= \frac{1}{h_r h_\theta h_z}\left[\frac{\partial(h_\theta h_z u_r)}{\partial r} + \frac{\partial(h_r h_z u_\theta)}{\partial \theta} + \frac{\partial(h_r h_\theta u_z)}{\partial z}\right]$$

But in cylindrical coordinates $h_r = h_z = 1$ and $h_\theta = r$

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$$\nabla . \vec{u} = \frac{1}{r}\left[\frac{\partial(ru_r)}{\partial r} + \frac{\partial(u_\theta)}{\partial \theta} + \frac{\partial(ru_z)}{\partial z}\right] = 0$$

$$= \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \tag{9}$$

This equation (9) is the continuity equation for a steady incompressible flow of blood plasma through the artery.

Again, returning to the momentum equation

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\nabla p + v\nabla^2 \vec{u} \tag{10}$$

But

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2\right) - \vec{u} \times (\nabla \times \vec{u})$$

Also

$$\nabla^2 \vec{u} = \nabla(\nabla . \vec{u}) - \nabla \times (\nabla \times \vec{u})$$

This in incompressible flow reduces to

$$\nabla^2 \vec{u} = -\nabla \times \omega$$

Where $\omega = \nabla \times \vec{u}$, the vorticity vector, (10) can be written as

$$\frac{\partial \vec{u}}{\partial t} - \vec{u} \times \omega = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}u^2\right) - v(\nabla \times \omega)$$

$$\frac{\partial \vec{u}}{\partial t} + \nabla\left(\frac{1}{2}u^2\right) - \vec{u} \times \omega = -\frac{1}{\rho}\nabla p - v(\nabla \times \omega) \tag{11}$$

In the orthogonal curvilinear coordinate, the components of the gradient and curl are:

$$\nabla = \left(\frac{1}{h_r}\frac{\partial}{\partial r}, \frac{1}{h_\theta}\frac{\partial}{\partial \theta}, \frac{1}{h_z}\frac{\partial}{\partial z}\right) = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right)$$

$$\nabla \times \vec{u} = \omega_r e_r + \omega_\theta e_\theta + \omega_z e_z$$

$$= \begin{vmatrix} e_r & e_\theta & e_z \\ \frac{1}{h_r}\frac{\partial}{\partial r} & \frac{1}{h_\theta}\frac{\partial}{\partial \theta} & \frac{1}{h_z}\frac{\partial}{\partial z} \\ u_r & u_\theta & u_z \end{vmatrix}$$

But $h_r = h_z = 1$ and $h_\theta = r$ in cylindrical coordinates

$$\nabla \times \vec{u} = \begin{vmatrix} e_r & e_\theta & e_z \\ \frac{\partial}{\partial r} & \frac{1}{r}\frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & u_\theta & u_z \end{vmatrix}$$

Hence ω_r, ω_θ and ω_z becomes

$$\omega_r = \frac{1}{r}\left(\frac{\partial}{\partial \theta}(u_z) - \frac{\partial}{\partial z}(ru_\theta)\right) = \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}$$

$$\omega_\theta = \frac{\partial}{\partial z}(u_r) - \frac{\partial}{\partial r}(u_z) = \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}$$

$$\omega_z = \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial}{\partial \theta}(u_r)\right) = \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}$$

Similarly for curl ω, that is $\nabla \times \omega$ is

$$\nabla \times \omega = \frac{1}{r}\begin{vmatrix} e_r & e_\theta & e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z} & r\frac{\partial u_r}{\partial z} - r\frac{\partial u_z}{\partial r} & \frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \end{vmatrix}$$

The component of $\nabla \times \omega$ in the e_r, e_θ and e_z direction is respectively

$$\frac{1}{r}\left[\frac{\partial}{\partial \theta}\left(\frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) - \frac{\partial}{\partial z}\left(r\frac{\partial u_r}{\partial z} - r\frac{\partial u_z}{\partial r}\right)\right]$$

$$\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) - \frac{\partial}{\partial r}\left(\frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial z} - r\frac{\partial u_z}{\partial r}\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\right]$$

In determining the term $-v(\nabla \times \omega)$ in (11), it is found that the analysis simplifies if the zero expression $v(\nabla(\nabla.\vec{u}))$ is added

Recall

$$\nabla.\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

$$= u_r\frac{\partial r}{\partial r} + r\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + r\frac{\partial u_z}{\partial z} = 0$$

$$= u_r + r\frac{\partial u_r}{\partial r} + \frac{\partial u_\theta}{\partial \theta} + r\frac{\partial u_z}{\partial z} = 0$$

Dividing through by r, we obtain

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$$\nabla . \vec{u} = \frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}$$

Therefore $\nabla(\nabla . \vec{u})$, has components

$$\frac{\partial}{\partial r}\left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{u_r}{r} + \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}\right)$$

Now, $v(\nabla(\nabla . \vec{u}) - \nabla \times \omega)$, has components

$$v\left(\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2} + \frac{\partial^2 u_r}{\partial z^2} - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right) = v\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)$$

$$v\left(\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_\theta}{\partial \theta^2} + \frac{\partial^2 u_\theta}{\partial z^2} - \frac{u_\theta}{r^2} - \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right) = v\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} - \frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right) \qquad (12)$$

$$v\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right) = v(\nabla^2 u_z)$$

Where $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$

Similarly, we obtain that $\nabla\left(\frac{1}{2}u^2\right) - u \times \omega$ has components

$$\frac{\partial}{\partial r}\left(\frac{1}{2}u_r{}^2 + \frac{1}{2}u_\theta{}^2 + \frac{1}{2}u_z{}^2\right) - u_\theta\left(\frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right) + u_z\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{2}u_r{}^2 + \frac{1}{2}u_\theta{}^2 + \frac{1}{2}u_z{}^2\right) - u_z\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) + u_r\left(\frac{u_\theta}{r} + \frac{\partial u_\theta}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

$$\frac{\partial}{\partial z}\left(\frac{1}{2}u_r{}^2 + \frac{1}{2}u_\theta{}^2 + \frac{1}{2}u_z{}^2\right) - u_r\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + u_\theta\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)$$

Simplifying to

$$u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} + u_z\frac{\partial u_r}{\partial z} - \frac{u_\theta{}^2}{r}$$

$$u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + u_z\frac{\partial u_\theta}{\partial z} - \frac{u_r u_\theta}{r} \qquad (13)$$

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$$u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z}$$

Substituting (12) and (13) into equation (11), the momentum equation becomes

$$\frac{\partial u_r}{\partial t} + \frac{u_r u_\theta}{r} \frac{\partial u_r}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_r}{\partial \theta} + u_z \frac{\partial u_r}{\partial z} - \frac{u_\theta^2}{r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + v \left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) \qquad (14)$$

$$\frac{\partial u_\theta}{\partial t} + u_r \frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_\theta}{\partial \theta} + u_z \frac{\partial u_\theta}{\partial z} + \frac{u_r u_\theta}{r} = -\frac{1}{\rho r} \frac{\partial p}{\partial \theta} + v \left(\nabla^2 u_\theta - \frac{u_\theta}{r^2} - \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right) \qquad (15)$$

$$\frac{\partial u_z}{\partial t} + u_r \frac{\partial u_z}{\partial r} + \frac{u_\theta}{r} \frac{\partial u_z}{\partial \theta} + u_z \frac{\partial u_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + v \nabla^2 u_z \qquad (16)$$

Equations (14), (15) and (16) are the r, θ and z coordinates respectively of the momentum equation in cylindrical coordinates.

METHOD OF SOLUTION

The motion of plasma in the artery is induced by axial pressure gradient. It is called a Hagen-Poiseuille flow.

Since the blood plasma flow parallel to the axis of the artery

$$u_\theta = u_r = 0, \quad u_z \neq 0$$

From the continuity equation (equation (9)):

$$\frac{1}{r} \frac{\partial (r u_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0$$

$$\frac{\partial u_z}{\partial z} = 0$$

Therefore u_z is a function of r alone $\left(u_z \neq u_z(z) \right)$

Since the flow is steady, it holds that $\frac{\partial \vec{u}}{\partial t} = 0$

From the momentum equation,

Equation (14) becomes

$$-\frac{1}{\rho} \frac{\partial p}{\partial r} = 0 \qquad (17)$$

Equation (15) becomes

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$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} = 0 \tag{18}$$

Equation (16) becomes

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + v(\nabla^2 u_z) = 0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + v\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2} + \frac{\partial^2 u_z}{\partial z^2}\right) = 0 \tag{19}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z} + v\left(\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r}\right) = 0$$

$$-\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right] = 0.$$

From (17) and (18)

$$p \neq p(r) \quad \text{and} \quad p \neq p(\theta)$$

Hence

$$p = p(z) \text{ and}$$

$$\frac{\partial p}{\partial z} = \frac{dp}{dz} = \text{constant}$$

From (19)

$$-\frac{dp}{dz} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right] = 0$$

$$\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right)\right] = \frac{dp}{dz}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) = \frac{r}{\mu}\frac{dp}{dz}$$

$$r\frac{\partial u_z}{\partial r} = \frac{r^2}{2\mu}\frac{dp}{dz} + A \tag{20}$$

$$\frac{\partial u_z}{\partial r} = \frac{r}{2\mu}\frac{dp}{dz} + \frac{A}{r}$$

$$u_z(r) = \frac{r^2}{4\mu}\frac{dp}{dz} + A\ln r + B \tag{21}$$

Boundary conditions

$$u_z \text{ is finite}, r = 0 \text{ (along the axis)}$$

$$u_z = 0, r = r_0 \text{ (no slip condition)}$$

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When $r = 0$, u_z is finite and $\frac{\partial u_z}{\partial r}$ is also finite

Applying boundary conditions

When $r = 0$, (20) becomes

$$A = 0$$

When $r = r_0$, (21) becomes

$$0 = \frac{r_0{}^2}{4\mu} \frac{dp}{dz} + B$$

$$B = -\frac{r_0{}^2}{4\mu} \frac{dp}{dz}$$

$$\therefore u_z = \frac{r^2}{4\mu} \frac{dp}{dz} - \frac{r_0{}^2}{4\mu} \frac{dp}{dz}$$

$$= \frac{1}{4\mu} \frac{dp}{dz} (r^2 - r_0{}^2)$$

But $\frac{dp}{dz} = \frac{p_2 - p_1}{L}$, $p = p_1$ at $z = 0$ and $p = p_2$ at $z = L$

Where L is the length of the cross-section of the artery and p_1, p_2 are the pressures at both ends of the cross-section of the artery.

$$u_z(r) = \frac{p_2 - p_1}{4\mu L} (r^2 - r_0{}^2), \quad 0 \leq r \leq r_0 \tag{22}$$

This equation (22) is the axial velocity of blood plasma through a cross-section of artery of length L.

RESULTS AND DISCUSSIONS

The axial velocity of blood plasma through a cross section of an artery of length L is defined as:

$$u_z(r) = \frac{p_2 - p_1}{4\mu L} (r^2 - r_0{}^2), \quad 0 \leq r \leq r_0$$

We consider a cross-section of a radial artery of

L = 10cm = 100mm

$r_0 = 1$mm

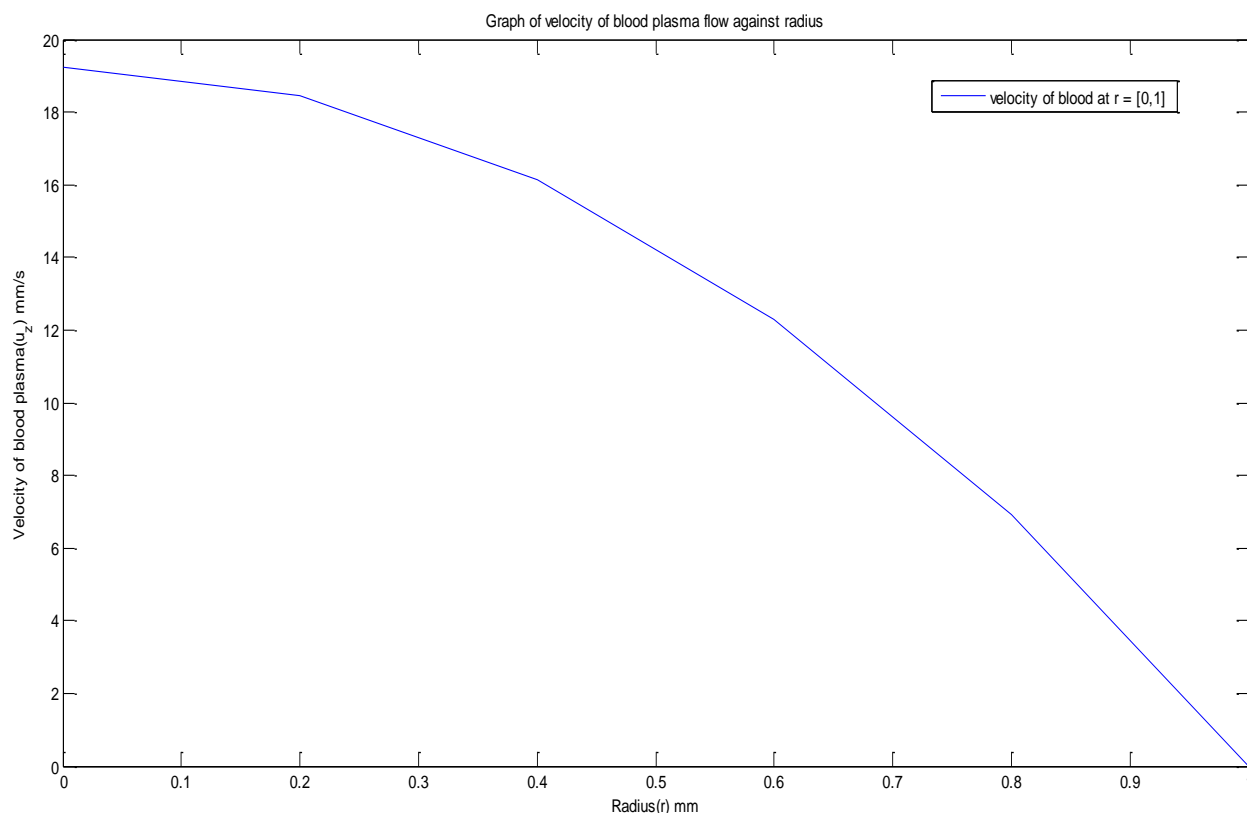$\mu = 1.30$mpa.s = 0.0013pa.s

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$p_1 = 50 \text{mmHg}$

$p_2 = 40 \text{mmHg}$

$\Delta p = -10 \text{mmHg} = -1333.32 \text{pa}$

Table 1 The relation between axial velocity of blood plasma and radius of the artery

| r (mm)	u_z (mm/s)
0	2564.079
0.2	2461.516
0.4	2153.827
0.6	1641.011
0.8	923.069
1	0

Data obtained from Nnamdi, Azikiwe, University Teaching
Hospital, Nnewi, Anambra State, Nigeria.

Fig 2 Variation of axial velocity of blood plasma (u_z) with radius of artery (r)

From figure 2, we see that as the radius of the artery increases, the axial velocity decreases and vice versa. The velocity is highest at the centre of the artery and reduces as $r \to r_0$, it is zero at $r = r_0$, which is at the walls of the artery. The zero velocity at the walls of the artery is as a result of viscous forces in the fluid, which are very high at the vicinity of the walls of the artery (Batchelor, 1967, p.149). This result is used to explain why the velocity of flow is at the highest value at the centre of a river and solid materials tend to move to the shore of the river because the velocity is low to move the materials. This can also be seen in turning tea in a tea cup, the velocity of the tea is highest at the centre, which is why there is a depression at the centre of the tea.

In medicine, this result can be used to explain the situation in stenosis, where the blood flow into a region of reduced radius in a blood vessel is lowered, that is the blood flow is strongly proportional to the blood vessel radius. The resistance (R) to blood flow into a region in a blood vessel is given by (Poiseuille's Law): $R = \frac{8\mu L}{\pi r^4}$.
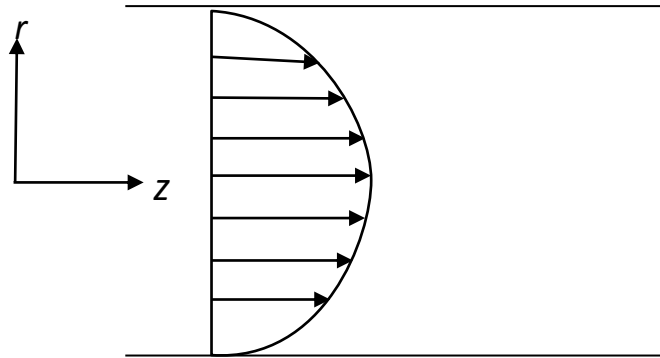
Fig 3 The velocity profile of blood plasma through a cross section of an un-deformed artery. The velocity is at the highest value at the centre $(r = 0)$ and zero at the walls $(r = r_0)$ of the artery because of viscous forces.

CONCLUSION

The model in equation 22 and the graph in figure 2 show the relationship of the arterial velocity of blood plasma flow to the radius of the artery. This result shed light into the dynamics of blood flow where blood flow into a region in a blood vessel depends on the radius of the vessel in that region.

Consumption of fatty foods should be lowered so as to avoid cholesterol building up on the walls of the artery. Smoking causes an increase in blood pressure since the chemicals in tobacco damages the blood vessel walls, causing inflammation and narrowing the arteries.

This work will be able to help readers and researchers understand the necessity of taking precautions to avoid self-inflicted causes that result in shrinkage of the arterial walls. The model helps to illustrate the relationship of the axial velocity of blood plasma and the radius of the artery. Preventive measures and perhaps prescriptive measures can be adopted to ensure that the rate of health issues resulting from shrinkage of arterial walls.

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