Original Research Article

A Generalized Topology from the Edge Set of Maximal Paths of Directed Graphs

Abstract

This study intends to provide a fundamental step towards studying the properties of directed graphs with their corresponding generalized topological spaces. A generalized topology (GT) μ on a nonempty set X is defined as a family of subsets of X such that $\varnothing$ and an arbitrary union of sets in μ is in μ. In this study, we introduce a new generalized topology generated by the set of edges of maximal paths of the directed graph D called the maximal path edge generalized topology (MPEΓ), denoted by $\Gamma_{MP}(D)$. The basic topological properties and connectedness in the context of this new structure are explored and illustrated.

Keywords: Directed Graph, Maximal Path, Maximal Path Edge Generalized Topological Space (MPEΓ space), Γ_{MP}-open sets, Γ_{MP}-closed sets.

2020 Mathematics Subject Classification: 05C10, 05C20, 05C40

1 Introduction

Graphs can be used to represent a variety of real-life situations. This makes graph theory one of the important structures in dicrete mathematics. It is a prominent mathematical tool in many subjects with applications in chemistry, operations research, social sciences, and computer science [1]. On the other hand, topology is also of great importance. Its main interests are the properties that remain unchanged by continuous deformations [14]. Topological structures are significant in data analysis even when the concept of distance is ignored for the extraction and processing of knowledge [1]. As a result, topology is a powerful tool that leads to ideas like connectedness, continuity, and homotopy [1]. In 2002, Császár [6] weakened the conditions of a topological space and introduced the idea of a generalized topological space. In this space, the set itself and the finite intersection of the members of the topology may not be in the topology which distinguishes this concept from that of a topological space.

Graph theory and topology, particularly the connection between topologies and digraphs, have been shown to have a significant link in numerous studies. In 1967, Evans J. W., et al. [8] found a correlation between the set of all topologies and the set of all transitive digraphs, that is, the family $B = \{\mathbb{Q}(v) : v \in V\}$ forms a base for a topology on the set V of nodes of a transitive directed graph $D = (V, E)$, where $\mathbb{Q}(v) = \{v\} \cup \{u \in V : (u, v) \in E(D)\}$. In the same year, Anderson and Chartrand [2] investigated the lattice-graph of the topologies of transitive directed graphs presented by Evans J. W. et al [8]. Other constructions of topology around the vertex set of directed graphs can be seen in the following studies: [4], [11], [13] and [3]. In addition, Abdu and Kilicman [1] in 2018 introduced a new approach in the construction of topology on directed graphs by using two subbasic families to generate two topologies on the edge set, namely, the compatible and incompatible edge topologies.

Many prior efforts on topologizing discrete structures have focused on the construction of topology around the vertex set of directed graphs while no one has tried to associate a generalized topology on the set of edges of a given directed graph. This motivated us to study some properties of directed graphs by their corresponding generalized topology. In this paper, we define a generalized topology on the edge sets of directed graphs. Here, we introduce a base consisting of the family of edge sets of the maximal paths in a directed graph to generate a generalized topology and establish its relevant properties. Also, the concept of connectedness is explored in the sense of this new space. The directed graphs considered in this study are nonempty, finite, and without loops.

2 Preliminaries

In this part, we review some basic notions from graph theory and generalized topology that are relevant to this paper. Also, some known results necessary for the study are also presented.

Definition 2.1. [1] **(Directed Graph (Digraph))** A *directed graph*, denoted by $D = (V(D), E(D), \phi_D)$, consists of a nonempty set $V(D)$ of vertices (or nodes), a set $E(D)$ of directed edges (or arcs), and an incidence function ϕ_D that joins each directed edge of D with an ordered pair of vertices of D.

It is defined in [16] that a graph with a finite number of vertices as well as a finite number of edges is called a *finite graph*; otherwise, it is called an *infinite graph.* If an edge e begins on a vertex u and terminates at a vertex v, then e is *incident out* of u and *incident into* v, and u is called the *initial vertex* and v is called the *terminal vertex* of e [16]. A *self-loop* (or simply *loop*) is an edge whose initial and terminal vertices are the same [16]. A *directed walk* is a finite sequence whose terms are alternately vertices and edges in D such that each edge is incident out of the vertex preceding it in the sequence and incident into the vertex following it; a directed walk in which no vertex is repeated is called a *directed path* (or briefly dipath); and given two vertices u and v, we say that v is *reachable* (or accessible) from u if there exists at least one directed path in D from u to v [16]. If D and H are two digraphs with vertex sets $V(H)$, $V(D)$, and edge sets $E(H)$, $E(D)$, respectively, such that $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$, then we call H as a *subdigraph* of D [5]. A member F of a family of subdigraphs $\mathcal{F}$ in D is *maximal* in $\mathcal{F}$ if no member of $\mathcal{F}$ properly contains F [5]. Other notions in graph theory that are used such as the *underlying graph*, *semi-walk* and *semi-path* can be seen in [16].

Example 2.2. Consider the finite digraphs in Figure 1. Here, H_1 and H_2 are subdigraphs of D. Let $\mathcal{F} = \{H_1, H_2\}$. Because H_1 is not a subdigraph of H_2 and H_2 is not a subdigraph of H_1, both H_1 and H_2 are maximal subdigraphs. In addition, $v_4e_5v_1e_1v_2e_2v_1e_4v_3$ is a directed walk and $v_4e_5v_1e_1v_2e_3v_3$ is a directed path. We can say that vertex v_4 is reachable from vertex v_3, but v_3 is not reachable from v_4.

As cited by Khayerri and Mohamadian in [10], Császár A. introduced the notion of generalized topological space in 2002 as follows:

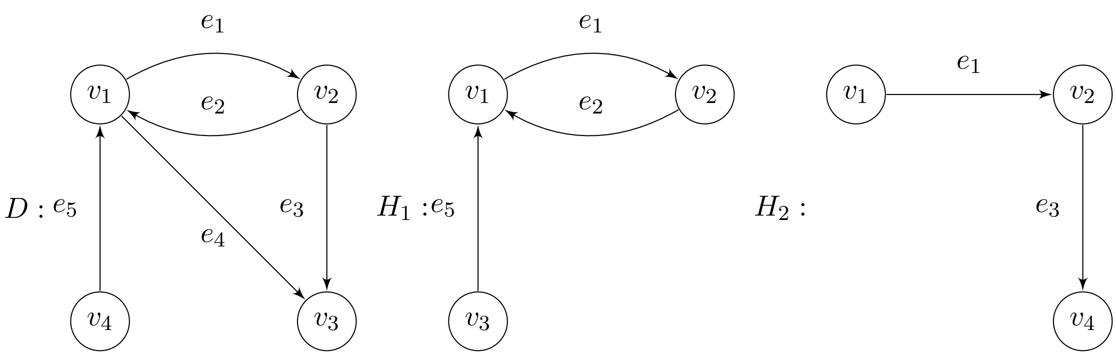

Figure 1: Subdigraphs of Directed Graph D

Definition 2.3. [10] **(Generalized Topological (GT) Space)** Let X be a nonempty set and $\mathcal{P}(X)$ the power set of X. A subset μ of $\mathcal{P}(X)$ is said to be a *generalized topology* (GT) on X if $\varnothing \in \mu$ and an arbitrary union of elements of μ belongs to μ. The elements of μ are called *μ-open sets* and the complemets of μ-open sets are called *μ-closed sets.* The ordered pair (X, μ) is called *generalized topological (GT) space.* A generalized topology is said to be *strong* if $X \in \mu$.

A point $x \in X$ is said to be an *interior point* of $B \subseteq X$ if x belongs to an open set O contained in B. The set of all interior points of B is referred to as the *interior* of B and is denoted by $int_\mu(B)$. Equivalently, $int_\mu(B)$ is the union of all μ-open subsets of B and $cl_\mu(B)$ is called the *closure* of B, the smallest μ-closed subset containing B. Equivalently, $cl_\mu(B)$ is the intersection of all μ-closed subsets which contains B and a point $x \in X$ is said to be an *exterior point* of B if $x \in int_\mu(X \setminus B)$. The set of all exterior points of B is referred to as the *exterior* of B and is denoted by $ext_\mu(B)$ and the *frontier* (or boundary) of B, denoted by $fr_\mu(B)$, is the set $fr_\mu(B) = X \setminus (int_\mu(B) \cup ext_\mu(B))$ [12]. A point $x \in X$ is called a *μ-cluster point* of $B \subseteq X$, if $U \cap (B \setminus \{x\}) \neq \varnothing$ for each $U \in \mu$ with $x \in U$. The set of all μ-cluster points of B, denoted by $d_\mu(B)$, is called *derived set* of B [10]. A subset B of X is said to be *μ-dense* in X if $X = cl_\mu(B)$ [7].

Example 2.4. Let $X = \{a, b, c\}$ and $\mu = \{\varnothing, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. Consider $B = \{b, c\} \subseteq X$. Then, $int_\mu(B) = int_\mu(\{b, c\}) = \{b, c\}$. Observe that the μ-closed sets are X, $\{c\}$, $\{b\}$, $\{a\}$ and $\varnothing$. Then, $cl_\mu(B) = X$. Also, $X \setminus B = X \setminus \{b, c\} = \{a\}$. So, $ext_\mu(B) = int_\mu(X \setminus B) = int_\mu(\{a\}) = \varnothing$. Moreover, $fr_\mu(B) = X \setminus (\{b, c\} \cup \varnothing) = \{a\}$. Now, if we consider $B = \{a, c\}$, then $d(B) = \{b\}$ since $\{a, b\} \cap \{a, c\} = \{a\}$, $\{b, c\} \cap \{a, c\} = \{c\}$ and $X \cap \{a, c\} = \{a, c\}$. Additionally, if $B = \{a, b\}$, then $cl_\mu = X$. Hence, B is μ-dense in X.

Definition 2.5. [10] **(Base of a Generalized Topology)** Let X be a nonempty set and $\beta \subseteq \mathcal{P}(X)$. Then β is called a *base* for a generalized topology μ if $\mu = \{\bigcup \beta' \mid \beta' \subseteq \beta\}$.

Several properties that are known to exist in a regular topological space were discovered as a result of Khayerri and Mohamadian's research. Some of these are as follows:

Proposition 2.6. [10] *Let B be a subset of a space X. Then the following hold:*

1. $int(B) \subseteq B \subseteq cl(B)$.
2. $int(int(B)) = int(B)$ *and* $cl(cl(B)) = cl(B)$.
3. *If* $B' \subseteq B$, *then* $int(B') \subseteq int(B)$ *and* $cl(B') \subseteq cl(B)$.
4. $int(B) = B$ *iff* B *is μ-open.*

5. *$cl(B) = B$ iff B is μ-closed.*
6. *$cl(B) = X \setminus int(X \setminus B)$ and $int(B) = X \setminus cl(X \setminus B)$.*
7. *$x \in cl(B)$ iff $U \cap B \neq \emptyset$ for each $U \in \mu_x$.*
8. *$x \in int(B)$ iff $U \subseteq B$ for some $U \in \mu_x$.*

Proposition 2.7. [10] *For any subset B of a generalized topological space X, we have:*

1. *$cl(B) = B \cup fr(B)$.*
2. *$cl(B) = int(B) \cup fr(B)$.*
3. *$int(B) = B \setminus fr(B)$.*
4. *$int(B) = cl(B) \setminus fr(B)$.*
5. *$X = int(B) \cup B \cup int(X \setminus B)$.*

In the same paper, Khayerri and Mohamadian [10] presented their findings regarding the basis of a GT space as follows:

Theorem 2.8. [10] *Any subset of $\mathcal{P}(X)$ is a base for some generalized topology on X.*

Theorem 2.9. [10] *β is a base for some strong generalized topology if and only if $X = \bigcup_{B \in \beta} B$.*

3 Maximal Path Edge Generalized Topological Spaces

First, we introduce the maximal path edge generalized topological space (MPEΓ) as follows:

Definition 3.1. (Maximal Path Edge Generalized Topology) Let $D = (V(D), E(D), \phi_D)$ be a nonempty finite directed graph without loops. Let $\{P_1, P_2, ..., P_n\}$ be the family of all distinct maximal paths in D and let $M_i = E(P_i)$ be referred to as the i^{th} edge set of D for $i \in \{1, 2, ..., n\}$. Define $\beta = \{M_i | i \in \{1, 2, ..., n\}\}$. By Theorem 2.8, β forms a base for a unique generalized topology $\Gamma_{MP}(D)$ on $E(D)$, called the *maximal path edge generalized topology* (or briefly, $MPE\Gamma$) of D. The pair $(E(D), \Gamma_{MP}(D))$ is called the maximal path edge generalized topological space (MPEΓ space).

Remark 3.2. In view of Definition 2.3, the elements of $\Gamma_{MP}(D)$ are called Γ_{MP}-open sets. In the same manner, the complements of these Γ_{MP}-open sets are called Γ_{MP}-closed sets. From this point forward, we denote $\overline{\Gamma}_{MP}(D)$ to be the set of all Γ_{MP}-closed sets. That is, $\overline{\Gamma}_{MP}(D) = \{E(D) \setminus O \mid O \in \Gamma_{MP}(D)\}$.

Definition 3.3. (Deleted i^{th} Edge Set) Let $D = (V(D), E(D), \phi_D)$ be a directed graph. For any $e \in E(D)$, suppose $e \in M_i$. Then, $E^i_M(e) = M_i \setminus \{e\}$ is called the *deleted i^{th} edge set* with respect to e.

Example 3.4. Consider the finite directed graph D without loops in Figure 2 with its corresponding maximal paths in Figure 3. The i^{th} edge sets of D are $M_1 = \{e_1, e_2, e_4\}$, $M_2 = \{e_1, e_5\}$, and $M_3 = \{e_3, e_4\}$. So, we have $\beta = \{\{e_1, e_2, e_4\}, \{e_1, e_5\}, \{e_3, e_4\}\}$. By taking the arbitrary unions of M_i, the maximal path edge generalized topology (MPEΓ) of D is given by

$$\Gamma_{MP}(D) = \{\varnothing, \{e_3, e_4\}, \{e_1, e_5\}, \{e_1, e_2, e_4\}, \{e_1, e_2, e_3, e_4\}, \{e_1, e_2, e_4, e_5\}, \{e_1, e_3, e_4, e_5\}, E(D)\}.$$

Also, $\overline{\Gamma}_{MP}(D) = \{E(D), \{e_1, e_2, e_5\}, \{e_2, e_3, e_4\}, \{e_3, e_5\}, \{e_5\}, \{e_3\}, \{e_2\}, \varnothing\}$. Additionally, the deleted i^{th} edge sets of the edges of D are as follows: $E^1_M(e_1) = M_1 \setminus \{e_1\} = \{e_2, e_4\}$ and $E^2_M(e_1) = M_2 \setminus \{e_1\} = \{e_5\}$, $E^1_M(e_2) = M_1 \setminus \{e_2\} = \{e_1, e_4\}$, $E^1_M(e_3) = M_3 \setminus \{e_3\} = \{e_4\}$, $E^1_M(e_4) = M_1 \setminus \{e_4\} = \{e_1, e_2\}$ and $E^2_M(e_4) = M_3 \setminus \{e_4\} = \{e_3\}$, and $E^1_M(e_5) = M_2 \setminus \{e_5\} = \{e_1\}$.

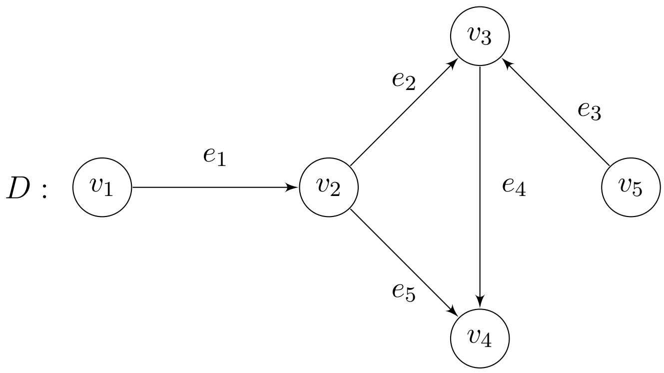

Figure 2: Directed Graph D for Example 3.4.

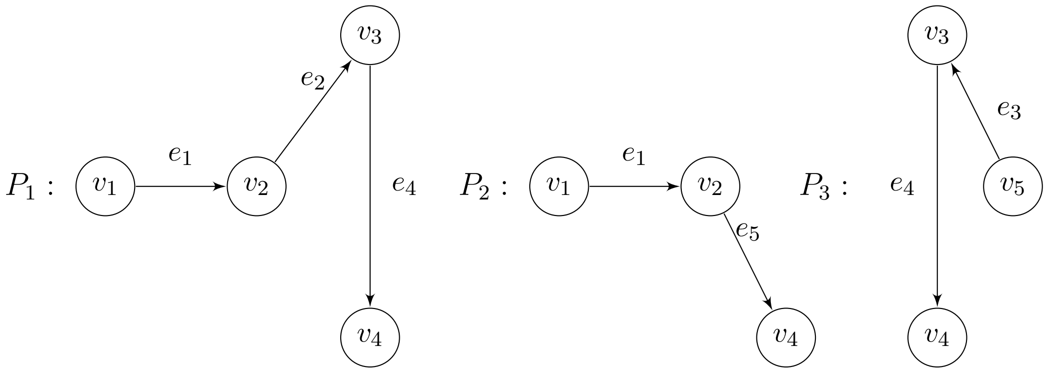

Figure 3: Maximal Paths of Directed Graph D for Example 3.4

Theorem 3.5. *The MPEΓ of a directed graph D is a strong generalized topology on $E(D)$.*

Proof: Let $\Gamma_{MP}(D)$ be the MPEΓ of the directed graph $D = (V(D), E(D), \phi_D)$. Since D does not have a loop, for every $e \in E(D)$, e belongs to some maximal path P_i. Hence, for every $e \in E(D)$, $e \in M_i$ for some $i \in \{1, 2, ..., n\}$. Thus, $E(D) \subseteq \bigcup_{i=1}^{n} M_i$. Since $\bigcup_{i=1}^{n} M_i \subseteq E(D)$, and so, $\bigcup_{i=1}^{n} M_i = E(D)$. Consequently, $E(D) \in \Gamma_{MP}(D)$. Therefore, $\Gamma_{MP}(D)$ is a strong generalized topology. $\square$

Theorem 3.6. *Let $\Gamma_{MP}(D)$ be the MPEΓ of the directed graph $D = (V(D), E(D), \phi_D)$. For any $e \in E(D)$, $E_M^i(e) = \varnothing$ for some $i \in \{1, 2, ..., n\}$ if and only if $\{e\} \in \Gamma_{MP}(D)$.*

Proof: Suppose $e \in E(D)$ such that $E_M^i(e) = \varnothing$ for some $i \in \{1, 2, ..., n\}$. Then, $M_i = \{e\}$ for some $i \in \{1, 2, ..., n\}$. Thus, $\{e\} \in \Gamma_{MP}(D)$. Conversely, let $\{e\} \in \Gamma_{MP}(D)$. Then for some $S \subseteq \{1, 2, ..., n\}$, $\{e\} = \bigcup_{i \in S} M_i$. So, for some $i \in \{1, 2, ..., n\}$, $M_i = \{e\}$; hence, $E_M^i(e) = M_i \setminus \{e\} = \varnothing$. $\square$

Corollary 3.7. *Let $D = (V(D), E(D), \phi_D)$ be a directed graph. If $E_M^i(e) = \varnothing$ for all $e \in E(D)$, then $\Gamma_{MP}(D) = \mathcal{P}(E(D))$.*

Proof: Suppose that $E_M^i(e) = \varnothing$ for all $e \in E(D)$. Then by Theorem 3.6, $\{e\} \in \Gamma_{MP}(D)$ for all $e \in E(D)$. Now, if $A \subseteq E(D)$, then $A = \bigcup_{e \in A} \{e\} \in \Gamma_{MP}(D)$. Hence, $\Gamma_{MP}(D)$ is a discrete generalized topology. □

4 Basic Properties of the Maximal Path Edge Generalized Topology

This section presents the following properties that were generated for $\Gamma_{MP}(D)$ of a finite directed graph D without loops.

Theorem 4.1. *Let $D = (V(D), E(D), \phi_D)$ be a directed graph. A subset O of $E(D)$ is Γ_{MP}-open if and only if for each $e \in O$, there exists $i \in \{1, 2, ..., n\}$ such that at least one of the following is satisfied:*

1. $E^i_M(e) = \varnothing$; *or*
2. $E^i_M(e) \neq \varnothing$ *and* $E^i_M(e) \subseteq O$.

Proof: Let $D = (V(D), E(D), \phi_D)$ be a directed graph. Suppose that O is Γ_{MP}-open. By definition of $\Gamma_{MP}(D)$, $O = \bigcup_{i \in S} M_i$ for some $S \subseteq \{1, 2, ..., n\}$. Let $e \in O$. Then, there exists $i \in \{1, 2, ..., n\}$ such that $e \in M_i$. If $M_i = \{e\}$, then $E^i_M(e) = M_i \setminus \{e\} = \varnothing$. Now, if $M_i \neq \{e\}$, then for each edge $g \in M_i$ such that $g \neq e$, $g \in O$. But, $g \in E^i_M(e)$ for all $g \in M_i$; hence, $E^i_M(e) \neq \varnothing$ and $E^i_M(e) \subseteq O$ for some $i \in \{1, 2, ..., n\}$.
Conversely, let $O \subseteq E(D)$ and suppose that for each $e \in O$, there exists $i \in \{1, 2, ..., n\}$ such that either $E^i_M(e) = \varnothing$ or $E^i_M(e) \neq \varnothing$ and $E^i_M(e) \subseteq O$. Then consider the following; Let $A \subseteq O$ such that for all $e \in A$, $E^i_M(e) = \varnothing$ for some $i \in \{1, 2, ..., n\}$. By Theorem 3.6, A is Γ_{MP}-open. Let $B = \{e \in O | E^i_M(e) \neq \varnothing \text{ and } E^i_M(e) \subseteq O\}$. Then for $e^{'} \in E^i_M(e) \subseteq O$, $E^i_M(e^{'}) = \{e\} \cup E^i_M(e) \setminus \{e^{'}\}$. It follows that $E^i_M(e') \subseteq O$. Hence, $e^{'} \in B$ for all $e^{'} \in E^i_M(e)$, and so, $E^i_M(e) \subseteq B$. Hence, for some $i \in \{1, 2, ..., n\}$, $M_i = \{e\} \cup E^i_M(e) \subseteq B$. It implies that $B = \bigcup_{e \in M_i} M_i$. Thus, B is Γ_{MP}-open. Therefore, $O = A \cup B$ implies that O is a Γ_{MP}-open. □

Corollary 4.2. *A subset F of $E(D)$ is Γ_{MP}-closed if and only if for each $e \in E(D) \setminus F$ there exists $i \in \{1, 2, ..., n\}$ that satisfies the following:*

1. $E^i_M(e) = \varnothing$; *or*
2. $E^i_M(e) \neq \varnothing$ *and* $E^i_M(e) \subseteq O$.

Proof: This is immediate from the fact that F is Γ_{MP}-closed if and only if $E(D) \setminus F$ is Γ_{MP}-open that was shown in Theorem 4.1. □

Theorem 4.3. *Let $D = (V(D), E(D), \phi_D)$ be a directed graph and $A \subseteq E(D)$. Then, $int_{MP}(A) = \{e \in A \mid E^i_M(e) = \varnothing$ or $E^i_M(e) \neq \varnothing$ and $E^i_M(e) \subseteq A$ for some $i \in \{1, 2, ..., n\}\}$.*

Proof: Let $A \subseteq E(D)$ and $A^* = \{e \in A \mid E^i_M(e) = \varnothing$ or $E^i_M(e) \neq \varnothing$ and $E^i_M(e) \subseteq A$ for some $i \in \{1, 2, ..., n\}\}$. Suppose $e \in int_{MP}(A)$. Then, there exists $O \in \Gamma_{MP}(D)$ such that $e \in O \subseteq A$. By Theorem 4.1, there exists $i \in \{1, 2, ..., n\}$ such that $E^i_M(e) = \varnothing$ or $E^i_M(e) \neq \varnothing$ and $E^i_M(e) \subseteq A$. Thus, $e \in A^*$. Therefore, $int_{MP}(A) \subseteq A^*$.
On the other hand, let $e \in A^*$. Then, $e \in A$ and either $E^i_M(e) = \varnothing$ or $E^i_M(e) \neq \varnothing$ and $E^i_M(e) \subseteq A$, for some $i \in \{1, 2, ..., n\}$. Thus, by Theorem 4.1, A^* is Γ_{MP}-open. Since $A^* \subseteq A$, it follows that $A^* \subseteq int_{MP}(A)$. Therefore, $int_{MP}(A) = A^*$. □

Theorem 4.4. *Let $D = (V(D), E(D), \phi_D)$ be a directed graph and $A \subseteq E(D)$. Then, $cl_{MP}(A) = A \cup \{e \in E(D) | E^i_M(e) \cap A \neq \varnothing$ for all $i \in \{1, 2, ..., n\}\}$.*

Proof: Let $A \subseteq E(D)$ and $A^{'} = \{e \in E(D) | E_M^i(e) \cap A \neq \varnothing$ for all $i \in \{1, 2, ..., n\}\}$. Suppose $e \in cl_{MP}(A)$. If $e \in A$, then $e \in A \cup A^{'}$. Now, suppose $e \notin A$. By Proposition 2.6 (6), it follows that $e \in E(D) \setminus int_{MP}(E(D) \setminus A)$. By Theorem 4.3, $int_{MP}(E(D) \setminus A) = \{e \in E(D) \setminus A | E_M^i(e) = \varnothing$ or $E_M^i(e) \neq \varnothing$ and $E_M^i(e) \subseteq E(D) \setminus A$ for some $i \in \{1, 2, ..., n\}\}$. So, for all $i \in \{1, 2, ..., n\}$, $E_M^i(e) \neq \varnothing$ and $E_M^i(e) \nsubseteq E(D) \setminus A$. Hence, there exists $g \neq e$ such that $g \in E_M^i(e)$ but $g \notin E(D) \setminus A$ which implies $g \in A$. So, $E_M^i(e) \cap A \neq \varnothing$ for all $i \in \{1, 2, ..., n\}$. Therefore, $e \in A \cup A^{'}$ and so, $cl_{MP}(A) \subseteq A \cup A^{'}$.
Now, suppose $e \in A \cup A^{'}$. If $e \in A$, then by Proposition 2.6 (1), $e \in cl_{MP}(A)$. On the other hand, if $e \in A^{'}$, then $E_M^i(e) \cap A \neq \varnothing$ for all $i \in \{1, 2, ..., n\}$. This implies that $E_M^i(e) \neq \varnothing$ and $E_M^i(e) \nsubseteq E(D) \setminus A$ for all $i \in \{1, 2, ..., n\}$. So, by Theorem 4.3, $e \notin int_{MP}(E(D) \setminus A)$; hence, $e \in E(D) \setminus int_{MP}(E(D) \setminus A)$. By Proposition 2.6 (6), $e \in cl_{MP}(A)$. Therefore, $A \cup A^{'} \subseteq cl_{MP}(A)$. Consequently, $cl_{MP}(A) = A \cup A^{'}$. □

Theorem 4.5. *Let $D = (V(D), E(D), \phi_D)$ be a directed graph and $A \subseteq E(D)$. Then, $ext_{MP}(A) = \{e \in E(D) \setminus A \mid E_M^i(e) \cap A = \varnothing$ for some $i \in \{1, 2, ..., n\}\}$.*

Proof: Let $A \subseteq E(D)$ and $A^{\diamond} = \{e \in E(D) \setminus A | E_M^i(e) \cap A = \varnothing$ for some $i \in \{1, 2, ..., n\}\}$. Suppose $e \in ext_{MP}(A)$. Then, $e \in int_{MP}(E(D) \setminus A)$, that is, $e \in E(D) \setminus A$ such that $E_M^i(e) = \varnothing$ or $E_M^i(e) \neq \varnothing$ and $E_M^i(e) \subseteq E(D) \setminus A$ for some $i \in \{1, 2, ..., n\}$ by Theorem 4.3. If $E_M^i(e) = \varnothing$, then $E_M^i(e) \cap A = \varnothing$ for some $i \in \{1, 2, ..., n\}$. Now, if $E_M^i(e) \neq \varnothing$ and $E_M^i(e) \subseteq E(D) \setminus A$, then $E_M^i(e) \cap A = \varnothing$ for some $i \in \{1, 2, ..., n\}$. Hence, $e \in A^{\diamond}$, and so, $ext_{MP}(A) \subseteq A^{\diamond}$.
Now, suppose $e \in A^{\diamond}$. Then, $e \in E(D) \setminus A$ such that $E_M^i(e) \cap A = \varnothing$ for some $i \in \{1, 2, ..., n\}$. If $E_M^i(e) \cap A = \varnothing$, then either $E_M^i(e) = \varnothing$ or $E_M^i(e) \neq \varnothing$ and $E_M^i(e) \subseteq E(D) \setminus A$. Thus, $e \in int_{MP}(E(D) \setminus A) = ext_{MP}(A)$. Therefore, $A^{\diamond} \subseteq ext_{MP}(A)$. Consequently, $ext_{MP}(A) = A^{\diamond}$. □

Theorem 4.6. *Let $D = (V(D), E(D), \phi_D)$ be a directed graph and $A \subseteq E(D)$. Then, $fr_{MP}(A) = (A \setminus int_{MP}(A)) \cup \{e \in E(D) \setminus A | E_M^i(e) \cap A \neq \varnothing$ for all $i \in \{1, 2, ..., n\}\}$.*

Proof: Let $A \subseteq E(D)$ and $A^{''} = \{e \in E(D) \setminus A | E_M^i(e) \cap A \neq \varnothing$ for all $i \in \{1, 2, ..., n\}\}$. Suppose $e \in fr_{MP}(A)$. Then, $e \in E(D) \setminus (int_{MP}(A) \cup ext_{MP}(A))$. If $e \in A$, then $e \in (A \setminus int_{MP}(A)) \cup A^{''}$. Now, if $e \notin A$, then $e \in E(D) \setminus A$. Note that, $e \notin int_{MP}(A)$, so $E_M^i(e) \neq \varnothing$ and $E_M^i(e) \nsubseteq A$ for all $i \in \{1, 2, ..., n\}$. Since $e \notin ext_{MP}(A)$, it follows that $E_M^i(e) \cap A \neq \varnothing$ for all $i \in \{1, 2, ..., n\}$. Hence, $e \in E(D) \setminus A$ such that for all $i \in \{1, 2, ..., n\}$, $E_M^i(e) \cap A \neq \varnothing$. Thus, $e \in A^{''}$ which implies that $e \in (A \setminus int_{MP}(A)) \cup A^{''}$. Therefore, $fr_{MP}(A) \subseteq (A \setminus int_{MP}(A)) \cup A^{''}$.
On the other hand, let $e \in (A \setminus int_{MP}(A)) \cup A^{''}$. By Proposition 2.6 (1), if $e \in A \setminus int_{MP}(A)$, then $e \in cl_{MP}(A)$, and by Proposition 2.7 (2), $e \in fr_{MP}(A)$. Now, if $e \in A^{''}$, then $e \in E(D) \setminus A$ such that $E_M^i(e) \cap A \neq \varnothing$ for all $i \in \{1, 2, ..., n\}$. So, by Theorem 4.4, $e \in cl_{MP}(A)$ which implies that $e \notin int_{MP}(E(D) \setminus A) = ext_{MP}(A)$ by Proposition 2.6 (6). Since $e \in A^{''}$ implies $e \notin A$, which further implies that $e \notin int_{MP}(A)$. Hence, $e \in E(D) \setminus (ext_{MP}(A) \cup int_{MP}(A)) = fr_{MP}(A)$. Therefore, $(A \setminus int_{MP}(A)) \cup A^{''} \subseteq fr_{MP}(A)$. Consequently, $(A \setminus int_{MP}(A)) \cup A^{''} = fr_{MP}(A)$. □

Theorem 4.7. *Let $D = (V(D), E(D), \phi_D)$ be a directed graph and $B \subseteq E(D)$. Any $e \in E(D)$ is a Γ_{MP}-cluster point of B if and only if for all $i \in \{1, 2, ..., n\}$, $E_M^i(e) \cap B \neq \varnothing$.*

Proof: Let $e \in E(D)$ and $B \subseteq E(D)$ such that $E_M^i(e) \cap B \neq \varnothing$ for all $i \in \{1, 2, ..., n\}$. This is true if and only if there exists $d \neq e$ such that $d \in E_M^i(e) \cap B$ for all $i \in \{1, 2, ..., n\}$. Equivalently, $d, e \in M_i$ for some $i \in \{1, 2, ...n\}$. So that for every $U \in \Gamma_{MP}(D)$ where $e \in U$, d is also in U. Hence, $d \in U \cap \{B \setminus \{e\}\} \neq \varnothing$ if and only if $E_M^i(e) \cap B \neq \varnothing$ for all $i \in \{1, 2, ..., n\}$. □

Remark 4.8. In view of Theorem 4.7, it is worthy to note of the following:

1. The set of all Γ_{MP}-cluster points of B is called the Γ_{MP}-derived set of B, denoted by $d_{MP}(B)$. That is, $d_{MP}(B) = \{e \in E(D) |\ E^i_M(e) \cap B \neq \varnothing$ for all $i \in \{1, 2, ..., n\}\}$.
2. Recall that in the proof of Theorem 4.4, $cl_{MP}(A) = A \cup A^{'}$ where $A^{'} = \{e \in E(D) | E^i_M(e) \cap A \neq \varnothing$ for all $i \in \{1, 2, ..., n\}\}$. Then, we see that $cl_{MP}(A) = A \cup d_{MP}(A)$.

Theorem 4.9. *Let $D = (V(D), E(D), \phi_D)$ be a directed graph and $A \subseteq E(D)$. If A consists of at least one edge of every maximal path of D, then A is Γ_{MP}-dense in $E(D)$.*

Proof: Let $D = (V(D), E(D), \phi_D)$ be a directed graph and $A \subseteq E(D)$. Suppose A consists of at least one element of M_i for all $i \in \{1, 2, ..., n\}$. Let $O \subseteq E(D) \setminus A$. Then, either $O = \varnothing$ or $O \neq \bigcup_{i \in S} M_i$ for any $S \subseteq \{1, 2, ..., n\}$. So, $E(D) \setminus A$ has no nonempty open subsets, that is, $int_{MP}(E(D) \setminus A) = \varnothing$. Hence, by Proposition 2.6 (6), $cl_{MP}(A) = X \setminus int_{MP}(E(D) \setminus A) = X$. Thus, by Definition **??**, A is Γ_{MP}-dense in $E(D)$. □

5 Connectedness

In this section, the topological notion of connectedness in the MPEΓ space with respect to a directed graph $D = (V(D), E(D), \phi_D)$ is introduced.

Definition 5.1. [9] **(Connectedness in Digraphs)** Let $D = (V(D), E(D))$ be a digraph. D is said to be *strongly connected, or strong*, if every two points are mutually reachable; it is *unilaterally connected, or unilateral*, if for any two points at least one is reachable from the other; and it is *weakly connected, or weak*, if every two points are joined by a semipath. A digraph is connected if it is either strongly, unilaterally or weakly connected. A digraph is *disconnected* if it is not a weakly connected digraph.

It is worthy to note that, every strongly connected digraph is a unilaterally connected digraph, and every unilaterally connected digraph is a weakly connected digraph. Hence, a digraph is connected if it is at least a weakly connected digraph.

Definition 5.2. [15] **(Connectedness)** A generalized topological space (X, μ) is *μ-connected* if it is not the union $X = O_0 \cup O_1$ of two disjoint non-empty μ-open subsets O_0 and O_1 in μ. Otherwise, (X, μ) is *μ-disconnected.*

Remark 5.3. Let $D = (V(D), E(D), \phi_D)$ be a directed graph and $\Gamma_{MP}(D)$ be the MPEΓ of D. By Corollary 3.7, the MPEΓ space $(E(D), \Gamma_{MP}(D))$ where $E^i_M(e) = \varnothing$ for all $e \in E(D)$ is Γ_{MP}-disconnected since it is the discrete generalized topological space on $E(D)$.

In the next result, consider a disconnected digraph $D = (V(D), E(D), \phi_D)$, that is, D has more than one connected components.

Theorem 5.4. *The MPEΓ space $(E(D), \Gamma_{MP}(D))$ of every disconnected digraph is Γ_{MP}-disconnected.*

Proof: Let $D = (V(D), E(D), \phi_D)$ be a disconnected digraph. Suppose $\{D_j | j \in \{1, 2, ..., k\}$ is the set of all connected components of D and $\{M_1, ..., M_n\}$ is the basis of $\Gamma_{MP}(D)$. Then for every component D_j, we have $E(D_j) = \bigcup_{i \in S} M_i$, where $S \subset \{1, 2, ..., n\}$. This implies that $E(D_j) \in \Gamma_{MP}(D)$ for each $j = 1, 2, ..., k$. Since $E(D) \setminus E(D_j)$ is the union of the edge sets of the other components of D, we see that $E(D) \setminus E(D_j) \in \Gamma_{MP}(D)$. So, we have $E(D) = E(D_j) \cup E(D) \setminus E(D_j)$. Therefore, $(E(D), \Gamma_{MP}(D))$ is Γ_{MP}-disconnected. □

For the next theorem, a characterization for connectedness of the MPEΓ space $(E(D), \Gamma_{MP}(D))$ of any connected digraphs is introduced.

Theorem 5.5. *Let $D = (V(D), E(D), \phi_D)$ be any connected digraph and $\Gamma_{MP}(D)$ be the MPEΓ of D with basis $\{M_1, ..., M_n\}$. Then, the MPEΓ space $(E(D), \Gamma_{MP}(D))$ is Γ_{MP}-connected if and only if $\{1, 2, ..., n\}$ has no partition $\{S_1, S_2\}$ such that $M_i \cap M_j = \varnothing$ for all $i \in S_1$ and $j \in S_2$.*

Proof: Suppose there exists a partition $\{S_1, S_2\}$ of $\{1, 2, ..., n\}$ such that $M_i \cap M_j = \varnothing$ for all $i \in S_1$ and $j \in S_2$. Then, $\left(\bigcup_{i\in S_1} M_i\right) \cap \left(\bigcup_{j\in S_2} M_j\right) = \varnothing$; but, $\left(\bigcup_{i\in S_1} M_i\right) \cup \left(\bigcup_{j\in S_2} M_j\right) = E(D)$ and $\bigcup_{i\in S_1} M_i, \bigcup_{j\in S_2} M_j \in \Gamma_{MP}(D)$. Hence, $(E(D), \Gamma_{MP}(D))$ is Γ_{MP}-disconnected.
Conversely, suppose $(E(D), \Gamma_{MP}(D))$ is Γ_{MP}-disconnected. Then, there exist O_1 and O_2 in $\Gamma_{MP}(D)$ such that $O_1 \cap O_2 = \varnothing$ and $O_1 \cup O_2 = E(D)$. This implies that, there exists a partition $\{S_1, S_2\}$ of $\{1, 2, ..., n\}$ such that $O_1 = \bigcup_{i\in S_1} M_i$ and $O_2 = \bigcup_{i\in S_2} M_j$. It follows that $\left(\bigcup_{i\in S_1} M_i\right) \cap \left(\bigcup_{j\in S_2} M_j\right) = \varnothing$. Thus, $M_i \cap M_j = \varnothing$ for all $i \in S_1$ and $j \in S_2$. □

Remark 5.6. In view of Theorem 5.5, the following are observed:

1. If for some $i \in \{1, 2, ..., n\}$, $M_i \cap M_j = \varnothing$ for all $j \neq i$, then the MPEΓ space $(E(D), \Gamma_{MP}(D))$ is Γ_{MP}-disconnected.
2. By (1), if $\bigcap_{i=1}^{n} M_i = \varnothing$, then the MPE$\Gamma$ space $(E(D), \Gamma_{MP}(D))$ is Γ_{MP}-disconnected.

Corollary 5.7. *Let $D = (V(D), E(D), \phi_D)$ be any connected digraph. If there exists an edge $e \in E(D)$ such that $E^i_M(e) = \varnothing$, then the MPEΓ space $(E(D), \Gamma_{MP}(D))$ is Γ_{MP}-disconnected.*

Proof: Let $D = (V(D), E(D), \phi_D)$ be any connected digraph. Suppose there exists $e \in E(D)$ such that $E^i_M(e) = \varnothing$. It follows that, there exists $i \in \{1, 2, ..., n\}$ such that $M_i = \{e\}$. Then, $e \notin M_j$ for all $j \neq i$. So, $M_i \cap M_j = \varnothing$. Hence, by Remark 5.6 (1), the MPEΓ space $(E(D), \Gamma_{MP}(D))$ is Γ_{MP}-disconnected. □

6 Concluding Remarks

In this paper, we have presented a synthesis between graph theory and topology. We introduced a new approach in the construction of a generalized topology on the edge set of directed graphs called the maximal path edge generalized topology (MPEΓ). The basic properties of this new space were established. The construction of the MPEΓ on directed graphs gives many possible research directions, one of which is to explore topological concepts as continuity, homeomorphism, compactness and separation axioms.

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