Original Research Article

Calculus of orthogonal projectors

Abstract

It is possible to express all geometric notions connected with closed linear subspaces in terms of algebraic properties of the orthoprojectors onto these linear spaces. In this paper we give the conditions for ; The sum of a family of Orthoprojectors, Product of orthoprojectors and difference of orthoprojectors to be a projector.

Keywords: Sum of Orthoprojector, Difference of Orthoprojector, Product of Orthoprojector

2010 Mathematics Subject Classification:

1 Introduction

We observe that there is a natural one-to-one correspondence between the set of all closed linear subspaces of a Hilbert space H and the set of all Orthoprojectors on H.

Proposition 1. *Let H be a Hilbert space and P, Q be orthogonal projectors on H onto the closed linear subspace M, N respectively. The following statements are equivalent.*
(i) $M \perp N$
(ii) $PQ = 0$
(iii) $QP = 0$
(iv) $Q(M) = \{\overline{0}\}$
(v) $P(N) = \{\overline{0}\}$ *(note* $Q \longleftrightarrow P$*)*

Proof. (i) $\Rightarrow (ii)$. Let $x \in H$. Then $Qx \in N$. Now $M \perp N \Rightarrow N \subseteq M^{\perp}\,(for, y \in N, \langle y, z\rangle = 0 \quad \forall z \in M$ $\Rightarrow y \in M^{\perp}$ i.e $N \subseteq M^{\perp}$. Thus $Qx \in M^{\perp} = \eta_P$. So $P(Qx) = \overline{0}$ and this holds for all $x \in y$ therefore $PQ = 0$
$(ii) \Rightarrow (iii)$ Take adjoints of both sides of $PQ = 0$, $(PQ)^* = 0^* = 0$ i.e $Q^*P^* = 0$ But $Q^* = 0, P^* = 0$ hence $QP = 0$
(iii) $\Rightarrow$ (ii) Obviously since $PQ = QP = 0$, $P \longleftrightarrow Q$
(ii) $\Rightarrow$ (i) Let $x \in N$. Then $Qx = x$ (for $\mathfrak{R}_P = N$). Now $PQ = 0 \Rightarrow PQx{=}0$ i.e $P(Qx) = 0$, $P(x) = 0 \Rightarrow x \in \eta_p = M^{\perp}$ therefore $N \subseteq M^{\perp}$ in other words $M \perp N$ which is (i). So (i) $\longleftrightarrow$ (ii) $\longleftrightarrow$ (iii)
(i) $\Rightarrow$ (iv) $M \perp N \Rightarrow M \subset N^{\perp} = \eta_Q$ therefore $Q(M) = \{\overline{0}\}$.
Conversely $(iv) \Rightarrow (i)$

For $Q(M)=\{\overline{0}\} \Rightarrow M \subseteq \eta_Q = N^{\perp} \Rightarrow M \perp N$. Similarly $(i) \Rightarrow (v) \Rightarrow (i)$

□

The definitions in this paper are standard and can be found in [3],[4],[5],[6],[7],[8],[9].

Definition 1. Let H be a Hilbert space and P, Q be orthogonal projectors on H. We say that P is orthogonal to Q. In symbols $P \perp Q$ if $M \perp N$, where M, N and ranges of P, Q respectively This is equivalent to saying that $PQ = 0$.

2 Product of Orthoprojectors

Proposition 2. *Let H be a Hilbert space and P, Q be orthoprojectors on H. Then PQ is an orthoprojector if and only if $P \longleftrightarrow Q$. In this case $\mathfrak{R}_{PQ} = M \cap N$, where M, N are the closed linear subspaces of H onto which P, Q project.*

Proof. Suppose $P \longleftrightarrow Q$ i.e $PQ = QP$. Now $(PQ)^* = Q^*P^* = PQ$ since P, Q are self-adjoint. $= PQ$ since $P \longleftrightarrow Q \Rightarrow$ P Q is self-adjoint. Note that $PQ \in B(H)$

$$(PQ)^2 = (PQ)(PQ) = P(QP)Q = P(PQ)Q = (PP)(QQ) = P^2Q^2 = PQ.$$

Since P, Q being orthogonal projectors are idempotent. Thus PQ is idempotent. PQ is self adjoint and idempotent implies PQ is an orthogonal projector. Conversely, let PQ be an orthoprojector. We must show that $P \longleftrightarrow Q$ therefore PQ must be self-adjoint $(PQ)^* = PQ$. But $(PQ)^* = Q^*P^* = QP$ therefore, $PQ = QP$ i.e $P \longleftrightarrow Q$. To show that $\mathfrak{R}_{PQ} = M \cap N$. Let $x \in M \cap N$. Then $x \in M$ and $x \in N$, $x \in M \Rightarrow Px = x$ (for $\mathfrak{R}_P = M$) $QPx = Qx = x$ for $x \in N$ i.e $PQx = Qx = x$ i.e $PQx = x$ i.e $x \in \mathfrak{R}_{PQ}$. Thus $M \cap N \subseteq \mathfrak{R}_{PQ}$. Conversely let $x \in \mathfrak{R}_{PQ}$. So $PQ(x) = x, P(Qx) = x$ implies $x \in \mathfrak{R}_P = M$. Similarly, $PQ = QP \Rightarrow QPx = x$ i.e $Q(Px) = x$, $QPx = x$ i.e $x \in \mathfrak{R}_P = N$, $x \in M$ and $x \in N \Rightarrow x \in M \cap N$.Therefore,

$$\mathfrak{R}_{PQ} \subseteq M \cap N$$

Thus,

$$\mathfrak{R}_{PQ} = M \cap N$$

□

Proposition 3. *Let H be a Hilbert space and P, Q be orthoprojectors and H onto the closed linear subspace M, N respectively.The following statements are equivalent.*
(i) $P \leqslant Q$
(ii) $\|Px\| \leqslant \|Qx\|$ *for all* $x \in H$
(iii) $PQ = P$
(iv) $QP = P$
(v) $M \subseteq N$

Proof. (i) $\Rightarrow$ (ii)

$$P \leqslant Q \Rightarrow \langle Px, x\rangle \leq \langle Qx, x\rangle \forall x \in H$$

But $\langle Px, x\rangle = \|Px\|^2$ for $\langle Px, x\rangle = \langle Px^2, x\rangle$ (P is idempotent)

$$= \langle PPx, x\rangle = \langle Px, Px^*\rangle = \langle Px, Px\rangle \left(\text{ P is self-adjoint } \right) \\ = \|Px\|^2$$

Similarly,

$$\langle Qx, x\rangle = \|Qx\|^2$$

Hence $(i) \Rightarrow \|Px\|^2 \leqslant \|Qx\|^2$ *i.e* $\|Px\| \leqslant \| Qx\|$ for all $x \in H$ which gives (ii)

(ii) $\Rightarrow$ (v) Let $x \in \eta_Q = N^{\perp}$. So $Qx = \overline{0}$ i.e $\|0x\| = 0$ But (ii) $\|Pz\| \leqslant \|Qz\|$ for all $z \in H$ therefore $\|Px\| = 0$ i.e $Px = \overline{0}$ i.e $x \in \eta_P = M^{\perp}, N^{\perp} \subseteq M^{\perp}$.
Taking orthogonal complements of both sides $(N^{\perp})^{\perp} \supseteq \left(M^{\perp}\right)^{\perp}$ i.e $N \supseteq M$ i.e $M \subseteq N$
Next we show that $(v) \Rightarrow$ (iv)
Let $x \in H$. Then $Px \in M \subseteq N$ (by (v))

$$QPx = Q\,(Px) = Px \quad (Px \in N = \mathfrak{R}_{\mathfrak{Q}})$$

$$QP = P$$

(iv)$\Rightarrow$(iii) Take adjoints of $QP = P$ we get

$$(QP)^* = P^*$$
$$P^*Q^* = P^*$$

i.e $PQ = P$ which is (iii) . Also (iii) $\Rightarrow$ (iv) obviously.

We finally show that (iii) $\Rightarrow$ (i) . For any $x \in H$

$$\langle Px, x\rangle = \|Px\,\|^2 = \|\, PQx\|^2 (PQ = P)$$

Now,

$$\|PQx\| = \|P(Qx)\| \leqslant \|P\|\|Qx\| \leqslant \|Qx\| \;(\|P\| \leqslant 1)$$

Therefore,

$$\|PQx\|^2 \leqslant \|Qx\|^2$$

Thus $\langle Px, x\rangle \leqslant \|\, Qx\|^2 = \langle\, Qx, x\rangle$ for all $x \in H$. Which shows that $P \leqslant Q$ and completes the proof.

□

Remark 1. $P \leq Q \Rightarrow P \longleftrightarrow Q$ *for* $QP = P$ *and* $PQ = P$

3 Differences of Orthoprojectors

Proposition 4. *Let H be a Hilbert space and P, Q be orthoprojectors on H. Then $P - Q$ is an orthoprojector if $P \geq Q$ i.e $Q \leq P$. In this case the range of $P - Q$ is $\mathrm{M} \cap \mathrm{N}^{\perp}$, where $M = \mathfrak{R}_P$ and $N = \mathfrak{R}_Q$.*

Proof. Suppose $Q \leqslant P$, we already know that $P \longleftrightarrow Q$ (for $QP = Q, PQ = Q$). To show that, $P - Q$ is an orthoprojector $(P - Q)^* = P^* - Q^* = P - Q$
Since P, Q are self-adjoint. $P - Q$ is self adjoint element of $B(H)$

$$(P - Q)^2 = (P - Q)(P - Q) = P^2 - PQ - QP + P^2$$
$$= P - Q - Q + Q \;(P, Q \text{ are idempotent } QP = Q \text{ and } PQ = Q)$$
$$= P - Q$$

Thus $P - Q$ is self -adjoint and idempotent. Hence $P - Q$ is an orthoprojector. To find the range of $P - Q$,

$$P - Q = P - PQ = P(I - Q)$$

where I is the identity operator. Since Q is an orthoprojector, so is $I-Q$

$$(I-Q)^{*}{=}I^{*}-Q^{*}=I-Q \text{ (self-adjoint)}$$
$$(I-Q)^{2}=I-Q-Q+Q^{2}=I-Q-Q+Q=I-P \text{ (idempotent).}$$

$\mathfrak{R}_{I-Q}=N^{\perp}$ where $N=\mathfrak{R}_{P}$ and $\mathfrak{R}_{I-Q}=N^{\perp}$ Since $P\longleftrightarrow Q, P\longleftrightarrow I-Q$. So $P, I-Q$ are orthoprojectors with ranges $M, N^{\perp}$ and $P\longleftrightarrow I-Q$. So $P(I-Q)$ is an orthoprojector and its range is $M\cap N^{\perp}$. □

Definition 2. Let X be a normed linear space and $\{T_{\alpha}:\alpha\in\wedge.\}$ be a family of bounded linear transformation on X into X. We say that $\{T_{\alpha}:\alpha\in\wedge\}$ is summable to $T\in B(x)$, if for each $x\in X$ the family $\{T_{\alpha}X:\alpha\in\wedge]$ is summable to Tx, In this case we write $\sum_{\alpha\in\wedge}T_{\alpha}=T$.

Proposition 5. *Let $T,S\in B(X)$ and $\{T_{\alpha}:\alpha\in\wedge\}$ be a summable family of elements of $B(X)$ such that $\sum\limits_{\alpha\in\wedge}T_{\alpha}=T$. Then $ST_{\alpha}:\alpha\in\wedge, T_{\alpha}S:\alpha\in\wedge\}$ are summable to ST and TS respectively.*

Proof. Since $\{T_{\alpha}:\alpha\in\wedge\}$ is sumable to T. So for each $x\in X$,$\{T_{\alpha}X:\alpha\in\wedge]$ is summable to Tx. Hence for each real $\varepsilon>0$, there exists a finite subset π_{ε} of $\wedge$ such that for each finite subset π of $\wedge$ satisfies $\pi\supseteq\pi_{\epsilon}$, we have $\left\|\sum\limits_{\alpha\in\pi}T_{\alpha}x-Tx\right\|<\frac{\varepsilon}{\|s\|}$. Where $S\neq0$ (If $S=0$, then the results are obvious). Now,

$$\|S(\sum_{\alpha\in\pi}T_{\alpha}x)-S(Tx)\|=\|\sum_{\alpha\in\pi}ST_{\alpha}x-STx\|$$
$$\text{therefore } \|\sum_{\alpha\in\pi}ST_{\alpha}x-STx\|=\|S(\sum_{\alpha\in\pi}T_{\alpha}x-Tx)\|\leq\|S\|\|\sum_{\alpha\in\pi}T_{\alpha}x-Tx\|\ <\varepsilon$$

Which shows that $\sum\limits_{\alpha\in\wedge}ST_{\alpha}=ST$ i.e $(ST_{\alpha})_{\alpha\in\wedge}$ is summable to ST. Likewise $\{T_{\alpha}S:\alpha\in\wedge\}$. □

4 Sum of Orthoprojectors

For a meaningful consideration of the sum of a family of orthoprojectors, we need first to introduce the notion of a sum of not necessarily finite family of operators in $B(H)$.

Proposition 6. *Let $P\in B(H)$ and $\{P_{\alpha}:\alpha\in\wedge\}$ be a family of orthogonal projectors on H which is summable to P, i.e $P=\sum\limits_{\alpha\in\wedge}P_{\alpha}$. Then P is an orthogonal projector if and only if $\{P_{\alpha}:\alpha\in\wedge\}$ is an orthogonal family i.e $P_{\alpha}\perp P_{\beta}$ whenever $\alpha\neq\beta(\alpha,\beta\in\wedge)$ in this case the range of P (i.e of $\sum_{\alpha\in\wedge}P_{\alpha}$) is $\vee_{\alpha\in\wedge}M_{\alpha}$ where $M_{\alpha}=$ range of P_{α} for each $\alpha\in\wedge$.*

Proof. Let $P_{\alpha}\perp P_{\beta}$ whenever $\alpha,\beta\in\wedge$ and $\alpha\neq\beta$ and $P=\sum\limits_{\alpha\in\wedge}P_{\alpha}$. We shall show that P is an orthogonal projector on H. We know that

$$P_{\alpha}\perp P_{\beta}\implies M_{\alpha}\perp M_{\beta}\implies P_{\alpha}P_{\beta}=P_{\beta}P_{\alpha}=0 \text{ (Zero operator).}$$

Now $\quad P^{2}=PP=(\sum\limits_{\alpha\in\wedge}P_{\alpha})(\sum\limits_{\beta\in\Lambda}P_{\beta})=\sum\limits_{\alpha\in\wedge}\sum\limits_{\beta\in\wedge}(P_{\alpha}P_{\beta})$

(Why?) For If $S\in B(H)$ and $\{T_{\alpha}:\alpha\in\wedge\}$ is a summable family of elements of $B(H)$ with sum T. Then

$$ST=S\left(\sum_{\alpha\in\wedge}T_{\alpha}\right)=\sum_{\alpha\in\wedge}ST_{\alpha}$$

Since $\sum\limits_{\beta\in\wedge} P_\beta = P$ therefore $P_\alpha \left(\sum\limits_{\beta\in\wedge} P_\beta\right) = \sum\limits_{\beta\in\wedge} P_\alpha P_\beta$ for any $\alpha \in \wedge$. Each $P_\alpha P_\beta \in B(H)$ and the family $\left\{\sum\limits_{\beta} P_\alpha P_\beta : \alpha \in \wedge\right\}$ is a family of bounded linear operators.

$$\sum_\alpha(\sum_\beta P_\alpha P_\beta) = \sum_\alpha\sum_\beta P_\alpha P_\beta$$
$$(\sum_\alpha P_\alpha)(\Sigma P_\beta) = \sum_\alpha\sum_\beta P_\alpha P_\beta$$

But $P_\alpha P_\beta = 0$ if $\alpha \neq \beta$ therefore $P^2 = \sum\limits_\alpha P_\alpha^2 = \sum_\alpha P_\alpha = P$ since each P_α is idempotent. Thus P is idempotent.We show that P is self-adjoint. Let $x, y \in H$. Then,

$$\langle Px, y\rangle = \left\langle \left(\sum_\alpha P_\alpha\right) x, y\right\rangle = \sum_\alpha \langle P_\alpha x, y\rangle = \sum_\alpha \langle x, P_\alpha y\rangle \text{ (since } P_\alpha \text{ is self-adjoint)}$$

$$= \left\langle x, \left(\sum_\alpha P_\alpha\right) y\right\rangle = \langle x, Py\rangle\, \forall x, y \in H \text{ therefore } P \text{ is self-adjoint.}$$

Thus $P \in B(H)$ is self-adjoint and idempotent. Hence P is an orthogonal projector.
Conversely, let P be an orthogonal projector , we must show that the family $\{P_\alpha : \alpha \in \wedge\}$ is orthogonal.
Take any $x \in M_\alpha$. Then $x \in H$. Since P is an orthogonal projector $\|P\| \leqslant 1$ and hence $\|x\| \geq \|Px\|$. Since $P = \sum\limits_{\alpha\in\wedge} P_\alpha$, we have $\|x\|^2 \geq \|Px\|^2 = \langle Px, Px\rangle = \langle P^2x, x\rangle = \langle Px, x\rangle = \left\langle \sum\limits_\beta P_\beta x, x\right\rangle = \sum_\beta \langle P_\beta x, x\rangle$. We know that an orthogonal projector is a positive operator $\langle Px, x\rangle = \|Px\|^2 \geq 0$ for all $x \in H$ each $\langle Px, x\rangle$ is real and non-negative $\geq \langle P_\alpha x, x\rangle = \|P_\alpha x\|^2 = \|x\|^2$ (since $x \in M_\alpha = \mathfrak{R}_{P_\alpha}$). So $P_\alpha = x$

$$\|P_\alpha x\| = \|x\|$$

Since we have $\|x\|^2$ at both ends of the above chain of inequalities it shows that equality must hold throughout. So if $x \in M_\alpha$ (α fixed arbitrary) then $\|Px\| = \|x\|$ and $\langle P_\beta x, x\rangle = 0 \quad \forall\beta \neq \alpha\langle P_\beta x, x\rangle = 0$ for all $x \in M_\alpha$ and $\beta \neq \alpha$ implies

$$\|P_\alpha x\|^2 = 0 \quad \forall x \in M_\alpha$$

So $P_\beta (M_\alpha) = \{\overline{0}\} \quad \forall\beta \neq \alpha$. This implies $P_\beta \perp P_\alpha$ for all $\beta \neq \alpha$. Since this is true for any $\alpha \in \wedge$, we get $P_\alpha \perp P_\beta\ \forall\beta \neq \alpha$ Since $\|Px\| = \|x\| \implies x \in \mathfrak{R}_P = M$

□

Lemma 1. *If H is a Hilbert space and P is an orthogonal projector then $\|Px\| = \|x\|$ if and only if $x \in M = \mathfrak{R}_P$.*

Proof. For if $x \in M = R_P$, then $Px = x$ and hence $\|Px\| = \|x\|$. Conversely let $\|Px\| = \|x\|$ for an $x \in H$. Then $\|Px - x\|^2 = \langle Px - x, Px - x\rangle = \langle Px, x\rangle - \langle Px, Px\rangle - \langle x, Px\rangle + \langle x, x\rangle = \|Px\|^2 - \|Px\|^2 - \|Px\|^2 + \|x\|^2 = \|x\|^2 - \|Px\|^2 = \|x\|^2 = \|x\|^2 \quad (\|Px\| = \|x\|) = 0$. Thus $Px - x = \overline{0}$ therefore $Px = x$ i.e $x \in \mathfrak{R}_P = M$. Thus we shown that $x \in M_\alpha$ then $x \in \mathfrak{R}_P = M \quad \forall\alpha \in \wedge$ therefore $M \supseteq M_\alpha$ for all $\alpha \in \wedge$ therefore $M \supseteq \left[\bigcup\limits_{\alpha\in\wedge} M_\alpha\right]$ therefore $M \supseteq [\bigcup\limits_{\alpha\in\wedge} M_\alpha] = \bigvee\limits_{\alpha\in\wedge} M_\alpha$. It remains to show $M \subseteq \bigvee\limits_{\alpha\in\wedge} M_\alpha$. Since $P = \sum\limits_{\alpha\in\wedge} P_\alpha$ for any $x \in H$

$$Px = (\sum_\alpha P_\alpha)x = \sum_\alpha P_\alpha x \quad \text{(But } P_\alpha x \in M_\alpha)$$

$$Px \in \sum_{\alpha\in\wedge} M_\alpha = \bigvee_{\alpha\in\Lambda} M_\alpha \text{ therefore } \quad \mathfrak{R}_P = \bigvee_{\alpha\in\wedge} M_\alpha$$

□

Proposition 7. *If P, Q are orthogonal projectors on M, N respectively and $P \longleftrightarrow Q$ then PQ is an orthogonal projector with range $M \cap N$ and $P + Q - PQ$ is an orthogonal projection with range $M \vee N$. Thus*

$$\left.\begin{array}{l} P \wedge Q = PQ \\ P \vee Q = P + Q - PQ \end{array}\right\} \text{ where } P \longleftrightarrow Q$$

Proof. We have already seen that PQ is an orthogonal projector if and only if $P \longleftrightarrow Q$ and then $\mathfrak{R}_{PQ} = M \cap N$. Let $\{M_\alpha : \alpha \in \wedge\}$ be a family of closed linear subspace of H $\bigvee_{\alpha\in\wedge} M_\alpha, \bigwedge_{\alpha\in\wedge} M_\alpha \left(\bigwedge_{\alpha\in\wedge} M_\alpha\right)$ are both closed linear subspace of H. If P_α represents the orthogonal projector on H onto M_α (for each $\alpha \in \wedge$) then we represent the orthogonal projectors onto $\bigvee_{\alpha\in\wedge} M_\alpha$ and $\bigwedge_{\alpha\in\wedge} M_\alpha$ by the symbol $\bigvee_{\alpha\in\wedge} P_\alpha$ and $\bigwedge_{\alpha\in\wedge} P_\alpha$. By definition $P \wedge Q$ is the projector on H onto $M \cap N (= M \wedge N)$ therefore $P \wedge Q = PQ$ when $P \longleftrightarrow Q$. Since $M = \mathfrak{R}_P, N = \mathfrak{R}_Q$ so PVQ is the orthogonal projector corresponding to $M \vee N$. Specifically when $P \longleftrightarrow Q$,

$$P \vee Q = P + Q - PQ.$$

We show this,

$$P + Q - PQ = P + (Q - PQ) = P + (I - P)Q$$

Since $P \longleftrightarrow Q$, so $I - P \longleftrightarrow Q, P$ is an orthogonal projector $\longleftrightarrow I - P$ is an orthogonal projector. Thus $(I - P), Q$ are orthogonal projectors and $(I - P) \longleftrightarrow Q$. Hence $(I - P)\ Q$ is an orthogonal projector with range $= \mathfrak{R}_{\text{I-P}}\ \cap R_Q = M^\perp \cap N$. For any $x \in H$, $Px \perp (I - P)Qx$. Indeed

$$\langle Px, (I-P)Qx\rangle = \langle (I-P)^* Px, Qx\rangle = \langle (I-P)Px, Qx\rangle \text{ for } (I-P)^* = I - P = \left\langle Px - Px^2, Qx\right\rangle$$
$$\text{but } P^2 = P = \langle Px - Px, Qx\rangle = \langle \overline{0}, Qx\rangle = 0$$

Thus,

$$P \perp (I - P)Q$$

Using the result $M \perp N \Rightarrow P_M + P_N$. is a projection with range $M \vee N$. Finite version of the theorem proved. We observe that $P + (I - P)Q$ is an orthogonal projector with range $MV(M^\perp \cap N)$. Now writing $P + Q - PQ$ as $Q + P(I - Q)$ (Note $Q \longleftrightarrow P$) and observing that $I - Q \longleftrightarrow P$, we note that the range of the projection $(I - Q)P$ is $N^\perp \cap M$. Since $Q \perp (I - Q)P$: we see that the range of the projection $P + Q - PQ$ is also $N \vee \left(N^\perp \cap M\right)$
Thus,

$$\left.\begin{array}{r} \mathfrak{R}_{P+Q-PQ} = M \vee \left(M^\perp \cap N\right) \\ = N \vee \left(N^\perp \cap M\right) \end{array}\right\} \tag{1}$$

Certainly $\mathfrak{R}_{P+Q-PQ} \supseteq M, N$ and therefore $\supseteq \overline{[M \cup N]} = M \vee N$. Since $\mathfrak{R}_{P+Q-PQ}$ is closed therefore $\mathfrak{R}_{P+Q-PQ} \supseteq M \vee N$. On the other hand (1) also reveals $\mathfrak{R}_{P+P-PQ} \subseteq M \vee N$ for $M \vee \left(N \cap M^\perp\right) \subseteq M \vee N, N \vee \left(M \cap N^\perp\right) \subseteq M \vee N$. Thus $\mathfrak{R}_{P+Q-PQ} = M \vee N$. Let $M \perp N$ and P, Q be orthogonal projectors onto M, N respectively

$$(P + Q)^* = P + Q$$
$$(P + Q)^2 = P^2 + PQ + QP + Q^2 \quad \text{But} \quad PQ = QP = 0$$

$$P^2 + Q^2 = P + Q$$

So $P + Q$ is an orthogonal projector for any $x \in H$

$$(P + Q)x = Px + Qx \in M + N$$
$$\text{therefore } \Re_{P+Q} \subseteq M + N = [\overline{M \cup N}] = M \vee N$$

On the other hand, if $x \in M$, then $(P + Q)x = Px + Qx = x + \bar{0} = x$ i.e $x \in \Re_{Q+P}$ therefore $M \subseteq \Re_{P+Q}$
Similarly,

$$N \subseteq \Re_{P+Q}$$
$$M \cup N \subseteq \Re_{P+Q}$$

therefore $[\overline{M \cup N}] \subseteq \Re_{P+Q}$ since R_{P+Q} is closed $M \vee N \subseteq R_{P+Q}$ therefore

$$\Re_{P+Q} = M \vee N$$

□

Remark 2. *In the infinite version as given ;* $x \in M_\alpha \Rightarrow x \in M = \Re_P \ \forall \alpha \in \wedge$ *therefore* $M_\alpha \subseteq \Re_P$

$$[\overline{U_\alpha M_\alpha}] \subseteq \Re_P \ (\Re_P \textit{ is closed })$$

$$V_{\alpha \in \wedge} M_\alpha \subseteq \Re_P \tag{2}$$

On the other hand for each $x \in H$

$$Px = \sum_\alpha P_\alpha x \in \sum_\alpha M_\alpha = M = \overline{M}$$

(Since $M_\alpha : \alpha \in \wedge$ *is an orthogonal family of subspace.)*

$$\Re_P \subseteq M = \bigvee_{\alpha \in \wedge} M_\alpha \tag{3}$$

(2) and (3) imply $\Re_P = \bigvee_{\alpha \in \wedge} M_\alpha$

4.1 Conclusion

The results has clearly discussed the sum, difference and product of orthogonal projectors. For the product its clear that if P and Q are projectors then $P \leq Q \Rightarrow P \longleftrightarrow Q$ for $QP = P$ and $PQ = P$.

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