On the Restrained Cost Effective Sets of Some Special Classes of Graphs

Original Research Article

Abstract

Let G be a nontrivial, undirected, simple graph. Let S be a subset of $V(G)$. S is a restrained cost effective set of G if for each vertex v in S, $\deg_S(v) \leq \deg_{V(G)\setminus S}(v)$ and the subgraph induced by the vertex set, $V(G) \setminus S$ has no isolated vertex. The maximum cardinality of a restrained cost effective set is the restrained cost effective number, $CE_r(G)$. In this paper, the restrained cost effective sets of paths, cycles, complete graphs, complete product of graphs and graphs resulting from line graph of graphs with maximum degree of 2 were characterized. As a direct consequence, the bounds or exact values for the restrained cost effective number were determined as well.

Keywords: Restrained cost effective set, Restrained cost effective number, Line graph

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1 Introduction

One of the fascinating discoveries on Graph Theory is the notion of cost effective set on graphs. Each element of this set compares the degree with respect to a vertex subset and its complement such that the degree of a vertex in a vertex subset is as least as much in the complement. It was first introduced by Hedetniemi and first called it as unfriendly partition of a graph. It was utilized to generate a self-stabilizing algorithm for two disjoint dominating sets in a graph [9]. The following study of Haynes and Hedetniemi formally coined the term, cost effective sets, as a basis for cost effectivity of servers to clients in a computer network. Thereafter, various variations of this set have been made in the recent years. Among of them are the very cost effective sets that follows a strict ineuality, k-cost effective sets where the inequality depends on the value k, and the cost effective dominating set that adds the domination attribute in graphs.

newpage On the other hand, the concept of restrained sets is introduced in 1999 by Hedetniemi et. al [6]. Restrained sets refers to a set where the subgraph induced by the complement of this set does not contain any isolated vertices. This sets are both defined for both connected and disconnected graphs. Furthermore, restrained set is renowned from the domination of graphs as one of its parameter variations.

In this paper, we combine these two ideas to have a restrained cost effective set of graphs. Features for a restrained cost effective set to be defined on some special classes of graphs are formulated. Graphs considered in this study are the path P_n, cycle C_n, complete graph K_n, graphs resulting from the complete product between a trivial graph and connected graphs with maximum degree of 2, graphs resulting from the complete product between two empty graphs, and the line graph of connected graphs with maximum degree of 2. Moreover, bounds or exact values for the restrained cost effective number $CE_r(G)$ were derived. The restrained cost effective number refers to the maximum cardinality of a restrained cost effective set for a given graph.

All graphs stated here are simple, connected, and undirected. The graph operations involved in this paper are limited to line graph, $L^k(G)$, and complete product of two graphs.

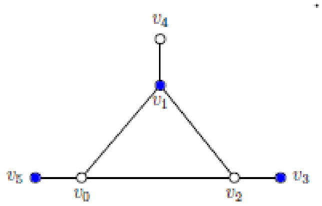
2 Preliminary Notes

For the sense of formality, we present below the definition of the concepts to be discussed on this paper.

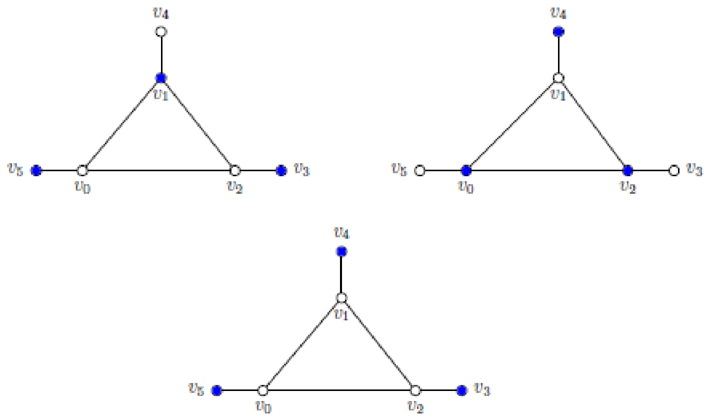
Definition 2.1. [9] Let G be a nontrivial, undirected, simple graph. A nonempty subset S of $V(G)$ is a ***cost effective set*** of G, if for every $v \in S$, $|N(v) \cap S| \leq |N(v) \cap V(G) \smallsetminus S|$. The ***cost effective number*** of G, denoted by $CE(G)$, is the maximum cardinality of a cost effective set of G.

For brevity, we denote $|N(v) \cap S|$ as $\mathbf{deg_S(v)}$ which is the degree of a vertex v in G with respect to a vertex subset S and $|N(v) \cap V(G) \smallsetminus S|$ as $\mathbf{deg_{V(G) \smallsetminus S}(v)}$ which is the degree of a vertex v in G with respect to a vertex subset $V(G) \smallsetminus S$ on the rest of the discussions.

Example of this set is given below:

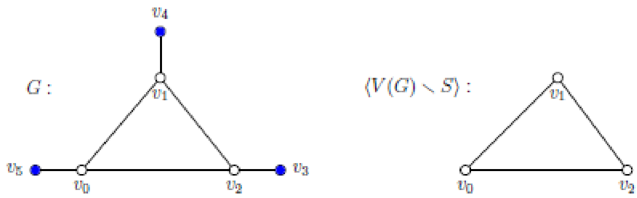


Furthermore, $CE(G) = 3$ from finding all maximum cost effective sets shown by the shaded vertices of the graph,



Definition 2.2. Let G be a nontrivial, undirected, simple graph. A nonempty subset S of $V(G)$ is a ***restrained cost effective set*** of G if it is a cost effective set and the subgraph induced by $V(G) \setminus S$ has no isolated vertices. The ***restrained cost effective number*** of G, denoted by $CE_r(G)$, is the maximum cardinality of a restrained cost effective set of G.

Example of this set is given below.

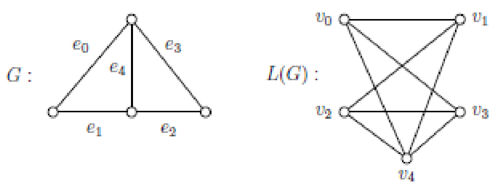


The shaded vertices above shows a restrained cost effective set of a graph. The subgraph induced by the set, $V(G) \setminus S$ is a cycle of order 3 having no isolated vertices. Moreover, $CE_r(G) = 3$ since the shaded vertices are the only maximum restrained cost effective set for the graph.

The definitions for the graph operations of graphs utilized in this study are given below.

Definition 2.3. [5] Let G be a connected graph. The ***line graph*** of G, denoted by $L(G)$ is a unary graph operation where $V(G)$ can be put in a one-to-one coressppondence to $E(G)$ in such a way that two vertices in $L(G)$ are adjacent if its corresponding edges of G are adjacent. Also, $V(L(G)) = E(G)$.

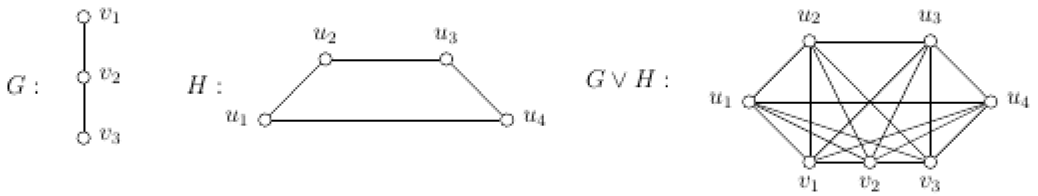
Example of this graph operation is given below.



Definition 2.4. [5] Let G and H be graphs. The ***complete product*** or the ***join*** of G and H, denoted by $G \vee H$, is a graph having a vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set,

$$E(G \vee H) = E(G) \cup E(H) \cup \{v_0 v_1 : v_0 \in V(G), v_1 \in V(H)\}$$

Example of this graph operation is given below.



3 Main Results

The main results of this study is divided into three parts. The first one provides the characterization of a restrained cost effective set for the considered graphs.

Theorem 3.1. *Let P_n be a path of order $n \geq 3$ with $V(P_n) = \{v_0, v_1, ..., v_{n-1}\}$. Let S be a nonempty subset of $V(P_n)$. Then S is a restrained cost effective set of P_n if and only if the following hold:*

(i) For each $v_i \in S,\ i \neq 1$ and $i \neq n-2$.

(ii) Every component of $\langle S \rangle_{P_n}$ is an isolated vertex or P_2.

(iii) For any two distinct vertices, u, v in S, $(N(u) \setminus S) \cap (N(v) \setminus S) = \varnothing$.

Proof: Let a nonempty subset S of $V(P_n)$ be a restrained cost effective set of P_n. Suppose $v_i \in S$ such that $i = 1$ or $i = n-2$. Then v_i is adjacent to v_0 or v_{n-1}. This implies that $\langle V(P_n) \setminus S \rangle_{P_n}$ has an isolated vertex v_0 or v_{n-1}. This is a contradiction to the assumption on S. This implies that $v_i \notin S$ for $i = 1, n-2$. Hence, $v \in S$ when $i \neq 1$ and $i \neq n-2$.

Now, suppose $\langle S \rangle_{P_n}$ has a component that is not an isolated vertex and not P_2. This indicates that $\langle S \rangle_{P_n}$ has a component of path P_n of order $n \geq 3$. This implies that at least one vertex in S, say $u \in S$, is not adjacent to any vertices of $V(P_n) \setminus S$. So, $\deg_S(u) = 2 > 0 = \deg_{V(P_n) \setminus S}(u)$ which is a contradiction. Hence, the components of $\langle S \rangle_{P_n}$ are either an isolated vertex or P_2.

Lastly, suppose there exist distinct vertices u and v such that $(N(u) \setminus S) \cap (N(v) \setminus S) \neq \varnothing$. Then these vertices are incident to two edges having a common vertex in $V(P_n) \setminus S$. Let this vertex be x. Note that $\langle V(P_n) \setminus S \rangle_{P_n}$ has an isolated vertex, exactly x. This is a contradiction on the assumption for S. Hence, $u, v \in S$ when $(N(u) \setminus S) \cap (N(v) \setminus S) = \varnothing$.

For the converse, it immediately follows. □

Theorem 3.2. *Let C_n be a cycle with $V(C_n) = \{v_0, v_1, ..., v_{n-1}\}$. Let S be a nonempty subset of $V(C_n)$. Then, S is a restrained cost effective set of C_n if and only if the following hold:*

(i) Every component of $\langle S \rangle_{C_n}$ is an isolated vertex or P_2.

(ii) For any two distinct vertices, u, v in S, $(N(u) \setminus S) \cap (N(v) \setminus S) = \varnothing$.

Proof: Let a nonempty subset S of $V(C_n)$ be a restrained cost effective set of C_n. Suppose $\langle S \rangle_{C_n}$ has a component that is not an isolated vertex and not P_2. This indicates that $\langle S \rangle_{C_n}$ has a component of path P_n of order $n \geq 3$. This implies that at least one vertex in S, say $v_k \in S$, is not adjacent to any vertices of $V(C_n) \setminus S$. So, $\deg_S(v_k) = 2 > 0 = \deg_{V(C_n) \setminus S}(v_k)$ which is a contradiction. Hence, the components of $\langle S \rangle_{C_n}$ are either an isolated vertex or P_2.

Now, suppose there exist distinct vertices u and v such that $(N(u) \setminus S) \cap (N(v) \setminus S) \neq \varnothing$. Then these vertices are incident to two edges having a common vertex in $V(C_n) \setminus S$. Let this vertex be v_k. Note that $\langle V(C_n) \setminus S \rangle_{C_n}$ has an isolated vertex, exactly v_k. This is a contradiction on the assumption for S. Hence, $v, u \in S$ when $(N(v) \setminus S) \cap (N(u) \setminus S) = \varnothing$.

For the converse, it follows immediately. □

Theorem 3.3. *Let K_n be a complete graph with $n \geq 4$ and S be a nonempty subset of $V(K_n)$. S is a restrained cost effective set of K_n if and only if $1 \leq |S| \leq \lceil \frac{n}{2} \rceil$.*

Proof: Let S be a restrained cost effective set of K_n. Then, for each $v \in S$, $\deg_S(v) \leq \deg_{V(K_n) \setminus S}(v)$. If $|S| = 1$, then $\deg_S(v) = 0 \leq n-1 = \deg_{V(K_n) \setminus S}(v)$. Clearly, $|S| \geq 1$. Next, we show that $|S| \leq \lceil \frac{n}{2} \rceil$. In contrary, suppose $|S| > \lceil \frac{n}{2} \rceil$. Then, for each $v \in S$, $\deg_S(v) \geq \lceil \frac{n}{2} \rceil$ and $\deg_{V(K_n) \setminus S}(v) \leq \lceil \frac{n}{2} \rceil - 2$. This implies that $\deg_S(v) > \deg_{V(K_n) \setminus S}(v)$. A contradiction to the assumption on S. Thus, $|S| \leq \lceil \frac{n}{2} \rceil$. Therefore, $1 \leq |S| \leq \lceil \frac{n}{2} \rceil$.

Conversely, suppose $1 \leq |S| \leq \lceil \frac{n}{2} \rceil$. It suffices to show that if $|S| = 1$ or $|S| = \lceil \frac{n}{2} \rceil$, then S is a restrained cost effective set. Now, if $|S| = 1$, then $\deg_S(v) = 0 \leq n - 1 = \deg_{V(K_n) \setminus S}(v)$. Clearly, S is a restrained cost effective set. On the other hand, if $|S| = \lceil \frac{n}{2} \rceil$, then for each $v \in S$, $\deg_S(v) \leq \lfloor \frac{n}{2} \rfloor$ and $\deg_{V(K_n) \setminus S}(v) \geq \lfloor \frac{n}{2} \rfloor$. Thus, $\deg_S(v) \leq \deg_{V(K_n) \setminus S}(v)$. So, S is a cost effective set. At this point, observe that $\langle V(K_n) \setminus S \rangle_{K_n}$ is a complete graph K_i of order $i < n$. Hence, S is a restrained cost effective set of K_n. □

Theorem 3.4. *Let G be a connected graph with maximum degree of 2 and H be a trivial graph. A nonempty subset S of $V(G \vee H)$ is a restrained cost effective set of $G \vee H$ if and only if one of the following holds:*

(i) $S \subseteq V(H)$, $S = V(H)$.

(ii) $S \subseteq V(G)$, *every component of* $\langle S \rangle_{G \vee H}$ *is* P_2 *or an isolated vertex in* G.

(iii) $S = S_1 \cup S_2$ *such that* $S_1 \subseteq V(G)$ *and* $S_2 \subseteq V(H)$. S_1 *is an independent set of* G *and* $\langle V(G) \setminus S_1 \rangle_{G \vee H}$ *has no isolated vertex.*

Proof: Let a nonempty subset S of $V(G \vee H)$ be a restrained cost effective set of $G \vee H$. Consider the following cases:
Case 1: $S \subseteq V(H)$. Clearly, $S = V(H)$ because H is a trivial graph.
Case 2: $S \subseteq V(G)$.

Suppose there exists a component in $\langle S \rangle_{G \vee H}$ that is a path P_n of order $n \geq 3$. Then, there exists at least one vertex $v \in S$ in this component such that $\deg_S(v) = 2 > 1 = \deg_{V(G \vee H) \setminus S}(v)$ making S not a cost effective set in $G \vee H$. A contradiction to the assumption for S. Hence, $\langle S \rangle_{G \vee H}$ has a component of P_2 or an isolated vertex in G.
Case 3: $S = S_1 \cup S_2$ such that $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$.

Suppose S_1 is not an independent set in G or $\langle V(G) \setminus S_1 \rangle_{G \vee H}$ has an isolated vertex. If S_1 is not an independent set in G, then there exist vertices, $v, u \in S$ such that v and u are adjacent. But, $\deg_S(v) = 2 > 1 = \deg_{V(G \vee H) \setminus S}(v)$. A contradiction. On the other hand, if $\langle V(G) \setminus S_1 \rangle_{G \vee H}$ has an isolated vertex then $\langle V(G \vee H) \setminus S \rangle_{G \vee H}$ has an isolated vertex since $V(G) \setminus S_1 \subseteq V(G \vee H) \setminus S$. Clearly, another contradiction on the assumption of S in $G \vee H$. Therefore, S_1 is an independent set of G and $\langle V(G) \setminus S_1 \rangle_{G \vee H}$ has no isolated vertex.

Conversely, if S satisfies (i), then S is obviously a restrained cost effective set in $G \vee H$. Now, if S satisfies (ii), let $v \in S$ where v is one of the isolated vertices in $\langle S \rangle_{G \vee H}$, then
$\deg_S(v) = 0 \leq 3 = \deg_{V(G \vee H) \setminus S}(v)$ when $G \vee H$ is a wheel and fan with $v \in V(F_n)$, $v \neq v_1, v_n$ and $\deg_S(v) = 0 \leq 2 = \deg_{V(G \vee H) \setminus S}(v)$ when $G \vee H$ is a fan with $v \in V(F_n)$, $v = v_1, v_n$. Otherwise, if $v \in V(P_2)$, then $\deg_S(v) = 1 \leq 1 = \deg_{V(G \vee H) \setminus S}(v)$ or $\deg_S(v) = 1 \leq 2 = \deg_{V(G \vee H) \setminus S}(v)$ when $G \vee H$ is fan or wheel. For all cases, S is a cost effective set. Moreover, $\langle V(G \vee H) \setminus S \rangle_{G \vee H}$ is a join between H and $\langle V(G) \setminus S \rangle_{G \vee H}$ that has no isolated vertices for all cases. Thus, S is a restrained cost effective set. If S satisfies (iii), let $v \in S$, then $v \in S_1$ or $v \in S_2$. If $v \in S_2$, then $\deg_S(v) \leq \deg_{V(G \vee H) \setminus S}(v)$ for all n in G. If $v \in S_1$, then $\deg_S(v) = 1$ since S_1 is an independent set in G. Thus,

$$\deg_S(v) = 1 \leq \deg_{V(G \vee H) \setminus S}(v) = \begin{cases} 1 & \text{if } \deg_{G \vee H}(v) = 2 \\ 2 & \text{if } \deg_{G \vee H}(v) = 3. \end{cases}$$

Therefore, S is a cost effective set. Now, let $\langle V(G)\setminus S_1\rangle_{G\vee H}$ has no isolated vertex. To prove that $\langle V(G\vee H)\setminus S\rangle_{G\vee H}$ has also no isolated vertex, note that the respective vertex sets, $V(G\vee H)\setminus S$ and $V(G)\setminus S_1$, are the same because,

$$\begin{aligned} V(G\vee H)\setminus S &= V(G\vee H)\setminus (S_1\cup S_2)\\ &= (V(G\vee H)\setminus S_1)\ \cap\ (V(G\vee H)\setminus S_2)\\ &= ((V(G)\cup V(H))\setminus S_1)\ \cap\ ((V(G)\cup V(H))\setminus V(H))\\ &= (V(G)\setminus S_1\cup V(H))\ \cap\ V(G)\\ &= V(G)\setminus S_1. \end{aligned}$$

This means that $\langle V(G)\setminus S_1\rangle_{G\vee H} = \langle V(G\vee H)\setminus S\rangle_{G\vee H}$ and $\langle V(G\vee H)\setminus S\rangle_{G\vee H}$ has no isolated vertex. Thus, S is a restrained cost effective set in $G\vee H$. □

Theorem 3.5. *Let G and H be empty graphs such that $|V(G)| = m$ and $|V(H)| = n$ where $n, m \in \mathbb{Z}^+$ and $n, m > 1$. A nonempty subset S of $V(G\vee H)$ is a restrained cost effective set of a complete bipartite graph, $K_{m,n}$ if and only if one of the following holds:*

(i) $S\subseteq V(G)$, $1\le |S|\le m-1$.

(ii) $S\subseteq V(H)$, $1\le |S|\le n-1$.

(iii) $S = S_1\cup S_2$ *such that* $S_1\subseteq V(G)$ *and* $S_2\subseteq V(H)$, $1\le |S_1|\le \lfloor \frac{m}{2}\rfloor$ *and* $1\le |S_2|\le \lfloor \frac{n}{2}\rfloor$.

Proof: Let a nonempty subset S of $V(G\vee H)$ be a restrained cost effective set of $K_{m,n}$.
Case 1: $S\subseteq V(G)$.

Suppose $|S| < 1$ or $|S| > m-1$. If $|S| < 1$, then clearly, S is not a restrained cost effective set. If $|S| > m-1$, then $|S| = m$ since $|S| \le |V(G)| = m$. Observe that $\langle V(K_{m,n})\setminus S\rangle_{K_{m,n}}$ is an empty graph of order n, which is exactly graph H. This means that there are n isolated vertices on $\langle V(K_{m,n})\setminus S\rangle_{K_{m,n}}$. A contradiction to the assumption for S. Similarly, this fact also holds when $S\subseteq V(H)$.
Case 2: $S = S_1\cup S_2$ such that $S_1\subseteq V(G)$ and $S_2\subseteq V(H)$.

Suppose $S = S_1\cup S_2$ such that $|S_1| > \lfloor \frac{m}{2}\rfloor$ or $|S_2| > \lfloor \frac{n}{2}\rfloor$. If $|S_1| > \lfloor \frac{m}{2}\rfloor$, then $|S_1| > |V(G)\setminus S_1|$ for all $m\in\mathbb{Z}^+$. With $\emptyset\ne S_2\subseteq V(H)$, there exist a vertex $v\in S_2$ such that $\deg_{S_1}(v) > \deg_{V(G)\setminus S_1}(v)$. Note that $S_1\subset S$ and $(V(G)\setminus S_1)\subset (V(K_{m,n})\setminus S)$. Hence, $\deg_S(v) > \deg_{V(K_{m,n})\setminus S}(v)$. This is a contradiction. Similarly, this fact also holds when $|S_2| > \lfloor \frac{n}{2}\rfloor$ for all $n\in\mathbb{Z}^+$.

Conversely, suppose $S\subseteq V(G)$ such that $1\le |S|\le m-1$. Let $v\in S$. Then, $\deg_S(v) = 0$ since G is an empty graph. By definition of $K_{m,n}$, v is adjacent to at least one of the vertices in H. This means that $\deg_S(v) = 0 < n = \deg_{V(K_{m,n})\setminus S}(v)$. Hence, S is a cost effective set. Similarly, this argument also holds when $S\subseteq V(H)$ such that $1\le |S|\le n-1$. Now, suppose $S = S_1\cup S_2$ such that $S_1\subseteq V(G)$ and $S_2\subseteq V(H)$ with $1\le |S_1|\le \lfloor \frac{m}{2}\rfloor$ and $1\le |S_2|\le \lfloor \frac{n}{2}\rfloor$. Let $v\in S$. Then $v\in S_1$ or $v\in S_2$. If $v\in S_1$, then $1\le \deg_S(v)\le \frac{n}{2}$ and $\frac{n}{2}\le \deg_{V(K_{m,n})\setminus S}(v)\le n-1$ when n is even. When n is odd, $1\le \deg_S(v)\le \frac{n-1}{2}$ and $\frac{n+1}{2}\le \deg_{V(K_{m,n})\setminus S}(v)\le n-1$. This implies $\deg_S(v)\le \deg_{V(K_{m,n})\setminus S}(v)$ for all $n\in\mathbb{Z}^+$. Similarly, this argument also holds if $v\in S_2$. Hence, S is a cost effective set. At this point, observe that $|V(G)\setminus S_1|, |V(H)\setminus S_2| > 0$. So, there is at least one vertex in G that is adjacent to every vertex of H and at least one vertex in H that is adjacent to every vertex of G. This implies that $\langle V(K_{m,n})\setminus S\rangle_{K_{m,n}}$ is also a complete bipartite graph, $K_{m-|S_1|,n-|S_2|}$. Hence, S is a restrained cost effective set in $K_{m,n}$. □

The second part of the result provides the bounds for the restrained cost effective number $CE_r(G)$.

Theorem 3.6. *For any path P_n of order $n \geq 3$, the restrained cost effective number is given by,*

$$CE_r(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n-1}{2} & \text{if } n \equiv 1 \pmod 4 \text{ or } n \equiv 3 \pmod 4 \\ \frac{n-2}{2} & \text{if } n \equiv 2 \pmod 4 \end{cases}$$

.

Proof: Let $S \subseteq V(P_n)$ such that $S = S_0 \cup S_1 \cup S_2$ where,

$$S_0 = \{v_0, v_{n-1}\}, \quad S_1 = \{v_3, v_7, \ldots, v_{4k-1}\}, \quad S_2 = \{v_4, v_8, \ldots, v_{4k}\}$$

for which $4k-1$, $4k < n$ for all $k \in \mathbb{Z}^+$. S is a restrained cost effective set of P_n because for $v \in S$ such that $v_i = v_0$ or $v_i = v_{n-1}$, $\deg_S(v) = 0 \leq 1 = \deg_{V(P_n) \setminus S}(v)$. Otherwise, $\deg_S(v) = 1 \leq 1 = \deg_{V(P_n) \setminus S}(v)$. Additionally, $\langle V(P_n) \setminus S \rangle$ consists family of P_2. Now,
Case 1: $n \equiv 0 \pmod 4$.

Let $v_{4k-1} \in S_1$ and $v_{4k} \in S_2$. Observe that v_{4k-1} and v_{4k} are adjacent in S for every $k \in \mathbb{Z}^+$. Thus, $|S_1| = |S_2|$. Let $A \subseteq \mathbb{Z}^+$. For every $n \equiv 0 \pmod 4$ such that $n > 4$, there exist an element k in the set: $A = \left\{k \in \mathbb{Z}^+ \mid 1 \leq k \leq \frac{n-4}{4}\right\}$. So, $|S_1|$ = $|S_2| = \max(A) = \frac{n-4}{4}$. Counting the elements of S, we have,

$$|S| = |S_0| + |S_1| + |S_2| = 2 + \frac{n-4}{4} + \frac{n-4}{4} = \frac{8+2n-8}{4} = \frac{2n}{4} = \frac{n}{2}.$$

Case 2: $n \equiv 1 \pmod 4$ or $n \equiv 3 \pmod 4$:

Let $n \equiv 1 \pmod 4$. Notice that v_{4k} and v_{4k-1} are adjacent in S for each $k \in \mathbb{Z}^+$. So, $|S_1| = |S_2|$. For every $n \equiv 1 \pmod 4$ such that $n > 5$, there exist an element k in the set: $A = \left\{k \in \mathbb{Z}^+ \mid 1 \leq k \leq \frac{n-5}{4}\right\}$. So, $|S_1| = |S_2| = \max(A) = \frac{n-5}{4}$. Counting the elements of S,

$$|S| = |S_0| + |S_1| + |S_2| = 2 + \frac{n-5}{4} + \frac{n-5}{4} = \frac{8+2n-10}{4} = \frac{2n-2}{4} = \frac{n-1}{2}.$$

Let $n \equiv 3 \pmod 4$. Observe that for each $n \equiv 3 \pmod 4$, $|S_1| > |S_2|$ for which $|S_1| = |S_2| + 1$. For $n \equiv 3 \pmod 4$ such that $n > 7$, there exist an element k in the set: $A = \left\{k \in \mathbb{Z}^+ \mid 1 \leq k \leq \frac{n-7}{4}\right\}$. So, $|S_2| = \max(A) = \frac{n-7}{4}$ and $|S_1| = \frac{n-7}{4} + 1 = \frac{n-3}{4}$. Counting the elements of S, we have,

$$|S| = |S_0| + |S_1| + |S_2| = 2 + \frac{n-3}{4} + \frac{n-7}{4} = \frac{8+2n-10}{4} = \frac{2n-2}{4} = \frac{n-1}{2}.$$

Case 3: $n \equiv 2 \pmod 4$:

In this case, v_{4k} and v_{4k-1} are still adjacent in S for each $k \in \mathbb{Z}^+$ so $|S_1| = |S_2|$. For every $n \equiv 2 \pmod 4$ such that $n > 6$, there exist an element k in the set: $A = \left\{k \in \mathbb{Z}^+ \mid 1 \leq k \leq \frac{n-6}{4}\right\}$. So, $|S_1|$ = $|S_2| = \max(A) = \frac{n-6}{4}$. Counting the elements of S, we have,

$$|S| = |S_0| + |S_1| + |S_2| = 2 + \frac{n-6}{4} + \frac{n-6}{4} = \frac{8+2n-12}{4} = \frac{2n-4}{4} = \frac{n-2}{2}.$$

For every case, suppose S is not a maximum restrained cost effective set such that $CE_r(P_n) = |S| + 1$. Then, there exist a restrained cost effective set, $T \subseteq V(P_n)$ such that $S \subset T$ with

$|T| = |S| + 1$. This implies a vertex $v_t \in T$ where $v_t \notin S$. Then, $v_t \in N(S)$ and $v_t \in N(V(P_n) \setminus S)$. From the leaf vertices of P_n, we have two cases for the placement of v_t: either $v_t = v_1$ or $v_t = v_{n-2}$ or $v_t \neq v_1$ and $v_t \neq v_{n-2}$. If $v_t = v_1$ or $v_t = v_{n-2}$, then by Theorem 3.1.1 (i), T is not a restrained cost effective set. A contradiction. On one hand, if $v_t \neq v_1$ and $v_t \neq v_{n-2}$, then $\langle V(P_n) \setminus T\rangle_{P_n}$ has a isolated vertex or $\langle T\rangle_{P_n}$ has a component of P_3. In all cases, this is a contradiction. So, S is a maximum restrained cost effective set. Hence, $CE_r(P_n) \neq |S| + 1$ implying $CE_r(P_n) \leq |S|$. Since a restrained cost effective set S with $|S|$ exists. Then, $CE_r(P_n) \geq |S|$. Therefore, $CE_r(P_n) = |S|$. □

The conditions for a restrained cost effective set to exist in P_n are also found in C_n by Theorem 3.2. This means that the structure of a restrained cost effective set for C_n is roughly the same as P_n. This indicates that the maximum restrained cost effective set S established by Theorem 3.6 is also the maximum restrained cost effective set for C_n. This leads to the following corollary,

Corollary 3.7. *Let* $n \geq 3$, $CE_r(C_n) = CE_r(P_n)$ *for all* $n \in \mathbb{Z}^+$.

For complete graphs, the bounds established for S by Theorem 3.3, we can infer the restrained cost effective number of K_n.

Corollary 3.8. *For any complete graph* K_n *of order* $n \geq 4$*. The restrained cost effective number is given by*

$$CE_r(K_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 2 \\ \\ \frac{n+1}{2} & \text{if } n \equiv 1 \pmod 2 \end{cases}$$

.

The next theorems describes the restrained cost effective number for fan and wheel graphs.

Theorem 3.9. *Let* F_n *be a fan graph with* $V(F_n) = V(G) \cup V(H) = \{v_0, v_1, v_2, ..., v_n\}$ *where* $V(H) = \{v_0\}$*. For any fan* F_n *of order* $n + 1 \geq 3$,

$$CE_r(F_n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod 3 \\ \\ \frac{2n+1}{3} & \text{if } n \equiv 1 \pmod 3 \\ \\ \frac{2n+2}{3} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

Proof: Let $S \subseteq V(F_n)$. Consider the set partition, $S = S_0 \cup S_1$ such that

$$S_0 = \{v_1, v_4, \ldots, v_{3k-2}\} \qquad S_1 = \{v_2, v_5, ..., v_{3k-1}\}$$

where $3k - 2, 3k - 1 \leq n$ for all $k \in \mathbb{Z}^+$. S is a restrained cost effective set of F_n because for all n, when $v \in S_0$ or $v \in S_1$, $\deg_S(v) = 0 \leq 2 = \deg_{V(F_n)\setminus S}(v)$ or $\deg_S(v) = 1 \leq 1 = \deg_{V(F_n)\setminus S}(v)$ or $\deg_S(v) = 1 \leq 2 = \deg_{V(F_n)\setminus S}(v)$. Moreover, the subgraph induced by $V(F_n) \setminus S$ is a star graph $K_{1,\frac{n}{3}}$ when $n \equiv 0 \pmod 3$, $K_{1,\frac{n-1}{3}}$ when $n \equiv 1 \pmod 3$, and $K_{1,\frac{n-2}{3}}$ when $n \equiv 2 \pmod 3$. Now,

Case 1: if $n \equiv 0 \pmod 3$

Let $v \in S_0$ and $u \in S_1$. Observe that v and u is adjacent in G for every $k \in \mathbb{Z}^+$. So, $|S_0| = |S_1|$. Let $A \subseteq Z^+$. For every $n \equiv 0 \pmod 3$, there exist an element k in the set: $A = \left\{k \in Z^+ \mid 1 \leq k \leq \frac{n}{3}\right\}$. This means that, $|S_0| = |S_1| = \max(A) = \frac{n}{3}$. Counting the elements of S,

$$|S| = |S_0| + |S_1| = \frac{n}{3} + \frac{n}{3} = \frac{2n}{3}.$$

Case 2: if $n \equiv 1 \pmod 3$

Let $v \in S_0$ and $u \in S_1$. Observe that for every $k \in \mathbb{Z}^+$, $|S_0| = |S_1| + 1$ implying $|S_0| > |S_1|$. Let $A \subseteq Z^+$. For every $n \equiv 1 \pmod 3$, there exist an element k in the set: $A = \left\{ k \in Z^+ \mid 1 \leq k \leq \frac{n-1}{3} \right\}$. So, $|S_1| = \max(A) = \frac{n-1}{3}$ and $|S_0| = \frac{n-1}{3} + 1 = \frac{n+2}{3}$. Counting the elements of S,

$$|S| = |S_0| + |S_1| = \frac{n+2}{3} + \frac{n-1}{3} = \frac{2n+1}{3}.$$

Case 3: if $n \equiv 2 \pmod 3$

Let $v \in S_0$ and $u \in S_1$. Observe that v and u is adjacent in G for every $k \in \mathbb{Z}^+$. So, $|S_0| = |S_1|$. Let $A \subseteq Z^+$. For every $n \equiv 2(\mod 3)$, there exist an element k in the set: $A = \left\{ k \in Z^+ \mid 1 \leq k \leq \frac{n+1}{3} \right\}$. So, $|S_0| = |S_1| = \max(A) = \frac{n+1}{3}$. Counting the elements of S,

$$|S| = |S_0| + |S_1| = \frac{n+1}{3} + \frac{n+1}{3} = \frac{2n+2}{3}.$$

Suppose S is not a maximum restrained cost effective set such that $CE_r(F_n) = |S| + 1$. Then, there exist a restrained cost effective set, $T \subseteq V(F_n)$ such that $S \subset T$ with $|T| = |S| + 1$. This means that there exist $v \in T$ where $v \notin S$. If $v \in T$, then either $v \in V(H)$ or $v \in V(G) \setminus S$. If $v \in V(H)$ or $v \in V(G) \setminus S$, notice that $\deg_T(v) > \deg_{V(F_n) \setminus T}(v))$ for every case. So, T is not a cost effective set. This means that T is impossible to exist. So, S is a maximum restrained cost effective set. Hence, $CE_r(F_n) \neq |S| + 1$. This implies that $CE_r(F_n) \leq |S|$. Since we found a restrained cost effective set S, with $|S|$, so $CE_r(F_n) \geq |S|$. Therefore, $CE_r(F_n) = |S|$. □

Theorem 3.10. *Let W_n be a wheel graph with $V(W_n) = V(G) \cup V(H) = \{v_0, v_1, v_2, ..., v_n\}$ where $V(H) = \{v_0\}$. For any wheel W_n of order $n + 1 \geq 3$,*

$$CE_r(W_n) = \begin{cases} \frac{2n}{3} & \text{if } n \equiv 0 \pmod 3 \\ \frac{2n-2}{3} & \text{if } n \equiv 1 \pmod 3 \\ \frac{2n-1}{3} & \text{if } n \equiv 2 \pmod 3 \end{cases}$$

.

Proof: Let $S \subseteq V(W_n)$. Consider the set partition, $S = S_0 \cup S_1$ such that

$$S_0 = \{v_1, v_4, \ldots, v_{3k-2}\} \qquad S_1 = \{v_2, v_5, ..., v_{3k-1}\}$$

where $3k-2, 3k-1 \leq n$ for all $k \in \mathbb{Z}^+$. S is a restrained cost effective set because whenever $v \in S_0$ or $v \in S_1$, $\deg_S(v) = 1 \leq 2 = \deg_{V(W_n) \setminus S}(v)$. Furthermore, the subgraph induced by $V(W_n) \setminus S$ is a star graph $K_{1,\frac{n}{3}}$ when $n \equiv 0 \pmod 3$, $K_{1,\frac{n+2}{3}}$ when $n \equiv 1 \pmod 3$, and $K_{1,\frac{n+1}{3}}$ when $n \equiv 2 \pmod 3$. Now,

Case 1: if $n \equiv 0 \pmod 3$

Let $v \in S_0$ and $u \in S_1$. Observe that v and u is adjacent in G for every $k \in \mathbb{Z}^+$. So, $|S_0| = |S_1|$. Let $A \subseteq Z^+$. For every $n \equiv 0 \pmod 3$, there exist an element k in the set: $A = \left\{ k \in Z^+ \mid 1 \leq k \leq \frac{n}{3} \right\}$. This means that, $|S_0| = |S_1| = \max(A) = \frac{n}{3}$. Counting the elements of S,

$$|S| = |S_0| + |S_1| = \frac{n}{3} + \frac{n}{3} = \frac{2n}{3}.$$

Case 2: if $n \equiv 1 \pmod 3$

Let $v \in S_0$ and $u \in S_1$. Observe that v and u is adjacent in G for every $k \in \mathbb{Z}^+$. So, $|S_0| = |S_1|$. Let $A \subseteq Z^+$. For every $n \equiv 2(\mod 3)$, there exist an element k in the set:
$A = \left\{k \in Z^+ \mid 1 \leq k \leq \frac{n-1}{3}\right\}$. So, $|S_0| = |S_1| = \max(A) = \frac{n-1}{3}$. Counting the elements of S,

$$|S| = |S_0| + |S_1| = \frac{n-1}{3} + \frac{n-1}{3} = \frac{2n-2}{3}.$$

Case 3: if $n \equiv 2 \pmod 3$

Let $v \in S_0$ and $u \in S_1$. Observe that for every $k \in \mathbb{Z}^+$, $|S_0| = |S_1| + 1$ implying $|S_0| > |S_1|$. Let $A \subseteq Z^+$. For every $n \equiv 1 \pmod 3$, there exist an element k in the set: $A = \left\{k \in Z^+ \mid 1 \leq k \leq \frac{n-2}{3}\right\}$. So, $|S_1| = \max(A) = \frac{n-2}{3}$ and $|S_0| = \frac{n-2}{3} + 1 = \frac{n+1}{3}$. Counting the elements of S,

$$|S| = |S_0| + |S_1| = \frac{n+1}{3} + \frac{n-2}{3} = \frac{2n-1}{3}.$$

Suppose S is not a maximum restrained cost effective set such that $CE_r(W_n) = |S| + 1$. Then, there exist a restrained cost effective set, $T \subseteq V(W_n)$ such that $S \subset T$ with $|T| = |S| + 1$. This means that there exist $v \in T$ where $v \notin S$. If $v \in T$, then either $v \in V(H)$ or $v \in V(G) \setminus S$. If $v \in V(H)$ or $v \in V(G) \setminus S$, notice that $\deg_T(v) > \deg_{V(W_n) \setminus T}(v))$ for every case. So, T is not a cost effective set. This means that T is impossible to exist. So, S is a maximum restrained cost effective set. Hence, $CE_r(W_n) \neq |S| + 1$. This implies that $CE_r(W_n) \leq |S|$. Since we found a restrained cost effective set S, with $|S|$, so $CE_r(W_n) \geq |S|$. Therefore, $CE_r(W_n) = |S|$. □

The following theorem describes the restrained cost effective number for complete bipartite graphs.

Theorem 3.11. *For any complete bipartite graph, $K_{m,n}$. The restrained cost effective number is given by*

$$CE_r(K_{m,n}) = \max\{m-1, n-1\}.$$

Proof: Let S be a maximum restrained cost effective set of $K_{m,n}$ and let X and Y be the partite sets of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. If $S \subseteq X$, then by Theorem 1.5 (i), $|S| \leq m-1$ and if $S \subseteq Y$, then by Theorem 1.5 (ii), $|S| \leq n-1$. Thus, $CE_r(K_{m,n}) = |S| = \max\{m-1.n-1\}$. □

The third part of the result presents the line graph operation on k^{th} iteration to connected graphs with maximum degree of 2 and its consequences to the restrained cost effective number, $CE_r(G)$.

Theorem 3.12. *Let P_n be a path of order $n \geq 3$. $L^k(P_n)$ has a restrained cost effective set only if $1 \leq k \leq n-3$.*

Proof: Let P_n be a path. Suppose there exist a restrained cost effective set of $L^k(P_n)$ on the k^{th} iterate where $k = (n-3) + 1 = n-2$. Observe that $L(P_n)$ is a path of order $n-1$ and size $(n-1) - 1 = n-2$. So, for all n, $L^{k\,=\,n-2}(P_n)$ is a path of order 2 and size 1, i.e. P_2, for which the restrained cost effective set is undefined. Thus, the restrained cost effective set of $L^k(P_n)$ is only defined until k^{th} iterate where $k = n-3$. □

The restrained cost effective number of $L^k(P_n)$ follows from the observation of the proof from the theorem above.

Corollary 3.13. *Let $1 \leq k \leq n-3$. For any path P_n of order $n \geq 3$,*

$$CE_r(L^k(P_n)) = CE_r(P_{n-k}), \text{ and}$$
$$CE_r(L^k(P_n)) < CE_r(P_n).$$

For cycles of order $n \geq 3$. $L(C_n)$ has the same order and size of C_n. So, $L^k(C_n) = C_n$ for all $k \in \mathbb{Z}^+$. This leads to the following remark,

Remark 3.14. *Let $k \geq 1$. For any cycle C_n of order $n \geq 3$,*

$$CE_r(L^k(C_n)) = CE_r(C_n).$$

4 Conclusions

The idea of restrained cost effective sets for graphs is a new variation from a cost effective set that adds the significance of an isolate-free graph formed by the complement of this set. In this paper, we established some results on the characterization of a restrained cost effective sets with finding the exact values of bounds for the restrained cost effective number of a path P_n, cycle C_n, complete graph K_n and the graphs resulting from graph operations, complete product and line graph.

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