Safe Sets in Some Graph Families

Original Research Article

Abstract

For a connected simple graph G, a non-empty set $S \subseteq V(G)$ of vertices is a safe set if, for every component A of $\langle S \rangle_G$ and every component B of $\langle V(G) - S \rangle_G$ adjacent to A, it holds that $| A | \geq | B |$. The safe number denoted by $s(G)$ of G is the minimum cardinality of a safe set G.
In this paper, it examines the characterization of a safe set in complete bipartite graph. It also discusses the minimum cardinality of a safe sets of path graph and cycle graph via modulus. Moreover, this study generates the possible exact values of the safe number of the complete graph, complete bipartite graph, and star graph.

Keywords: Safe Sets; Safe Number

1 Introduction

The concept of safe set and connected safe set was introduced by Fujita et al.,(6). Their ideas come from the type of facility location problem in which the goal is to determine a ”safe” subset of nodes in a network where facilities can be placed. They established that obtaining a minimal safe set and a minimum connected safe set are both NP-hard tasks in general. They also demonstrated that in linear time, a minimum connected safe set in a tree can be discovered (5). It was shown in (6) that $s(P_n) = cs(P_n) = \lceil \frac{n}{3} \rceil$ and $s(C_n) = cs(C_n) = \lceil \frac{n}{2} \rceil$, where P_n and C_n are the path and cycle graph.

The study of Fujita et al.,(6) has motivated this study. In this paper, we extended the study of safe sets in some graph families. We characterize the safe sets in graphs resulting from complete bipartite graph. We also present a new method in computing the minimum cardinality of path graph and cycle graph, this method is via modulus. Lastly, we generate the exact values of the safe number of complete graph, complete bipartite graph, and star graph.

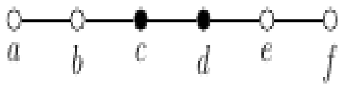
All graphs under considered here are undirected and nontrivial connected simple graph.

2 Preliminary Notes

The definitions of the concepts covered in this study are included below.

Definition 2.1. (5) The subgraph of a graph G induced by $S \subseteq V(G)$ is denoted by $\langle S \rangle_G$. A ***component*** of G is a connected induced subgraph of G with an inclusionwise maximal vertex set. For vertex-disjoint subgraphs A and B of G, if there is an edge between A and B, then A and B are ***adjacent***. A non-empty set $S \subseteq V(G)$ of vertices is a ***safe set*** if, for every component A of $\langle S \rangle_G$ and every component B of $\langle V(G) - S \rangle_G$ adjacent to A, it holds that $\mid A \mid \geq \mid B \mid$. The safe number denoted by $s(G)$ of G is the minimum cardinality of a safe set of G.

Consider the path P_6 in the given figure below. It can be seen that set $S = \{c, d\}$ is a safe set and a minimum safe set.



Observe that $\langle V(P_6) - S \rangle_{P_6} = \{B_1, B_2\}$, where $B_1 = \{a, b\}$ and $B_2 = \{e, f\}$, each of which with $\mid B_1 \mid = 2$ and $\mid B_2 \mid = 2$. It shows that $\mid S \mid \geq \mid B_1 \mid$ and $\mid S \mid \geq \mid B_2 \mid$. Hence, $s(P_6) = 2$.

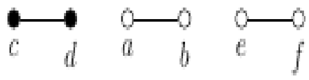


Figure 1: The path P_6 with a safe set $S = \{c, d\}$ and $s(P_6) = 2$.

3 Main Results

In this section, the safe number of path, cycle, complete graph, complete bipartite graph, and star graph are shown. As well as, the characteristics of a safe set of complete bipartite graph.

Theorem 3.1. *Let $G = (V, E)$ be a nontrivial connected graph. If $G = P_n$, $n \geq 2$, then*

$$s(P_n) = \begin{cases} \frac{n}{3} & \text{if } n \equiv 0 (\text{mod } 3) \\ \frac{n+2}{3} & \text{if } n \equiv 1 (\text{mod } 3) \\ \frac{n+1}{3} & \text{if } n \equiv 2 (\text{mod } 3) \end{cases}$$

Proof. Let $P_n = \{v_0, v_1, ..., v_{n-1}\}$, $n \geq 2$, and S be a non-empty subset of $V(P_n)$. Consider the following cases:
Case 1: $n \equiv 0(\text{mod } 3)$

Choose $S = \{v_k, ..., v_{2k-1}\}$ where $k = \frac{n}{3}$. Then the component of $\langle S \rangle_{P_n}$ is S itself with $\mid S \mid= \frac{n}{3}$. Now, $V(P_n) - S = \{v_0, ..., v_{k-1}\} \cup \{v_{2k}, ..., v_{n-1}\}$. Clearly, $\langle V(P_n) - S \rangle_{P_n}$ has two components, $B_1 = \{v_0, ..., v_{k-1}\}$ and $B_2 = \{v_{2k}, ..., v_{n-1}\}$, each of which with $\mid B_1 \mid= \frac{n}{3}$ and $\mid B_2 \mid= \frac{n}{3}$. Hence, $\mid S \mid \geq \mid B_1 \mid$ and $\mid S \mid \geq \mid B_2 \mid$. Thus, S is a safe set. Now, we want to show that S is a minimum safe set of P_n. Suppose S is not a minimum safe set of P_n. Then there exist a safe set $S_0 \subseteq V(P_n)$ such that $\mid S_0 \mid < \mid S \mid$. Thus, $\mid S_0 \mid < \frac{n}{3}$ and $\mid V(P_n) - S_0 \mid > 2\left(\frac{n}{3}\right)$. Then for each component A in $\langle S_0 \rangle_{P_n}$, $\mid A \mid < \mid S_0 \mid < \frac{n}{3}$ and there exist a component B in $\langle V(P_n) - S_0 \rangle_{P_n}$ such that $\mid B \mid > \frac{n}{3}$. Thus, $\mid A \mid < \mid B \mid$. A contradiction. Thus, S is a minimum safe set. Hence, $s(P_n) = \frac{n}{3}$.

Case 2: $n \equiv 1$(mod 3)

Choose $S = \{v_k, ..., v_{2k}\}$, where $k = \frac{n-1}{3}$. Then the component of $\langle S \rangle_{P_n}$ is S itself with $\mid S \mid= \frac{n+2}{3}$. Now, $V(P_n) - S = \{v_0, ..., v_{k-1}\} \cup \{v_{2k+1}, ..., v_{n-1}\}$. Clearly, $\langle V(P_n) - S \rangle_{P_n}$ has two components, $B_1 = \{v_0, ..., v_{k-1}\}$ and $B_2 = \{v_{2k+1}, ..., v_{n-1}\}$, each of which with $\mid B_1 \mid= \frac{n-1}{3}$ and $\mid B_2 \mid= \frac{n-1}{3}$. Hence, $\mid S \mid \geq \mid B_1 \mid$ and $\mid S \mid \geq \mid B_2 \mid$. Thus, S is a safe set. Now, we want to show that S is a minimum safe set of P_n. Suppose S is not a minimum safe set of P_n. Then there exist a safe set $S_0 \subseteq V(P_n)$ such that $\mid S_0 \mid < \mid S \mid$. Thus, $\mid S_0 \mid < \frac{n+2}{3}$ and $\mid V(P_n) - S_0 \mid > 2\left(\frac{n-1}{3}\right)$. Now for each component A in $\langle S_0 \rangle_{P_n}$, $\mid A \mid < \mid S_0 \mid < \frac{n+2}{3}$ and there exist a component B in $\langle V(P_n) - S_0 \rangle_{P_n}$ such that $\mid B \mid > \frac{n+2}{3}$. Thus, $\mid A \mid < \mid B \mid$. A contradiction. Thus, S is a minimum safe set. Hence, $s(P_n) = \frac{n+2}{3}$.

Case 3: $n \equiv 2$(mod 3)

Choose $S = \{v_k, ..., v_{2k}\}$, where $k = \frac{n-2}{3}$. Then the component of $\langle S \rangle_{P_n}$ is S itself with $\mid S \mid= \frac{n+1}{3}$. Now, $V(P_n) - S = \{v_0, ..., v_{k-1}\} \cup \{v_{2k+1}, ..., v_{n-1}\}$. Clearly, $\langle V(P_n) - S_0 \rangle_{P_n}$ has two components, $B_1 = \{v_0, ..., v_{k-1}\}$ and $B_2 = \{v_{2k+1}, ..., v_{n-1}\}$, each of which with $\mid B_1 \mid= \frac{n-2}{3}$ and $\mid B_2 \mid= \frac{n+1}{3}$. Hence, $\mid S \mid \geq \mid B_1 \mid$ and $\mid S \mid \geq \mid B_2 \mid$. Thus, S is a safe set. Now, we want to show that S is a minimum safe set of P_n. Suppose S is not a minimum safe set of P_n. Then there exist a safe set $S_0 \subseteq V(P_n)$ such that $\mid S_0 \mid < \mid S \mid$. Thus, $\mid S_0 \mid < \frac{n+1}{3}$ and $\mid V(P_n) - S_0 \mid > \frac{n-2}{3}$, $\mid V(P_n) - S_0 \mid > \frac{n+1}{3}$. Now for each component A in $\langle S_0 \rangle_{P_n}$, $\mid A \mid < \mid S_0 \mid < \frac{n+1}{3}$ and there exist a component B in $\langle V(P_n) - S_0 \rangle_{P_n}$ such that $\mid B \mid > \frac{n+1}{3}$. Thus, $\mid A \mid < \mid B \mid$. A contradiction. Thus, S is a minimum safe set. Hence, $s(P_n) = \frac{n+1}{3}$. □

Theorem 3.2. *For a cycle graph G of order $n \geq 3$, the following holds:*

$$s(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \text{(mod 2)} \\ \frac{n+1}{2} & \text{if } n \equiv 1 \text{(mod 2)} \end{cases}$$

Proof. Let $C_n = \{v_0, ..., v_{n-1}\}$, $n \geq 3$, S be a non-empty subset of $V(C_n)$. Consider the following cases:

Case 1: $n \equiv 0$(mod 2)

Choose $S = \{v_k, ..., v_{2k}\}$, where $k = \frac{n-2}{2}$. Then the component of $\langle S \rangle_{C_n}$ is S itself with $\mid S \mid= \frac{n}{2}$. Now, $V(C_n) - S = \{v_0, ..., v_{k-1}, v_{2k+1}\}$, the component of $\langle V(C_n) - S \rangle_{C_n}$ is $V(C_n) - S$ itself with $\mid V(C_n) - S \mid= \frac{n}{2}$. Hence, $\mid S \mid \geq \mid V(C_n) - S \mid$. Thus, S is a safe set. Now, we want to show that S is a minimum safe set of C_n. Suppose S is not a minimum safe set of C_n. Then there exist a safe set $S_0 \subseteq V(C_n)$ such that $\mid S_0 \mid < \mid S \mid$. Thus, $\mid S_0 \mid < \frac{n}{2}$ and and $\mid V(C_n) - S_0 \mid > \frac{n}{2}$. Then the component A in $\langle S_0 \rangle_{C_n}$, $\mid A \mid < \mid S_0 \mid < \frac{n}{2}$ and the component B in $\langle V(C_n) - S \rangle_{C_n}$ such that $\mid B \mid > \frac{n}{2}$. Thus, $\mid A \mid < \mid B \mid$. A contradiction. Thus, S is a minimum safe set. Hence, $s(C_n) = \frac{n}{2}$.

Case 2: $n \equiv 1$(mod 2)

Choose $S = \{v_k, ..., v_{2k}\}$, where $k = \frac{n-1}{2}$. Then the component of $\langle S \rangle_{C_n}$ is S itself with $\mid S \mid= \frac{n+1}{2}$. Now, $V(C_n) - S = \{v_0, ..., v_{k-1}\}$, the component of $\langle V(C_n) - S \rangle_{C_n}$ is $V(C_n) - S$ itself with $\mid V(C_n) - S \mid= \frac{n-1}{2}$. Hence,

$| S | \geq | V(C_n) - S |$. Thus, S is a safe set. Now, we want to show that S is a minimum safe set of C_n. Suppose S is not a minimum safe set of C_n. Then there exist a safe set $S_0 \subseteq V(C_n)$ such that $| S_0 | < | S |$. Thus, $| S_0 | < \frac{n+1}{2}$ and and $| V(C_n) - S_0 | > \frac{n-1}{2}$. Then the component A in $\langle S_0 \rangle_{C_n}$, $| A | < | S_0 | < \frac{n+1}{2}$ and the component B in $\langle V(C_n) - S_0 \rangle_{C_n}$ such that $| B | > \frac{n-1}{2}$. Thus, $| A | < | B |$. A contradiction. Thus, S is a minimum safe set. Hence, $s(C_n) = \frac{n+1}{2}$. □

Let G be a complete graph K_n with $n \geq 3$. Since the component of $\langle S \rangle_{K_n}$ is S itself and the component of $\langle V(G) - S \rangle_{K_n}$ is $V(K_n) - S$ itself, such that
$| S | \geq | V(G) - S |$. Hence, S is a safe set.

Remark 1. Let G be a complete graph K_n, $n \geq 3$. Then,

$$s(G) = s(K_n) \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (\text{mod } 2) \\ \frac{n+1}{2} & \text{if } n \equiv 1 (\text{mod } 2) \end{cases}$$

Theorem 3.3. *Let G and H be empty graphs with $| V(G) | = m$ and $| V(H) | = n$. Then a non-empty subset $S \subseteq V(G \vee H)$ is a safe set of $G \vee H$ if and only if one of the following holds:*

- ***(i)*** $S \subseteq V(G)$, $S = V(G)$.
- ***(ii)*** $S \subseteq V(H)$, $S = V(H)$.
- ***(iii)*** $S = S_1 \cup S_2$, *where S_1 is a non-empty subset of $V(G)$ and S_2 is a non-empty subset of $V(H)$, such that $| S | \geq | V(G \vee H) - S |$ where $1 \leq | S_1 | \leq m$ and $1 \leq | S_2 | \leq n$.*

Proof. Let S be a non-empty subset of $V(G \vee H)$ be a safe set of $G \vee H$. For the case (i), suppose $S \subseteq V(G)$, clearly $S = V(G)$. Similarly, for the case (ii), if $S \subseteq V(H)$. Now for the case (iii). Suppose $S = S_1 \cup S_2$ such that $S_1 \subseteq V(G)$ and $S_2 \subseteq V(H)$. If either of S_1 or S_2 are empty, then S is in case (i) or (ii). Since $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$, then $| S_1 | \geq 1$ and $| S_2 | \geq 1$. Suppose further that
$| S | < | V(G \vee H) - S |$. Note that, $\langle S \rangle_{G \vee H}$ is connected and
$\langle V(G \vee H) - S \rangle_{G \vee H}$ is also connected. Thus, their can only be one component of S and one component of $V(G \vee H) - S$. A contradiction, since S is a safe set. Thus, $| S | \geq | V(G \vee H) - S |$. Obviously, $1 \leq | S_1 | \leq m$ and $1 \leq | S_2 | \leq n$.

The converse immediate follows. □

Corollary 3.4. *Let G be a complete bipartite graph, $K_{m,n}$, then*
$S(K_{m,n}) = min\{m, n\}$.

Proof. Let X and Y be the partite set of $K_{m,n}$ where $| X | = m$ and $| Y | = n$. Let S be a non-empty subset of $V(K_{m,n})$. By Theorem 3.3 (i), if $S \subseteq X$ then $| S | = | X | = m$ and S is a safe set of $K_{m,n}$. Now if $S \subseteq Y$, then by Theorem 3.3 (ii) $| S | = | Y | = n$ and S is a safe set of $K_{m,n}$. Thus, $s(K_{m,n}) = \min\{m, n\}$. □

Corollary 3.5. *For any star graph, $K_{1,n}$, $n \geq 1$, $s(K_{1,n}) = 1$.*

Proof. This proof immediately follows from Corollary 3.4. □

4 CONCLUSIONS

In this article, the safe set resulting from complete bipartite graph and safe number of path, cycle, complete graph, complete bipartite graph, and star graph are studied. As future line of research, we plan to investigate the safe set and safe number for some other graph families that has not been studied.

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