Original Research Article

Some Elementary Properties of Kurzweil-Henstock-Stieltjes Integral on $\mathbb{R}^n$

Abstract

Kurzweil-Henstock integral is a generalization of the Reimann integral. In this paper, we established the definition of Kurzweil-Henstock-Stieltjes integral on $\mathbb{R}^n$ via gauge type approach where integrand and integrator are all real-valued functions defined on a compact interval in $\mathbb{R}^n$. Moreover, the Cauchy Criterion is established. To this end, some underlying simple properties of this integral are studied, specifically, uniqueness, linearity, monotonocity, integrability over a subset, and additivity. Results gathered in this paper may serve as a foundation to some related studies such as the notion of convergence with respect to this integral, and the formulation of the Saks-Henstock Lemma.

Keywords: Stieltjes, Perron partition, δ-fine, Cauchy Criterion.

1 Introduction

In 1854, Bernhard Riemann introduced the first formal definition of integral called the Riemann integral which served as the basis in solving mathematical problems in elementary calculus. Nevertheless, at the end of the nineteenth century, mathematicians discovered several shortcomings [8].

In 1902, Henri Lebesgue augmented the shortcomings of the Riemann integral and defined an integral called the Lebesgue integral. However, with respect to its rigor, its formulation was not sufficient enough to integrate all finite derivatives [8]. In 1912, Arnaud Denjoy resolved the weakness of the Lebesgue integral and introduced a new integral which can integrate all finite derivatives. Two years later Oskar Perron separately established his integral called the Perron integral which can also integrate all finite derivatives. Later on, in 1925, it was determined that the integral defined by Arnaud Denjoy and Oskar Perron are equivalent, and this integral is called the Denjoy-Perron integral [5].

In 1957, Joraslav Kurzweil introduced a new integral which is used to study ordinary differential equations [13]. On the other hand, four years later Ralph Henstock introduced his integral which is surprisingly similar to the work of Jaroslav Kurzweil. Nowadays, the integral of Jaroslav Kurzweil and Ralph Henstock is now called the Henstock-Kurzweil integral and apparently, it turns out that it is equivalent to Denjoy-Perron integral.

The idea of integrating the function with respect to another function was authored by Thomas Stieltjes. Originally, his ideas were developed as an extension of the Riemann integral, known as the Riemann-Stieltjes integral [2]. In addition, Jong Sul Lim, Ju Han Yoon and Gwang Sik Eun defined the Kurzweil-Henstock-Stieltjes integral on $\mathbb{R}$ in which the integrator is an increasing function. This integral is more general compared to the Kurzweil-Henstock integral; in fact, the Kurzweil-Henstock integral is a special case of Kurzweil-Henstock-Stieltjes integral, whenever the integrator is an identity function. Various Henstock-Stieltjes type of definitions had been worked. For instance, Flores and Benitez [3, 4] provided a Henstock-Stieltjes integral in Banach Space using the notion of a partition of unity.

In this paper, we established the definition of Kurzweil-Henstock-Stieltjes integral on $\mathbb{R}^n$ via gauge type approach where integrand and integrator are all real-valued functions defined on a compact interval in $\mathbb{R}^n$. Further, a characterization of this integral is established via Cauchy Criterion.

2 Preliminaries

Definition 2.1. [8] A **compact interval** in $\mathbb{R}^n$ is a set of the form $[\boldsymbol{a}, \boldsymbol{b}] = \prod_{i=1}^{n} [a_i, b_i]$, where $-\infty < a_i < b_i < +\infty$ for all $i = 1, 2, \cdots, n$.

Definition 2.2. [8] Two intervals $[\boldsymbol{a}, \boldsymbol{b}]$ and $[\boldsymbol{c}, \boldsymbol{d}]$ in $\mathbb{R}^n$ are said to be **non-overlapping** if $\prod_{i=1}^{n}(a_i, b_i) \bigcap \prod_{i=1}^{n}(c_i, d_i) = \varnothing$.

Definition 2.3. [8] A **partition** of $[\boldsymbol{a}, \boldsymbol{b}]$ is a finite collection of pairwise non-overlapping intervals in $\mathbb{R}^n$ whose union is $[\boldsymbol{a}, \boldsymbol{b}]$.

Definition 2.4. [8] A function $\delta : [\boldsymbol{a}, \boldsymbol{b}] \longrightarrow \mathbb{R}^+$ is known as **gauge** on $[\boldsymbol{a}, \boldsymbol{b}]$.

Definition 2.5. [8] Given $\boldsymbol{x} \in \mathbb{R}^n$ and $r > 0$, we set

$$B(\boldsymbol{x}, r) = \Big\{ \boldsymbol{y} \in \mathbb{R}^n : |||\boldsymbol{x} - \boldsymbol{y}||| < r \Big\},$$

where $|||\boldsymbol{x} - \boldsymbol{y}||| = \max\big\{|x_i - y_i| : i = 1, 2, \cdots, n\big\}$, $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$.

Definition 2.6. [8] A **point-interval pair** $(\boldsymbol{t}, [\boldsymbol{a}, \boldsymbol{b}])$ consists of a point $\boldsymbol{t} \in \mathbb{R}^n$ and an interval $[\boldsymbol{a}, \boldsymbol{b}]$ in $\mathbb{R}^n$. Here $\boldsymbol{t}$ is known as a **tag** of $[\boldsymbol{a}, \boldsymbol{b}]$.

Definition 2.7. [8] A **Perron partition** of $[\boldsymbol{a}, \boldsymbol{b}]$ is a finite collection $\{(\boldsymbol{t}_1, [\boldsymbol{u}_1, \boldsymbol{v}_1]), ..., (\boldsymbol{t}_p, [\boldsymbol{u}_p, \boldsymbol{v}_p])\}$ of point-interval pairs, where $\{[\boldsymbol{u}_1, \boldsymbol{v}_1], ..., [\boldsymbol{u}_p, \boldsymbol{v}_p]\}$ is a partition of $[\boldsymbol{a}, \boldsymbol{b}]$ and $\boldsymbol{t}_k \in [\boldsymbol{u}_k, \boldsymbol{v}_k]$ for $k = 1, \cdots, p$.

Definition 2.8. [8] Let $P = \{(\boldsymbol{t}_1, [\boldsymbol{u}_1, \boldsymbol{v}_1), ..., (\boldsymbol{t}_p, [\boldsymbol{u}_p, \boldsymbol{v}_p])\}$ be a Perron partition of $[\boldsymbol{a}, \boldsymbol{b}]$ and let δ be a gauge defined on $\{\boldsymbol{t}_1, \cdots, \boldsymbol{t}_p\}$. The Perron partition P is said to be **δ-fine** if for every $\boldsymbol{x_k} \in [\boldsymbol{u}_k, \boldsymbol{v}_k], |||\boldsymbol{t}_k - \boldsymbol{x}_k||| < \delta(\boldsymbol{t}_k)$ for $k = 1, \cdots, p$.

Theorem 2.1. [8] **(Cousin's Lemma)** *If δ is a gauge on $[\boldsymbol{a}, \boldsymbol{b}]$, then there exists a δ-fine Perron partition of $[\boldsymbol{a}, \boldsymbol{b}]$.*

Definition 2.9. [8] Let $f : [\boldsymbol{a}, \boldsymbol{b}] \longrightarrow \mathbb{R}$. The **total variation** of f over $[\boldsymbol{a}, \boldsymbol{b}]$ is given by

$$Var(f, [\boldsymbol{a}, \boldsymbol{b}]) = \sup \Bigg\{ \sum_{[\boldsymbol{u}, \boldsymbol{v}] \in P} \big|\Delta_f([\boldsymbol{u}, \boldsymbol{v}])\big| : P \text{ is a partition of } [\boldsymbol{a}, \boldsymbol{b}] \Bigg\}$$

such that

$$\Delta_f([\boldsymbol{u},\boldsymbol{v}]) = \sum_{\boldsymbol{t}\in\mathcal{V}[\boldsymbol{u},\boldsymbol{v}]} f(\boldsymbol{t}) \prod_{k=1}^{n} (-1)^{\chi_{\{u_k\}}(t_k)},$$

where $[\boldsymbol{u},\boldsymbol{v}] \in \mathcal{I}_n([\boldsymbol{a},\boldsymbol{b}])$.

Lemma 2.2. [8] *If $\boldsymbol{I} \in \mathcal{I}_n[\boldsymbol{a},\boldsymbol{b}]$, then there exists a net D of $[\boldsymbol{a},\boldsymbol{b}]$ such that $\boldsymbol{I} \in D$ and the cardinality of D is not more than 3^n.*

Lemma 2.3. [8] *If $\{\boldsymbol{I}_1,\cdots,\boldsymbol{I}_p\} \subset \mathcal{I}_n[\boldsymbol{a},\boldsymbol{b}]$ is finite collection of non-overlapping intervals in $\mathbb{R}^n$, then there exists a net D_0 of $[\boldsymbol{a},\boldsymbol{b}]$ with the following property: if $J \in D_0$ and $J \cap \boldsymbol{I}_r \in \mathcal{I}_n[\boldsymbol{a},\boldsymbol{b}]$ for some $r \in \{1,2,\cdots,p\}$, then $J \subseteq \boldsymbol{I}_r$.*

3 Main Results

Definition 3.1. Let f and g be two real-valued functions defined on $[\boldsymbol{a},\boldsymbol{b}]$. A function f is said to be **Kurzweil-Henstock-Stieltjes** integrable, or simply **KHS**-integrable, with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ if there exists $A \in \mathbb{R}$ with the following property: for each $\varepsilon > 0$ there exists a gauge δ such that

$$\left| \sum_{(\boldsymbol{t},[\boldsymbol{u},\boldsymbol{v}])\in P} f(\boldsymbol{t})\Delta_g([\boldsymbol{u},\boldsymbol{v}]) - A \right| < \varepsilon$$

for each δ-fine Perron partition P of $[\boldsymbol{a},\boldsymbol{b}]$. In this case, $A = (KHS)\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg$. Moreover, for brevity, denote $S(f;g;P) = \sum_{(\boldsymbol{t},[\boldsymbol{u},\boldsymbol{v}])\in P} f(\boldsymbol{t})\Delta_g([\boldsymbol{u},\boldsymbol{v}])$.

Following to the Definition 3.1, we have the uniqueness of the value of the integral.

Theorem 3.1. Let f and g be two real-valued functions defined on $[\boldsymbol{a},\boldsymbol{b}]$. Suppose that f is **KHS**-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$, then the value of the integral is unique.

Proof. Let $\epsilon > 0$. Let $A_1 = \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg$. There exists a gauge δ_1 on $[\boldsymbol{a},\boldsymbol{b}]$ such that

$$\left|S(f;g;P_1) - A_1\ \right| < \frac{\varepsilon}{2}$$

for every δ_1-fine Perron partition P_1 of $[\boldsymbol{a},\boldsymbol{b}]$. Suppose, on the other hand, $A_2 \in \mathbb{R}$ such that $A_2 = \int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg$. Similarly, there exists a gauge δ_2 on $[\boldsymbol{a},\boldsymbol{b}]$ such that

$$\left|S(f;g;P_2) - A_2\ \right| < \frac{\varepsilon}{2}$$

for every δ_2-fine Perron partition P_2 of $[\boldsymbol{a},\boldsymbol{b}]$. It remains to show that $A_1 = A_2$. Define δ on $[\boldsymbol{a},\boldsymbol{b}]$ by

$$\delta = \min\{\delta_1, \delta_2\}.$$

In this case, δ is a gauge on $[\boldsymbol{a},\boldsymbol{b}]$. In view of Cousin's Lemma, we may fix a δ-fine Perron partition P of $[\boldsymbol{a},\boldsymbol{b}]$. In this case, P is both δ_1-fine and δ_2-fine. Observe that,

$$\left|A_1 - A_2\right| \le \left|S(f;g;P) - A_1\ \right| + \left|S(f;g;P) - A_2\ \right| < \varepsilon.$$

This means that, $0 \le \left|A_1 - A_2\right| < \varepsilon$. Therefore, $\left|A_1 - A_2\right| = 0$, that is $A_1 = A_2$. □

Theorem 3.2. *If f_1 and f_2 are **KHS**-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$, then for all $\alpha,\beta\in\mathbb{R}$, $\alpha f_1+\beta f_2$ is **KHS**-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]}(\alpha f_1+\beta f_2)\ dg=\alpha\int_{[\boldsymbol{a},\boldsymbol{b}]}f_1\ dg+\beta\int_{[\boldsymbol{a},\boldsymbol{b}]}f_2\ dg.$$

Proof. Let $\alpha,\beta\in\mathbb{R}$. Fix $\varepsilon>0$. Since f_1 is KHS-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$, choose δ_1 as gauge on $[\boldsymbol{a},\boldsymbol{b}]$ such that

$$\left|S(f_1;g;P_1)-\int_{[\boldsymbol{a},\boldsymbol{b}]}f_1\ dg\right|<\frac{\varepsilon}{2(|\alpha|+1)}$$

for every δ_1-fine Perron partition P_1 of $[\boldsymbol{a},\boldsymbol{b}]$. Similarly, since f_2 is **KHS**-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$, choose gauge δ_2 such that

$$\left|S(f_2;g;P_2)-\int_{[\boldsymbol{a},\boldsymbol{b}]}f_2\ dg\right|<\frac{\varepsilon}{2(|\beta|+1)}$$

for every δ_2-fine Perron partition P_2 of $[\boldsymbol{a},\boldsymbol{b}]$. Define δ on $[\boldsymbol{a},\boldsymbol{b}]$ by setting

$$\delta=\min\{\delta_1,\delta_2\}.$$

Then δ is a gauge on $[\boldsymbol{a},\boldsymbol{b}]$. Now, let P be δ-fine Perron partition on $[\boldsymbol{a},\boldsymbol{b}]$. Here, P is both δ_1-fine and δ_2-fine. Notice that by the Definition 3.1, we have

$$S\big((\alpha f_1+\beta f_2);g;P\big)=\alpha\ S(f_1;g;P)+\beta\ S(f_2;g;P),$$

and so

$$\begin{aligned}
&\left|S\big((\alpha f_1+\beta f_2);g;P\big)-\left\{\alpha\int_{[\boldsymbol{a},\boldsymbol{b}]}f_1\ dg+\beta\int_{[\boldsymbol{a},\boldsymbol{b}]}f_2\ dg\right\}\right|\\
&\qquad\le\left|\alpha\ S(f_1;g;P)-\alpha\int_{[\boldsymbol{a},\boldsymbol{b}]}f_1\ dg\right|+\left|\beta\ S(f_2;g;P)-\beta\int_{[\boldsymbol{a},\boldsymbol{b}]}f_2\ dg\right|\\
&\qquad=|\alpha|\left|S(f_1;g;P)-\int_{[\boldsymbol{a},\boldsymbol{b}]}f_1\ dg\right|+|\beta|\left|S(f_2;g;P)-\int_{[\boldsymbol{a},\boldsymbol{b}]}f_2\ dg\right|\\
&\qquad<(|\alpha|+1)\ \frac{\varepsilon}{2(|\alpha|+1)}+(|\beta|+1)\ \frac{\varepsilon}{2(|\beta|+1)}\\
&\qquad=\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we conclude that $\alpha f_1+\beta f_2$ is **KHS**-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and

$$\int_{[\boldsymbol{a},\boldsymbol{b}]}(\alpha f_1+\beta f_2)\ dg=\alpha\int_{[\boldsymbol{a},\boldsymbol{b}]}f_1\ dg+\beta\int_{[\boldsymbol{a},\boldsymbol{b}]}f_2\ dg.$$ □

Proposition 3.1. *Let g_1 and g_2 be real-valued functions defined on compact interval $[\boldsymbol{u},\boldsymbol{v}]$ on $\mathbb{R}^n$. Then for all $\alpha,\beta\in\mathbb{R}$*

$$\Delta_{\alpha g_1+\beta g_2}([\boldsymbol{u},\boldsymbol{v}])=\alpha\Delta_{g_1}([\boldsymbol{u},\boldsymbol{v}])+\beta\Delta_{g_2}([\boldsymbol{u},\boldsymbol{v}]).$$

Proposition 3.2. *If f,g_1 and g_2 are real-valued functions defined on a compact interval $[\boldsymbol{a},\boldsymbol{b}]$ on $\mathbb{R}^n$, then for all $\alpha,\beta\in\mathbb{R}$ and for all Perron partition P of $[\boldsymbol{a},\boldsymbol{b}]$,*

$$S\big(f;(\alpha g_1+\beta g_2);P\big)=\alpha S(f;g_1;P)+\beta S(f;g_2;P).$$

Theorem 3.3. *If f is **KHS**-integrable with respect to g_1 and g_2 on $[\boldsymbol{a},\boldsymbol{b}]$, then for all $\alpha,\beta\in\mathbb{R}$, f is **KHS**-integrable with respect to $\alpha g_1+\beta g_2$ on $[\boldsymbol{a},\boldsymbol{b}]$ and*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ d(\alpha g_1+\beta g_2)=\alpha\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_1+\beta\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_2.$$

The proof is similar to the Theorem 3.2.

Theorem 3.4. *If f_1 and f_2 are **KHS**-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ such that $f_1(\boldsymbol{x})\leq f_2(\boldsymbol{x})$ for all $\boldsymbol{x}\in[\boldsymbol{a},\boldsymbol{b}]$, then*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f_1\ dg\leq\int_{[\boldsymbol{a},\boldsymbol{b}]} f_2\ dg.$$

Proof. Let $\varepsilon>0$. Choose δ_1 and δ_2 as gauges on $[\boldsymbol{a},\boldsymbol{b}]$ so that

$$\left|S(f_1;g;P_1)-\int_{[\boldsymbol{a},\boldsymbol{b}]} f_1\ dg\right|<\frac{\varepsilon}{2}$$

and

$$\left|S(f_2;g;P_2)-\int_{[\boldsymbol{a},\boldsymbol{b}]} f_2\ dg\right|<\frac{\varepsilon}{2}$$

for all δ_1-fine Perron partition P_1 and δ_2-fine Perron partition P_2 of $[\boldsymbol{a},\boldsymbol{b}]$. Next define δ on $[\boldsymbol{a},\boldsymbol{b}]$ by setting

$$\delta=\min\{\delta_1,\delta_2\}$$

so that we can fix a δ-fine Perron partition P on $[\boldsymbol{a},\boldsymbol{b}]$. In this case, P is both δ_1-fine and δ_2-fine. Notice that,

$$S(f_1;g;P)\leq S(f_2;g;P).$$

Since

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f_1\ dg<S(f_1;g;P)+\frac{\varepsilon}{2}$$

and

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f_2\ dg+\varepsilon>S(f_2;g;P)+\frac{\varepsilon}{2},$$

thus

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f_1\ dg<S(f_1;g;P)+\frac{\varepsilon}{2}\leq S(f_2;g;P)+\frac{\varepsilon}{2}<\int_{[\boldsymbol{a},\boldsymbol{b}]} f_2\ dg+\varepsilon.$$

Therefore, by the arbitrary nature of $\varepsilon>0$,

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f_1\ dg\leq\int_{[\boldsymbol{a},\boldsymbol{b}]} f_2\ dg.$$

□

Proposition 3.3. *If g_1 and g_2 are real-valued functions defined on compact interval $[\boldsymbol{u},\boldsymbol{v}]$ on $\mathbb{R}^n$ such that $g_1(\boldsymbol{x})\leq g_2(\boldsymbol{x})$ for all $\boldsymbol{x}\in[\boldsymbol{u},\boldsymbol{v}]$, then*

$$\Delta_{g_1}([\boldsymbol{u},\boldsymbol{v}])\leq\Delta_{g_2}([\boldsymbol{u},\boldsymbol{v}]).$$

Proposition 3.4. *If f,g_1 and g_2 are real-valued functions defined on a compact interval $[\boldsymbol{a},\boldsymbol{b}]$ on $\mathbb{R}^n$ such that $g_1(\boldsymbol{x})\leq g_2(\boldsymbol{x})$ for all $\boldsymbol{x}\in[\boldsymbol{a},\boldsymbol{b}]$, then for all Perron partition P of $[\boldsymbol{a},\boldsymbol{b}]$*

$$S(f;g_1;P)\leq S(f;g_2;P).$$

Theorem 3.5. *If f is **KHS**-integrable with respect to g_1 and g_2 on $[\boldsymbol{a},\boldsymbol{b}]$ such that $g_1(\boldsymbol{x})\leq g_2(\boldsymbol{x})$ for all $\boldsymbol{x}\in[\boldsymbol{a},\boldsymbol{b}]$ then*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_1\leq\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg_2.$$

The proof is similar to the Theorem 3.4.

3.1 Cauchy Criterion

Theorem 3.6. *A function f is said to be **KHS**-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ if and only if for each $\varepsilon>0$ there exists a gauge δ such that*

$$\left|S(f;g;P)-S(f;g;Q)\right|<\varepsilon$$

for every δ-fine Perron partition P and Q of $[\boldsymbol{a},\boldsymbol{b}]$.

Proof. ($\Rightarrow$) Let $\varepsilon>0$. Since f is **KHS**-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$, there exists a gauge δ such that

$$\left|S(f;g;P)-\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\right|<\frac{\varepsilon}{2}$$

for every δ-fine Perron partition P of $[\boldsymbol{a},\boldsymbol{b}]$. Let P and Q be a δ-fine Perron partition of $[\boldsymbol{a},\boldsymbol{b}]$. Observe that,

$$\left|S(f;g;P)-S(f;g;Q)\right|\leq\left|S(f;g;P)-\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\right|+\left|S(f;g;Q)-\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg\right|<\varepsilon.$$

($\Leftarrow$) For each $n\in\mathbb{N}$, let δ_n be a gauge on $[\boldsymbol{a},\boldsymbol{b}]$ so that

$$\left|S(f;g;Q_n)-S(f;g;R_n)\right|<\frac{1}{n}$$

for every pair of δ_n-fine Perron partition Q_n and R_n of $[\boldsymbol{a},\boldsymbol{b}]$. Define Φ_n on $[\boldsymbol{a},\boldsymbol{b}]$ by setting

$$\Phi_n=\min\{\delta_1,\delta_2,\cdots,\delta_n\}.$$

Then Φ_n is a gauge on $[\boldsymbol{a},\boldsymbol{b}]$. In view of Cousin's Lemma, we can choose P_n to be Φ_n-fine Perron partition of $[\boldsymbol{a},\boldsymbol{b}]$. We further show that $\left\{S(f;g;P_n)\right\}_{n=1}^{\infty}$ is a Cauchy sequence. To this end, let $\varepsilon>0$. Choose $N\in\mathbb{N}$ such that $\dfrac{1}{N}<\varepsilon$. If n_1 and n_2 are positive integers such that $\min\{n_1,n_2\}\geq N$, then we see that P_{n_1} and P_{n_2} are both $\Phi_{\min\{n_1,n_2\}}$-fine Perron partition of $[\boldsymbol{a},\boldsymbol{b}]$ and so

$$\left|S(f;g;P_{n_1})-S(f;g;P_{n_2})\right|<\frac{1}{\min\{n_1,n_2\}}\leq\frac{1}{N}<\varepsilon.$$

Hence, $\left\{S(f;g;P_n)\right\}_{n=1}^{\infty}$ is a Cauchy sequence. Note that, $\left\{S(f;g;P_n)\right\}_{n=1}^{\infty}\subseteq\mathbb{R}$. Since $\mathbb{R}$ is complete, there exist $A\in\mathbb{R}$ such that $\left\{S(f;g;P_n)\right\}_{n=1}^{\infty}\longrightarrow A$. Here, it remains to show that f is **KHS**-integrable with respect to g and $\displaystyle\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg=A$. Let P be Φ_N-fine Perron partition of $[\boldsymbol{a},\boldsymbol{b}]$. Since $\{\Phi_n\}_{n=1}^{\infty}$ is decreasing, we see that the Φ_n-fine Perron partition P_n is Φ_N-fine for every integer $n\geq N$. Thus,

$$\begin{aligned}\left|S(f;g;P)-A\right|&=\left|S(f;g;P)-\lim_{n\to\infty}S(f;g;P_n)\right|\\&=\lim_{n\to\infty}\left|S(f;g;P)-S(f;g;P_n)\right|\\&<\lim_{n\to\infty}\frac{1}{N}\\&<\lim_{n\to\infty}\varepsilon\\&=\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we conclude that f is **KHS**-integrable with respect to g and

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg=A.$$

□

Proposition 3.5. *Let f and g be real-valued functions defined on a compact interval of $[\boldsymbol{a},\boldsymbol{b}] \subset \mathbb{R}^n$ and let $\{\ I_k \mid k = 1, 2, \cdots, m\}$ be a partition of $[\boldsymbol{a},\boldsymbol{b}]$. For each $k = 1, 2, \cdots, m$, assume that P_k is a Perron partition of I_k, then $\bigcup_{k=1}^m P_k$ is a Perron partition of $[\boldsymbol{a},\boldsymbol{b}]$ and*

$$\sum_{k=1}^{m} S\big(f;g;P_k\big) = S\Big(f;g;\bigcup_{k=1}^{m} P_k\Big).$$

Proof. Let P_k be a Perron partition of $\boldsymbol{I}_k$ for all $k = 1, 2, \cdots, m$. For convenience, let $\mathscr{F}_k = \{\boldsymbol{I}^{(k)} : (\boldsymbol{t}^{(k)}, \boldsymbol{I}^{(k)}) \in P_k\}$, for all $k \leq m$. Here, for each $k = 1, 2, \cdots, m$, $\mathscr{F}_k$ is finite and $\bigcup_{\boldsymbol{I}\in\mathscr{F}_k} \boldsymbol{I} = \boldsymbol{I}_k$. Observe that,

$$\begin{aligned}\bigcup_{k=1}^{m} P_k &= P_1 \cup P_2 \cup \cdots \cup P_m \\ &= \big\{(\boldsymbol{t}^{(1)}, \boldsymbol{I}^{(1)})\big\} \cup \big\{(\boldsymbol{t}^{(2)}, \boldsymbol{I}^{(2)})\big\} \cup \cdots \cup \big\{(\boldsymbol{t}^{(m)}, \boldsymbol{I}^{(m)})\big\}.\end{aligned}$$

In this case, we show that $\bigcup_{k=1}^m \mathscr{F}_k$ partitions $[\boldsymbol{a},\boldsymbol{b}]$. Notice that,

$$\bigcup_{k=1}^{m} \bigcup_{\boldsymbol{I}\in\mathscr{F}_k} \boldsymbol{I} = \bigcup_{k=1}^{m} \boldsymbol{I}_k = [\boldsymbol{a},\boldsymbol{b}].$$

Let $\boldsymbol{K}, \boldsymbol{J} \in \bigcup_{k=1}^m \mathscr{F}_k$ such that $\boldsymbol{K} \neq \boldsymbol{J}$. We further show that $int(\boldsymbol{K}) \cap int(\boldsymbol{J}) = \varnothing$. To this end, choose $s, s' \in \{1, 2, \cdots, m\}$ such that $\boldsymbol{K} \in \mathscr{F}_s$ and $\boldsymbol{J} \in \mathscr{F}_{s'}$. Here, there exists $\boldsymbol{I} \in \mathscr{F}_s$ such that $\boldsymbol{K} = \boldsymbol{I}$. Similarly, there exists $\boldsymbol{I}' \in \mathscr{F}_{s'}$ such that $\boldsymbol{J} = \boldsymbol{I}'$. Since $\boldsymbol{K} \neq \boldsymbol{J}$, it follows that $\boldsymbol{I} \neq \boldsymbol{I}'$, and so $int(\boldsymbol{K}) \cap int(\boldsymbol{J}) = int(\boldsymbol{I}) \cap int(\boldsymbol{I}') = \varnothing$. Thus, $\bigcup_{k=1}^m \mathscr{F}_k$ partitions $[\boldsymbol{a},\boldsymbol{b}]$; hence this makes the $\bigcup_{k=1}^m P_k$ is a Perron partition of $[\boldsymbol{a},\boldsymbol{b}]$. Now,

$$\begin{aligned}\sum_{k=1}^{m} S\big(f;g;P_k\big) &= S\big(f;g;P_1\big) + S\big(f;g;P_2\big) + \cdots + S\big(f;g;P_m\big) \\ &= \sum_{(\boldsymbol{t}^{(1)},\boldsymbol{I}^{(1)})\in P_1} f(\boldsymbol{t}^{(1)})\Delta_g(\boldsymbol{I}^{(1)}) + \sum_{(\boldsymbol{t}^{(2)},\boldsymbol{I}^{(2)})\in P_2} f(\boldsymbol{t}^{(2)})\Delta_g(\boldsymbol{I}^{(2)}) + \cdots + \\ &\qquad \sum_{(\boldsymbol{t}^{(m)},\boldsymbol{I}^{(m)})\in P_m} f(\boldsymbol{t}^{(m)})\Delta_g(\boldsymbol{I}^{(m)}) \\ &= \sum_{\substack{(\boldsymbol{t},\boldsymbol{I})\in P_k \\ 0<k\leq m}} f(\boldsymbol{t})\Delta_g(\boldsymbol{I}) \\ &= \sum_{\boldsymbol{I}\in\bigcup_{k=1}^m \mathscr{F}_k, \boldsymbol{t}\in\boldsymbol{I}} f(\boldsymbol{t})\Delta_g(\boldsymbol{I}) \\ &= \sum_{(\boldsymbol{t},\boldsymbol{I})\in\bigcup_{k=1}^m P_k} f(\boldsymbol{t})\Delta_g(\boldsymbol{I}) \\ &= S\big(f;g;\bigcup_{k=1}^{m} P_k\big).\end{aligned}$$

□

The following Theorem is a corollary of the Cauchy Criterion.

Theorem 3.7. *If f is **KHS**-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$, then f is **KHS**-integrable with respect g on $\boldsymbol{I} \in \mathcal{I}_n[\boldsymbol{a},\boldsymbol{b}]$.*

Proof. Let $\varepsilon > 0$. By Cauchy Criterion, choose a gauge δ on $[\boldsymbol{a},\boldsymbol{b}]$ such that

$$\left|S(f;g;P) - S(f;g;Q)\right| < \varepsilon$$

for all δ-fine Perron partitions P and Q of $[\boldsymbol{a},\boldsymbol{b}]$. If $\boldsymbol{I} = [\boldsymbol{a},\boldsymbol{b}]$, then we are done. Suppose $\boldsymbol{I} \subset [\boldsymbol{a},\boldsymbol{b}]$. Then by Lemma 2.2.12, there exists a finite collection of pairwise non-overlapping subintervals of $[\boldsymbol{a},\boldsymbol{b}]$, say $\{\boldsymbol{I}_1,\boldsymbol{I}_2,\cdots,\boldsymbol{I}_N\}$ such that $\boldsymbol{I} \notin \{\boldsymbol{I}_1,\boldsymbol{I}_2,\cdots,\boldsymbol{I}_N\}$ and $\boldsymbol{I} \cup \bigcup_{k=1}^N \boldsymbol{I}_k$ is a net on $[\boldsymbol{a},\boldsymbol{b}]$. For each $k = 1,2,\cdots,N$, $\delta_{|\boldsymbol{I}_k}$ is a gauge on $\boldsymbol{I}_k$. Let P_k be a $\delta_{|\boldsymbol{I}_k}$-fine Perron partition of $\boldsymbol{I}_k$ for all $k = 1,2,\cdots,N$. Similarly, $\delta_{|\boldsymbol{I}}$ is a gauge on $\boldsymbol{I}$. Fix $P_{\boldsymbol{I}}$ and $Q_{\boldsymbol{I}}$ be $\delta_{|\boldsymbol{I}}$-fine Perron partitions of $\boldsymbol{I}$. In this case, $P_{\boldsymbol{I}} \cup \bigcup_{k=1}^N P_k$ and $Q_{\boldsymbol{I}} \cup \bigcup_{k=1}^N P_k$ are δ-fine Perron partitions of $[\boldsymbol{a},\boldsymbol{b}]$. By Proposition 3.5, observe that

$$\begin{aligned}
\left|S(f;g;P_{\boldsymbol{I}}) - S(f;g;Q_{\boldsymbol{I}})\right| &= \Bigg|S(f;g;P_{\boldsymbol{I}}) + \sum_{k=1}^N S(f;g;P_k) \\
&\qquad\qquad - \sum_{k=1}^N S(f;g;P_k) - S(f;g;Q_{\boldsymbol{I}})\Bigg| \\
&= \Bigg|S(f;g;P_{\boldsymbol{I}}) + S\Big(f;g;\bigcup_{k=1}^N P_k\Big) \\
&\qquad\qquad - \Big\{S\Big(f;g;\bigcup_{k=1}^N P_k\Big) + S(f;g;Q_{\boldsymbol{I}})\Big\}\Bigg| \\
&= \Bigg|S\Big(f;g;P_{\boldsymbol{I}} \cup \bigcup_{k=1}^N P_k\Big) - S\Big(f;g;Q_{\boldsymbol{I}} \cup \bigcup_{k=1}^N P_k\Big)\Bigg| \\
&< \varepsilon.
\end{aligned}$$

Therefore, the theorem holds. □

Theorem 3.8. *Let $\{\boldsymbol{I},\boldsymbol{J}\}$ be a partition of $[\boldsymbol{a},\boldsymbol{b}]$. If f is **KHS**-integrable with respect to g over $\boldsymbol{I}$ and $\boldsymbol{J}$, then f is **KHS**-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg = \int_{\boldsymbol{I}} f\ dg + \int_{\boldsymbol{J}} f\ dg.$$

Proof. Let $\varepsilon > 0$. Next, choose gauges δ_1 and δ_2 on $[\boldsymbol{a},\boldsymbol{b}]$ so that

$$\left|S(f;g;P_{\boldsymbol{I}}) - \int_{\boldsymbol{I}} f\ dg\right| < \frac{\varepsilon}{2}$$

and

$$\left|S(f;g;P_{\boldsymbol{J}}) - \int_{\boldsymbol{J}} f\ dg\right| < \frac{\varepsilon}{2}$$

for all δ_1-fine Perron partition $P_{\boldsymbol{I}}$ of $\boldsymbol{I}$ and δ_2-fine Perron partition $P_{\boldsymbol{J}}$ of $\boldsymbol{J}$, respectively. Define δ on $[\boldsymbol{a},\boldsymbol{b}]$ by setting,

$$\delta(\boldsymbol{x}) = \begin{cases} \min\{\delta_1(\boldsymbol{x}),\delta_2(\boldsymbol{x})\}, & \text{if } \boldsymbol{x} \in \boldsymbol{I} \cap \boldsymbol{J}, \\ \min\{\delta_1(\boldsymbol{x}),\mathrm{dist}(\boldsymbol{x},\boldsymbol{J})\}, & \text{if } \boldsymbol{x} \in \boldsymbol{I} \setminus \boldsymbol{J}, \\ \min\{\delta_2(\boldsymbol{x}),\mathrm{dist}(\boldsymbol{x},\boldsymbol{I})\}, & \text{if } \boldsymbol{x} \in \boldsymbol{J} \setminus \boldsymbol{I}, \end{cases}$$

In this case, δ is a gauge on $[\boldsymbol{a},\boldsymbol{b}]$. Let P be δ-fine Perron partition of $[\boldsymbol{a},\boldsymbol{b}]$. For convenience, write $P = \{(\boldsymbol{x},\boldsymbol{H})\}$. Let $P_1 = \{(\boldsymbol{x},\boldsymbol{K}) \in P : \boldsymbol{x} \in \boldsymbol{I},\ \boldsymbol{H} \cap \boldsymbol{I} = \boldsymbol{K} \text{ and } \mathrm{vol}(\boldsymbol{K}) > 0\}$.
Let $P_2 = \{(\boldsymbol{x},\boldsymbol{L}) \in P : \boldsymbol{x} \in \boldsymbol{J},\ \boldsymbol{H} \cap \boldsymbol{J} = \boldsymbol{L} \text{ and } \mathrm{vol}(\boldsymbol{L}) > 0\}$. Here, P_1 is both δ-fine and δ_1-fine

of $\boldsymbol{I}$. Similarly, P_2 is both δ-fine and δ_2-fine of J. By Proposition 3.5, $P_1 \cup P_2$ is a δ-fine Perron partition of $[\boldsymbol{a},\boldsymbol{b}]$ and so

$$S(f;g;P) = S(f;g;P_1 \cup P_2) = S(f;g;P_1) + S(f;g;P_2).$$

Thus,

$$\left| S(f;g;P) - \left\{ \int_{\boldsymbol{I}} f\ dg + \int_{\boldsymbol{J}} f\ dg \right\} \right| \leq \left| S(f;g;P_1) - \int_{\boldsymbol{I}} f\ dg \right| + \left| S(f;g;P_2) - \int_{\boldsymbol{J}} f\ dg \right|$$
$$< \varepsilon.$$

Therefore, the theorem holds. □

Proposition 3.6. *Suppose f is* ***KHS****-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$. If $\{\boldsymbol{I},\boldsymbol{J}\}$ is a partition of $[\boldsymbol{a},\boldsymbol{b}]$, then f* ***KHS****-integrable with respect to g over $\boldsymbol{I}$ and $\boldsymbol{J}$ and*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg = \int_{\boldsymbol{I}} f\ dg + \int_{\boldsymbol{J}} f\ dg.$$

Theorem 3.9. *Let D be a partition of $[\boldsymbol{a},\boldsymbol{b}]$. If f is* ***KHS****-integrable with respect to g on $\boldsymbol{J}$ for all $\boldsymbol{J} \in D$, then f is* ***KHS****-integrable with respect to g on $[\boldsymbol{a},\boldsymbol{b}]$ and*

$$\int_{[\boldsymbol{a},\boldsymbol{b}]} f\ dg = \sum_{\boldsymbol{J} \in D} \int_{\boldsymbol{J}} f\ dg.$$

Proof. Let $\boldsymbol{J} \in D$. Suppose that f is **KHS**-integrable with respect to g on $\boldsymbol{J}$. By Theorem 3.7 and Lemma 2.3, we may view D as a net on $[\boldsymbol{a},\boldsymbol{b}]$. In this case, we repeatedly apply the Theorem 3.8 to get the result. □

4 Conclusion and Recommendation

Results gathered in the literature show that the Definition of Kurzweil-Henstock-Stieltjes integral on $\mathbb{R}^n$ is elegant that the simplicity of its definition, in most cases, is more powerful than the Lebesgue integral. Further, the Cauchy Criterion is another way to characterize functions that are KHS-integrable serving as a convenient tool for some results. As a recommendation, further convegence theorems and the Saks-Henstock Lemma and its corollary results are yet to be established.

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