Original Research Article

The differential equations of gravity-free double pendulum: Lauricella hypergeometric solutions and their inversion.

Abstract

This paper solves in closed form the system of ODEs ruling the 2D motion of a gravity free double pendulum (GFDP), not subjected to any force. In such a way its movement is governed by the initial conditions only. The relevant strongly non linear ODEs have been put back to hyperelliptic quadratures which, through the Integral Representation Theorem (IRT), have been driven to the Lauricella hypergeometric functions.

We compute time laws and trajectories of both point masses forming the GFDP in explicit closed form. Suitable sample problems are carried out in order to prove the method effectiveness.

Keywords: Double pendulum; hypergeometric Lauricella functions; Fourier series; non linear systems; functional inversion

1 Introduction

The system consists of two point-masses m_1 and m_2 moving over an horizontal fixed plane, where the distance between a fixed point P (called pivot) and m_1 and the distance between m_1 and m_2 are fixed and equal to l_1 and l_2 respectively: we can think to a couple of massless beams. In absence of gravity and any other external force, the motion is driven by the four initial conditions only. Assuming the pivot as a pole and as a polar axis the horizontal line passing through it, the planar line Γ referred to such a frame will denote the m_2 absolute trajectory given in polar coordinates ρ and μ.

The selected lagrangian coordinates are the angle θ (between the polar axis and the first beam), and ψ (the angle between the first and second beam) both positive counterclockwise.

Within the nonlinear systems there is a hierarchy of complexity: the simple pendulum of small amplitude has harmonic oscillations and a period independent from the amplitude; in the case of large amplitudes, such dependence occurs, but the regularity of motion holds. The heavy double pendulum

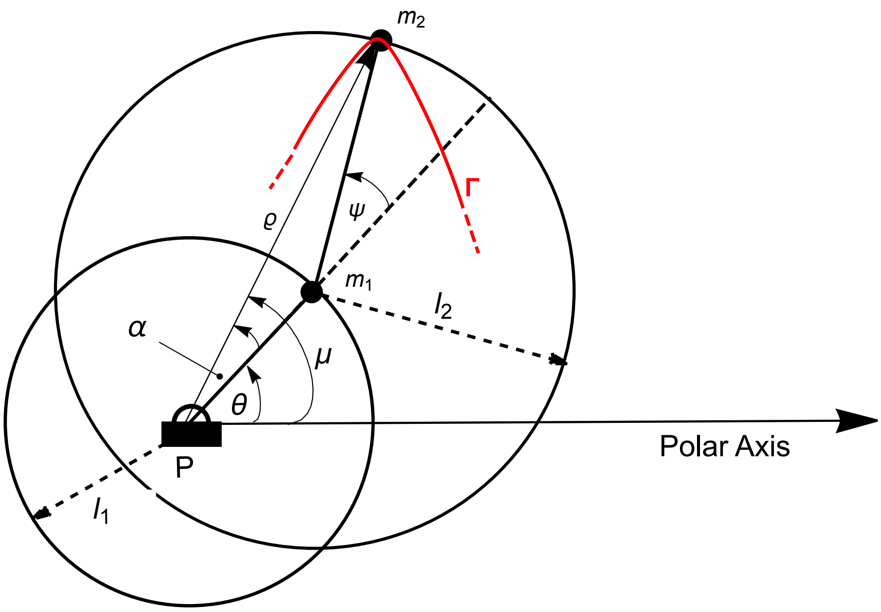

Figure 1: GFDP circles of motion and Γ trajectory of m_2.

in the case of large amplitudes changes radically with increasing energy and the oscillations become chaotic.

The GFDP problem has been treated by Enolsky [(2)] employing an analytical approach fully different from our and founded on Theta Riemann series [(6)]. As far as we are concerned, the literature on the GFDP non linear ODE system does not appear to have further items.

2 The motion differential equations

Since no potential energy is present, lagrangian (L) and the hamiltonian (H) functions will coincide with kinetic energy:

$$L = H = \frac{1}{2}m_1 l_1^2\dot{\theta}^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}^2 + l_2^2\dot{\phi}^2 + 2l_1l_2\cos(\phi-\theta)\dot{\phi}\dot{\theta}\right]$$

Setting $\psi = \phi - \theta$, $\nu = m_1/m_2$, $\beta = l_1/l_2$ and $\gamma = (\nu+1)\beta^2 + 1$ we get:

$$L = H = \frac{1}{2}m_2 l_2^2\left[\gamma + 2\beta\cos(\psi)\right]\dot{\theta}^2 + m_2 l_2^2[1+\beta\cos(\psi)]\dot{\theta}\dot{\psi} + \frac{1}{2}m_2 l_2^2\dot{\psi}^2$$

The conjugate moment relevant to the coordinate θ will be a motion first integral, say A:

$$\frac{p_\theta}{m_2 l_2^2} = \frac{1}{m_2 l_2^2}\frac{\partial L}{\partial\dot{\theta}} = \left[\gamma + 2\beta\cos(\psi)\right]\dot{\theta} + [1+\beta\cos(\psi)]\dot{\psi} = A$$

Since no dissipation is present, then the hamiltonian is also constant, say B; the coupled differential system is then:

$$\begin{cases} \left[\gamma + 2\beta\cos(\psi)\right]\dot{\theta} + [1+\beta\cos(\psi)]\dot{\psi} = A & A \in \mathbb{R} \\ \left[\gamma + 2\beta\cos(\psi)\right]\dot{\theta}^2 + 2[1+\beta\cos(\psi)]\dot{\theta}\dot{\psi} + \dot{\psi}^2 = B & B \in \mathbb{R} \end{cases} \tag{2.1}$$

Solving (2.1) for $\dot{\psi}$ and $\dot{\theta}$ we obtain the differential equations of motion. The first is:

$$\begin{cases} \dot{\psi} = \mp\sqrt{\dfrac{A^2 - 2\beta B\cos(\psi) - B\gamma}{\beta^2\cos^2(\psi) - \gamma + 1}} \\ \psi(t_0) = \psi_0 \end{cases} \tag{2.2}$$

and:

$$\begin{cases} \dot{\theta} = \pm\dfrac{\beta\cos(\psi) + 1}{2\beta\cos(\psi) + \gamma}\dot{\psi} \mp \dfrac{A}{2\beta\cos(\psi) + \gamma} \\ \theta(t_0) = \theta_0 \end{cases} \tag{2.3}$$

2.1 The ψ solution

Performing the following change of variable:

$$\cos(\psi) = y \rightarrow d\psi = -\frac{1}{\sqrt{1-y^2}}dy \tag{2.4}$$

(2.2) becomes:

$$t - t_0 = \pm\sqrt{\frac{\beta}{2B}}\int_{y_0}^{y}\sqrt{\frac{(s-\delta)(s+\delta)}{(\epsilon - s)\,(1+s)(1-s)}}\mathrm{d}s = \pm\sqrt{\frac{\beta}{2B}}\int_{y_0}^{y} f(s)\mathrm{d}s \tag{2.5}$$

with:

$$\epsilon = \frac{A^2 - B\gamma}{2\beta B}, \quad \delta = \sqrt{\nu+1} > 1, \quad f(s) = \sqrt{\frac{(s-\delta)(s+\delta)}{(\epsilon - s)(1+s)(1-s)}} \tag{2.6}$$

2.2 The θ solution

Dividing (2.3) to (2.2) and making the same change of variable (2.4) we obtain θ as a function of $y = \cos\psi$:

$$\theta - \theta_0 = I_1(y) + I_2(y)$$

where:

$$\begin{aligned} I_1(y) &= \mp\frac{1}{2}\int_{y_0}^{y}\frac{s + 1/\beta}{(s+\zeta)\sqrt{1-s^2}}\mathrm{d}s = \\ &= \mp\frac{1}{2}\left[\sin^{-1}(s) + \frac{1/\beta - \zeta}{\sqrt{\zeta^2 - 1}}\tan^{-1}\left(\frac{\zeta s + 1}{\sqrt{(1-s^2)(\zeta^2-1)}}\right)\right]_{y_0}^{y} \end{aligned}$$

with:

$$\zeta = \frac{\gamma}{2\beta} > 1$$

Notice that $\zeta > 1$ always; indeed the contrary would lead to:

$$(\nu+1)\beta^2 - 2\beta + 1 < 0$$

which cannot be true.

In order to complete the evaluation of the function θ, the further integral to be carried out is:

$$I_2(y) = \pm \frac{A}{2\sqrt{2B\beta}} \int_{y_0}^{y} \frac{f(s)}{s+\zeta} \mathrm{d}s \tag{2.7}$$

Our main problem is due to its upper limit y, namely the inverse of the hyper-elliptic integral (2.5) which therefore becomes our next step.

3 The three motions allowed

The highest possible span of y is of course $y \in [-1, 1]$; nevertheless it will be narrowed in some cases. Furthermore, by its definition $\delta > 1$: then these roots lie outside the y-domain. As a consequence, the only root really influencing all the features of the whole GFDP motion is the non dimensional parameter ϵ.

We see 4 cases at all:

1. If $0 \leq \epsilon \leq 1$ then: $y \in [\epsilon, 1]$, so that the angle ψ cannot assume all the possible values. Anyway y is time-periodical and its period will be given by:

$$T_1 = \frac{2\beta}{\sqrt{2B\beta}} \int_{\epsilon}^{1} f(s)\mathrm{d}s$$

2. If $-1 \leq \epsilon \leq 0$ then $y \in [-|\epsilon|, 1]$, namely the angle ψ has some different restrictions and the y time-period will be:

$$T_2 = \frac{2\beta}{\sqrt{2B\beta}} \int_{-|\epsilon|}^{1} f(s)\mathrm{d}s$$

3. If $\epsilon < -1$ then $y \in [-1, 1]$ and this is the only occurrence where the angle ψ is free of taking all the possible values.
   The $y(t)$ time-period will be:

$$T_3 = \frac{2\beta}{\sqrt{2B\beta}} \int_{-1}^{1} f(s)\mathrm{d}s$$

4. If $\epsilon > 1$ the motion cannot exist, being the argument of the square root in (2.2) less than zero $\forall y \in [-1, 1]$.

This analysis shows ϵ as the crucial parameter ruling the whole system behavior.

4 First motion: $0 < \epsilon < 1$

4.1 Integration

In such a case $y \in [\epsilon, 1]$ and we have:

$$t - t_0 = \pm \sqrt{\frac{\beta}{2B}} \int_{y_0}^{y} f(s)\mathrm{d}s, \quad \text{with} \quad y \in]\epsilon, 1[, \quad \epsilon \in]0, 1[\tag{4.1}$$

in order to solve (4.1) with $\epsilon \in [0, 1]$ in closed form, let us pass from s to u through a first change of variable:

$$s = u + \epsilon$$

i.e. we move to ϵ the origin of the variable of integration; let us now define:

$$g(u)=f(u+\epsilon)=\sqrt{\frac{(q_1-u)(q_2+u)}{u(q_3-u)(q_4+u)}}$$

with:

$$q_1=\delta-\epsilon,\quad q_2=\delta+\epsilon,\quad q_3=1-\epsilon,\quad q_4=1+\epsilon$$

So that:

$$\int_{y_0}^{y} f(s)\mathrm{d}s=\int_{y_0-\epsilon}^{y-\epsilon} g(u)\mathrm{d}u=$$
$$=\int_0^{y-\epsilon} g(u)\mathrm{d}u-\int_0^{y_0-\epsilon} g(u)\mathrm{d}u=G(y-\epsilon)-G(y_0-\epsilon)$$

where:

$$G(\lambda)=\int_0^{\lambda} g(u)\mathrm{d}u,\quad \text{with}\quad \lambda=y-\epsilon,\quad \text{so that:}\quad \lambda\in]0,1-\epsilon[\tag{4.2}$$

In order to solve (4.2), let us resort to a further change from u to v:

$$u=\lambda v$$

leading to:

$$G(\lambda)=\lambda\int_0^1 g(\lambda v)dv$$

After some algebraic manipulations we arrive at:

$$G(\lambda)=\sqrt{\lambda\hat{q}}\int_0^1 v^{-\frac{1}{2}}\left(1-\hat{\lambda}_1 v\right)^{-\frac{1}{2}}\left(1-\hat{\lambda}_2 v\right)^{\frac{1}{2}}\left(1-\hat{\lambda}_3 v\right)^{-\frac{1}{2}}\left(1-\hat{\lambda}_4 v\right)^{\frac{1}{2}}\mathrm{d}v \tag{4.3}$$

with:

$$\hat{\lambda}_1=\frac{\lambda}{q_3},\quad \hat{\lambda}_2=\frac{\lambda}{q_1},\quad \hat{\lambda}_3=-\frac{\lambda}{q_4},\quad \hat{\lambda}_4=-\frac{\lambda}{q_2},\quad \hat{q}=\frac{q_1q_2}{q_3q_4}$$

The reader (4.3) is referred to the IRT (8.1) for the Lauricella hypergeometric functions. We are faced to a $F_D^{(4)}$:

$$G(\lambda)=2\sqrt{\frac{\lambda q_1q_2}{q_3q_4}}F_D^{(4)}\left(\begin{array}{c|c}\frac{1}{2};\quad \frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2} & \\ & \frac{\lambda}{q_3},\frac{\lambda}{q_1},-\frac{\lambda}{q_4},-\frac{\lambda}{q_2}\\ \frac{3}{2} & \end{array}\right) \tag{4.4}$$

By (4.4) one can check that Lauricella's arguments are-as required by the representation theorem-less than unity. So that, going back,we could infer that:

$$t-t_0=\pm\left[\eta(s)F_D^{(4)}\left(\begin{array}{c|c}\frac{1}{2};\quad \frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2} & \\ & \frac{s-\epsilon}{1-\epsilon},\frac{s-\epsilon}{\delta-\epsilon},\frac{\epsilon-s}{1+\epsilon},\frac{\epsilon-s}{\delta+\epsilon}\\ \frac{3}{2} & \end{array}\right)\right]_{y_0}^{y} \tag{4.5}$$

where:

$$\eta(s)=\sqrt{\frac{2\beta(s-\epsilon)(\delta^2-\epsilon^2)}{B(1-\epsilon^2)}}$$

What above provides time as a hypergeometric function of four variable-ratios all related to y and then to ψ: this is the required $t=t(y)$. But (2.7) requires y, so that the last outcome has to be inverted.

4.2 Period evaluation of $y(t)$

Of course the period T_1 could be carried out by means of (4.4), namely $F_D^{(4)}$, nevertheless we preferred the following path employing $F_D^{(3)}$ which is simpler and faster. We have:

$$\lambda = 1 - \epsilon = q_3$$

By means of (4.3) we get:

$$T_1 = \pi\sqrt{\frac{2\beta(\delta^2-\epsilon^2)}{B(1+\epsilon)}}F_D^{(3)}\left(\begin{array}{c} \frac{1}{2}; \quad -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \\ 1 \end{array}\middle| \frac{1-\epsilon}{\delta-\epsilon}, \frac{\epsilon-1}{1+\epsilon}, \frac{\epsilon-1}{\delta+\epsilon}\right)$$

4.3 Fourier coefficients of $y(t)$ expansion

The Fourier-approach followed hereinafter has been already successfully tested in our previous paper [(1)].

In order to invert the $t(y)$ given by (4.5), let us refer to the $y(t)$ Fourier expansion of the form:

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left[a_n \cos(\xi_n t) + b_n \sin(\xi_n t)\right], \quad \xi_n = \frac{2\pi n}{T}$$

where T is the period of the function to be expanded.

The a.m. coefficients have to be computed as:

$$a_0 = \frac{2}{T}\int_0^T y(t)\mathrm{d}t, \quad a_n = \frac{2}{T}\int_0^T y(t)\cos(\xi_n t)\,\mathrm{d}t, \quad b_n = \frac{2}{T}\int_0^T y(t)\sin(\xi_n t)\,\mathrm{d}t$$

Let us define the function:

$$\hat{f}(s) = \sqrt{\frac{\beta}{2B}}f(s)$$

If $\epsilon > 1/2$ then:

$$\begin{aligned} a_0 &= \frac{4}{T_1}\int_\epsilon^1 s\hat{f}(s)ds = \\ &= \frac{4\pi\epsilon}{T_1}\sqrt{\frac{\beta(\delta^2-\epsilon^2)}{2B(1+\epsilon)}}F_D^{(4)}\left(\begin{array}{c} \frac{1}{2}; \quad -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -1 \\ 1 \end{array}\middle| \frac{1-\epsilon}{\delta-\epsilon}, \frac{\epsilon-1}{1+\epsilon}, \frac{\epsilon-1}{\delta+\epsilon}, \frac{\epsilon-1}{\epsilon}\right) \end{aligned}$$

To compute a_0 we do the change of variable:

$$s = u + \sigma, \quad \sigma = \frac{1+\epsilon}{2}$$

Proceeding as previously shown we have:

$$a_0 = \frac{8\sigma}{T}\sqrt{\frac{\beta(\delta^2-\sigma^2)}{2B(\sigma+1)}}\sum_{i=1}^{2}F_D^{(5)}\left(\begin{array}{c} 1; \quad -1, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{array}\middle| \hat{\sigma}_{1,i}, \hat{\sigma}_{2,i}, \hat{\sigma}_{3,i}, \hat{\sigma}_{4,i}, -1\right)$$

with:

$$\hat{\sigma}_{1,i} = (-1)^i \frac{1-\sigma}{\sigma}, \quad \hat{\sigma}_{2,i} = (-1)^{i+1} \frac{1-\sigma}{\delta-\sigma}, \quad \hat{\sigma}_{3,i} = (-1)^i \frac{1-\sigma}{\delta+\sigma}, \quad \hat{\sigma}_{4,i} = (-1)^i \frac{1-\sigma}{1+\sigma}$$

Now we define the function:

$$w(y) = \eta(y) F_D^{(4)} \left(\begin{array}{c} \frac{1}{2}; \quad \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \\ \\ \frac{3}{2} \end{array} \middle| \frac{y-\epsilon}{1-\epsilon}, \frac{y-\epsilon}{\delta-\epsilon}, \frac{\epsilon-y}{1+\epsilon}, \frac{\epsilon-y}{\delta+\epsilon} \right) \tag{4.6}$$

Let it be:

$$\hat{t} = w(y_0), \quad y_0 = \cos(\psi_0) \quad \text{and} \quad \dot{y}_0 = -\sin(\psi_0)\dot{\psi}_0 < 0$$

Considering a period, the piece-wise defined function which describes the $t = t(y)$ is:

$$t(y) = \begin{cases} t_1(y) = \hat{t} - w(y) & y \in [y_0, \epsilon] \\ t_2(y) = \hat{t} + w(y) & y \in [\epsilon, 1] \\ t_3(y) = \hat{t} + T - w(y) & y \in [1, y_0] \end{cases}$$

Marking as t' the prime derivative of t with respect to y, we get:

$$t'(y) = \begin{cases} t_1'(y) = -\hat{f}(y) & y \in [y_0, \epsilon] \\ t_2'(y) = \hat{f}(y) & y \in [\epsilon, 1] \\ t_3'(y) = -\hat{f}(y) & y \in [1, y_0] \end{cases}$$

Defining $\hat{a} = Ta_n/2$, we have:

$$\hat{a}_n = \int_0^T y(t)\cos(\xi_n t)dt = -\int_{y_0}^{\epsilon} s\hat{f}(s)\cos(\xi_n t_1(s))\mathrm{d}s + \\ + \int_{\epsilon}^{1} s\hat{f}(s)\cos(\xi_n t_2(s))ds - \int_1^{y_0} s\hat{f}(s)\cos(\xi_n t_3(s))\mathrm{d}s$$

where has been done the change $t = y^{-1}(s)$ and then $\mathrm{d}t = t'(s)\mathrm{d}s$.
Evaluating apart the three terms of the above formula, we get:

$$-\int_{y_0}^{\epsilon} s\hat{f}(s)\cos(\xi_n t_1(s))ds = \left[\frac{s}{\xi_n}\sin(\xi_n t_1(s))\right]_{y_0}^{\epsilon} - \frac{1}{\xi_n}\int_{y_0}^{\epsilon}\sin(\xi_n t_1(s))\mathrm{d}s$$
$$+\int_{\epsilon}^{1} s\hat{f}(s)\cos(\xi_n t_2(s))\mathrm{d}s = \left[\frac{s}{\xi_n}\sin(\xi_n t_2(s))\right]_{\epsilon}^{1} - \frac{1}{\xi_n}\int_{\epsilon}^{1}\sin(\xi_n t_2(s))\mathrm{d}s$$
$$-\int_1^{y_0} s\hat{f}(s)\cos(\xi_n t_3(s))\mathrm{d}s = \left[\frac{s}{\xi_n}\sin(\xi_n t_3(s))\right]_1^{y_0} - \frac{1}{\xi_n}\int_1^{y_0}\sin(\xi_n t_3(s))\mathrm{d}s$$

It is easy to see that adding the three equations, the sum of the square brackets is zero. Furthermore:

$$\sin(\xi_n t_1(s)) = \sin(\xi_n\hat{t})\cos(\xi_n w(s)) - \cos(\xi_n\hat{t})\sin(\xi_n w(s))$$
$$\sin(\xi_n t_2(s)) = \sin(\xi_n\hat{t})\cos(\xi_n w(s)) + \cos(\xi_n\hat{t})\sin(\xi_n w(s))$$
$$\sin(\xi_n t_3(s)) = \sin(\xi_n(\hat{t}+T))\cos(\xi_n w(s)) - \cos(\xi_n(\hat{t}+T))\sin(\xi_n w(s))$$

But:

$$\sin(\xi_n(\hat{t}+T)) = \sin(\xi_n\hat{t})\cos(\xi_n T) + \cos(\xi_n\hat{t})\sin(\xi_n T) = \sin(\xi_n\hat{t})$$
$$\cos(\xi_n(\hat{t}+T)) = \cos(\xi_n\hat{t})\cos(\xi_n T) - \sin(\xi_n\hat{t})\sin(\xi_n T) = \cos(\xi_n\hat{t})$$

so that:

$$\sin(\xi_n t_3(s)) = \sin(\xi_n\hat{t})\cos(\xi_n w(s)) - \cos(\xi_n\hat{t})\sin(\xi_n w(s))$$

By substitution we then obtain:

$$\hat{a}_n = -2\frac{\cos(\xi_n\hat{t})}{\xi_n}\int_\epsilon^1 \sin(\xi_n w(s))\mathrm{d}s$$

Minding that $w(s)$ is the hypergeometric Lauricella function (4.6), the previous formula provides a *numerical* recipe in order to compute the a_n coefficients as required. After defining $\hat{b}_n = Tb_n/2$, an analogous procedure provides:

$$\hat{b}_n = -2\frac{\sin(\xi_n\hat{t})}{\xi_n}\int_\epsilon^1 \sin(\xi_n w(s))\mathrm{d}s$$

so that a link between the coefficients is found as:

$$\hat{b}_n = \tan(\xi_n\hat{t})\hat{a}_n$$

The expansion of $y(t)$ in Fourier series therefore results finally to be:

$$y(t) = \frac{a_0}{2} + \frac{2}{T}\sum_{n=1}^{+\infty}\frac{\hat{a}_n}{\cos(\xi_n\hat{t})}\cos(\xi_n(t-\hat{t}))$$

In such a way $\cos\psi = y$ has been found as a function of time.

4.4 Detection of $\theta(t)$

By (2.7) we can evaluate the integral $I_2(y)$ We get:

$$I_2(y) = \left[\tilde{\eta}(s)F_D^{(5)}\left(\begin{array}{c} \frac{1}{2};\quad -\frac{1}{2},-\frac{1}{2},1,\frac{1}{2},\frac{1}{2} \\ \frac{3}{2}\end{array}\middle| \frac{\epsilon-s}{\delta+\epsilon},\frac{s-\epsilon}{\delta-\epsilon},\frac{\epsilon-s}{\zeta+\epsilon},\frac{\epsilon-s}{1+\epsilon},\frac{s-\epsilon}{1-\epsilon}\right)\right]_{y_0}^{y}$$

where:

$$\tilde{\eta}(s) = \frac{A}{\zeta+\epsilon}\sqrt{\frac{(s-\epsilon)(\delta^2-\epsilon^2)}{2\beta B(1-\epsilon^2)}}$$

In such a way θ has been found as a function of time.

4.5 Sample Problem

For the sample problem we assume:

$$\nu = \frac{1}{2},\quad \beta = \frac{1}{2},\quad \psi_0 = 70,\quad \theta_0 = 22.5,\quad \dot{\psi}_0 = 2\ \frac{\text{rad}}{\text{s}},\quad \dot{\theta}_0 = 1\ \frac{\text{rad}}{\text{s}}$$

from which we have $\epsilon = 0.209051$.

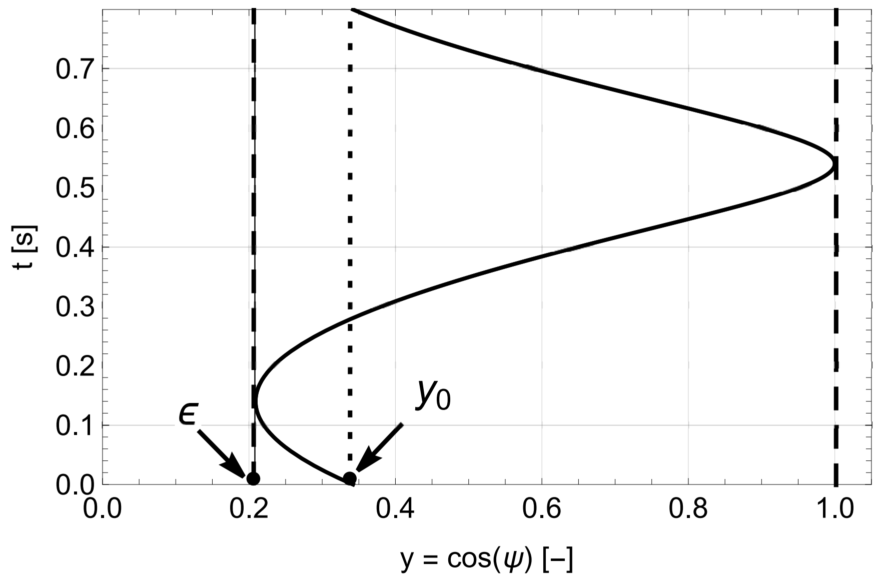

Figure 2: Time given as Lauricella function of y, eq.(4.5)

Starting from its initial value of y at $t_0 = 0$ the time line is run in the sense decided by the sign of dt arriving to its minimum value ϵ and afterwards grows to unity as already stated as characteristic of the first possible motion.

With reference to Figure 1, the polar trajectory of the bob m_2 is:

$$\begin{cases} \rho = \sqrt{l_1^2 + l_2^2 + 2 l_1 l_2 \cos(\psi)} \\ \mu = \theta + \alpha \end{cases}$$

where α comes from:

$$\begin{cases} \sin(\alpha) = l_2 \sin(\psi)/\rho \\ \cos(\alpha) = \dfrac{\rho^2 + l_1^2 - l_2^2}{2 l_1 \rho} \end{cases}$$

We assumed:

$$l_1 = 1 \quad \text{m}, \quad l_2 = 2 \quad \text{m}$$

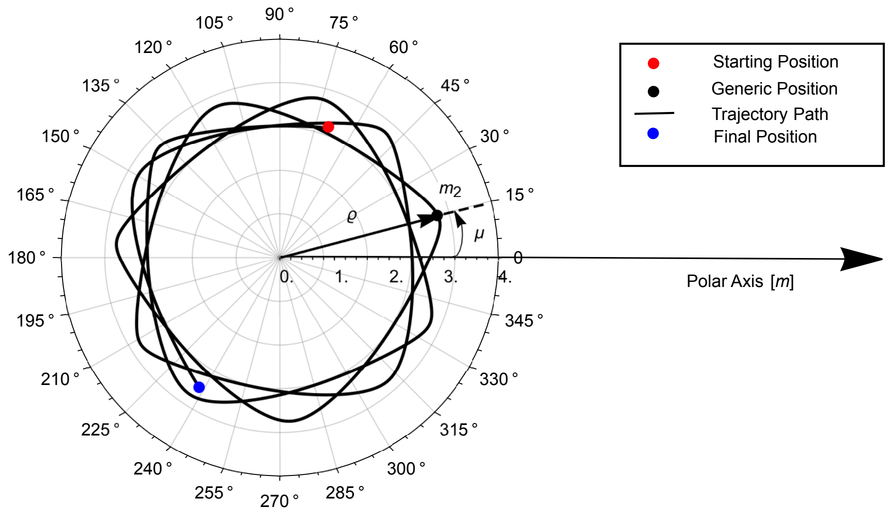

Figure 3: First motion: GFDP (ρ, μ) orbit

Of course the above formulae for ρ, μ, α allow the construction of polar orbit for each case. In such a way the m_2 -trajectory has been analytically caught.

5 Second motion: $-1 < \epsilon < 0$

For clarity's sake, in this and next case we will not use the notation ϵ, but the symbol $-|\epsilon|$. Furthermore all the treatments will be shortened, restricting them to the main analytical things.

If $\epsilon \in [-1, 0]$ we have $y \in]-|\epsilon|, 1[$ and the change of variable is:

$$s = u + \frac{1 - |\epsilon|}{2}$$

and it centres the new reference at the half of the interval $]-|\epsilon|, 1[$.

If now we define:

$$\sigma = \frac{1-|\epsilon|}{2}, \quad \hat{\sigma}(y) = \frac{y-\sigma}{1-\sigma}\sqrt{\frac{\beta(\delta^2-\sigma^2)}{2B(1+\sigma)}}, \quad \bar{\sigma} = 4\hat{\sigma}(1), \quad \tilde{\sigma}(y) = \frac{A\hat{\sigma}(y)}{2\beta(\zeta+\sigma)}$$

then we get:

$$t - t_0 = \pm \left[\hat{\sigma}(s) F_D^{(5)} \left(\begin{array}{cc|c} 1; & -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} & \frac{\sigma-s}{\sigma-\delta}, \frac{\sigma-s}{\sigma+\delta}, \frac{s-\sigma}{1-\sigma}, \frac{\sigma-s}{1-\sigma}, \frac{\sigma-s}{1+\sigma} \\ & 2 & \end{array} \right) \right]_{y_0}^{y}$$

As before we can found on IRT computing the period though a $F_D^{(4)}$ as:

$$T_2 = \bar{\sigma} \sum_{i=1}^{2} F_D^{(4)} \left(\begin{array}{cc|c} 1; & -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} & (-1)^i \frac{\sigma-1}{\sigma-\delta}, (-1)^i \frac{\sigma-1}{\sigma+\delta}, (-1)^i \frac{\sigma-1}{1+\sigma}, -1 \\ & \frac{3}{2} & \end{array} \right)$$

For $I_2(y)$ we have:

$$I_2(y) = \left[\tilde{\sigma}(s) F_D^{(6)} \left(\begin{array}{c} 1; \quad -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \\ 2 \end{array} \middle| \frac{\sigma - s}{\sigma - \delta}, \frac{\sigma - s}{\sigma + \delta}, \frac{s - \sigma}{1 - \sigma}, \frac{\sigma - s}{1 - \sigma}, \frac{\sigma - s}{1 + \sigma}, \frac{\sigma - s}{\zeta + s} \right) \right]_{y_0}^{y}$$

5.1 Sample Problem

The relevant choice of parameters is:

$$\nu = \frac{1}{2}, \quad \beta = \frac{1}{2}, \quad \psi_0 = 88, \quad \theta_0 = 22.5, \quad \dot{\psi}_0 = 2 \ \frac{\text{rad}}{s}, \quad \dot{\theta}_0 = 1 \ \frac{\text{rad}}{\text{s}}$$

from which $\epsilon = -0.123205$.

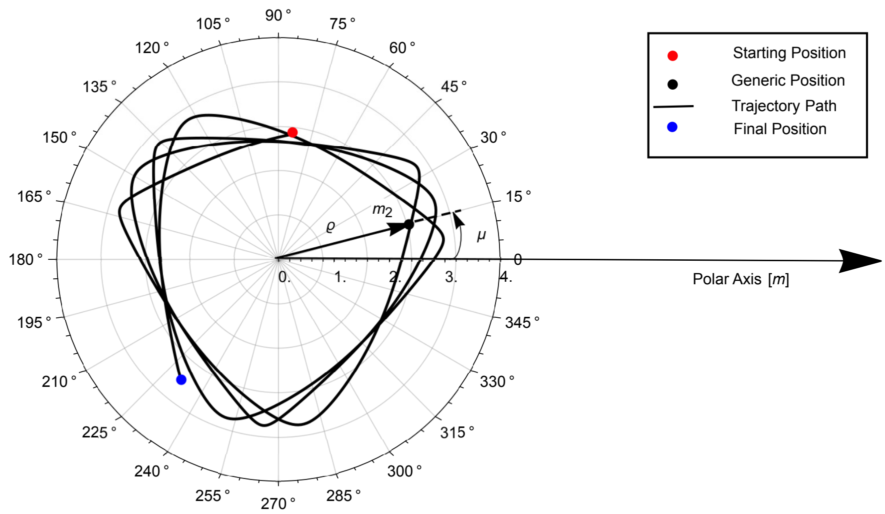

Figure 4: Second motion: GFDP (ρ, μ) orbit

6 Third motion: case $\epsilon < -1$

As it concerns the case $\epsilon < -1$ no change is necessary. In analogy to before we get:

$$t - t_0 = \pm\delta \sqrt{\frac{\beta}{2B|\epsilon|}} \left[s F_D^{(5)} \left(\begin{array}{c} 1; \quad -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 2 \end{array} \middle| \frac{s}{\delta}, -\frac{s}{\delta}, -\frac{s}{|\epsilon|}, -s, s \right) \right]_{y_0}^{y}$$

with the period given by:

$$T_3 = 2\delta \sqrt{\frac{2\beta}{B|\epsilon|}} \sum_{i=1}^{2} F_D^{(4)} \left(\begin{array}{c} 1; \quad -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{array} \middle| \frac{1}{\delta}, -\frac{1}{\delta}, \frac{(-1)^i}{|\epsilon|}, -1 \right)$$

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and:

$$I_2(y) = \frac{A\delta}{2\zeta}\sqrt{\frac{1}{2\beta B|\epsilon|}}\left[sF_D^{(6)}\left(\begin{array}{c} 1; \quad -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1 \\ 2 \end{array}\middle| \frac{s}{\delta}, -\frac{s}{\delta}, -\frac{s}{|\epsilon|}, -s, s, -\frac{s}{\zeta}\right)\right]_{y_0}^{y}$$

6.1 Fourier coefficients

First of all we have:

$$a_0 = \frac{16\delta}{3T}\sqrt{\frac{\beta}{2B|\epsilon|}}\sum_{i=1}^{2}(-1)^{i+1}F_D^{(4)}\left(\begin{array}{c} 2; \quad -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{5}{2} \end{array}\middle| \frac{1}{\delta}, -\frac{1}{\delta}, \frac{(-1)^i}{|\epsilon|}, -1\right)$$

We define now:

$$w(s) = \delta\sqrt{\frac{\beta}{2B|\epsilon|}}sF_D^{(5)}\left(\begin{array}{c} 1; \quad -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ 2 \end{array}\middle| \frac{s}{\delta}, -\frac{s}{\delta}, -\frac{s}{|\epsilon|}, -s, s\right)$$

Let it be:

$$\hat{t} = w(y_0), \quad \tilde{t} = w(-1)$$

Considering a period and assuming $\dot{y}_0 < 0$, the piecewise defined function which describes the $t = t^{-1}(y)$ ad its derivative are:

$$t(y) = \begin{cases} \hat{t} - w(y) & y \in [y_0, -1] \\ \hat{t} - 2\tilde{t} + w(y) & y \in [-1, 1] \\ \hat{t} + T - w(y) & y \in [1, y_0] \end{cases}, \quad t'(y) = \begin{cases} -\hat{f}(y) & y \in [y_0, -1] \\ \hat{f}(y) & y \in [-1, 1] \\ -\hat{f}(y) & y \in [1, y_0] \end{cases}$$

Then following the same approach we have seen before we obtain:

$$\hat{a}_n = \frac{2}{\xi_n}\cos(\xi_n(\tilde{t} - \hat{t}))\int_{-1}^{1}\sin(\xi_n(\tilde{t} - w(s)))ds$$

And:

$$\hat{b}_n = -\tan(\xi_n(\tilde{t} - \hat{t}))\hat{a}_n$$

Then:

$$y(t) = \frac{a_0}{2} + \frac{2}{T}\sum_{n=1}^{+\infty}\frac{\hat{a}_n}{\cos(\xi_n(\tilde{t} - \hat{t}))}\cos(\xi_n(t + \tilde{t} - \hat{t}))$$

6.2 Sample Problem

$$\nu = \frac{1}{2}, \quad \beta = \frac{1}{2}, \quad \psi_0 = 160, \quad \theta_0 = 22.5, \quad \dot{\psi}_0 = 2 \ \frac{\text{rad}}{s}, \quad \dot{\theta}_0 = 1 \ \frac{\text{rad}}{\text{s}}$$

from which $\epsilon = -1.0338$.

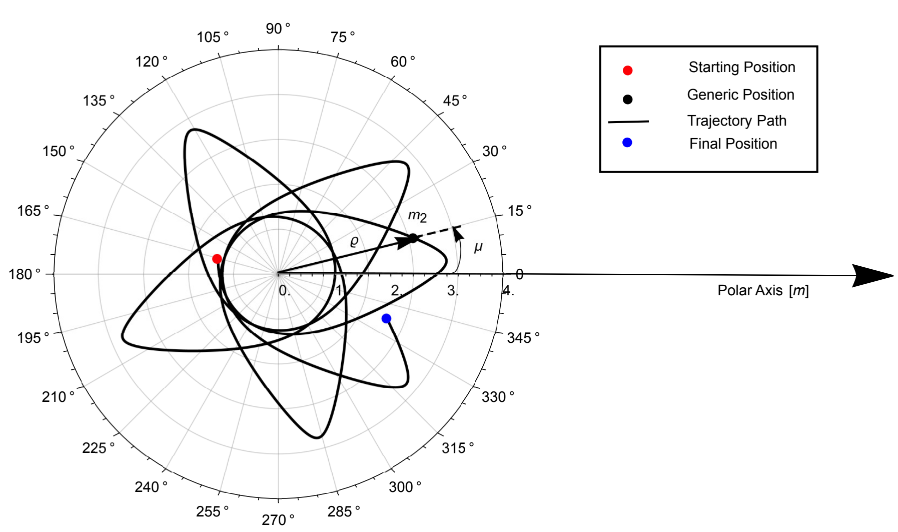

Figure 5: Third motion: GFDP (ρ, μ) orbit

7 Conclusions

The gravity free double pendulum has two degrees of freedom, say the functions $\theta(t)$ and $\psi(t)$ whose detection led us to hyper-elliptic integrals with the critical role of a non dimensional parameter ϵ which holds all the system data.

In such a way the GFDP will behave differently according to the ϵ spans.

Three possible ranges have been found and for only one of them the angle ψ will be free of describing all values of its turn. The a.m. integrals, thanks to the hypergeometric functions IRT, have been expressed in terms of different Lauricella functions. Such a representation is notoriously true if and only if a special constraint on the state variables is met; then, by means of a suitable change of variable, we make sure that such a condition is fully satisfied. Once the function $t = t(\psi)$ has been inverted via the Fourier series, we arrive at $\cos\psi(t)$ as a Fourier series of time. The function θ is obtained solving a further hyper-elliptic integral by means of a higher degree Lauricella function holding ψ too; in such a way θ comes as function of ψ and -thanks to the a.m. Fourier expansion- of time. These Fourier coefficients have been computed numerically, and mean the only one aspect of our work not in analytic closed form. All kinematic elements of the system are then computable, providing a complete knowledge of the time laws for m_1 and m_2 and then of their trajectories, Figure 3, Figure 4, Figure 5.

Let us analyze the Γ polar plots relevant to the proposed sample problems, obtained by means of the closed form formulae given above and tested by the ODEs numerical treatment.

The common value selected for each of the motions is $\theta_0 = \pi/8$. The starting point is red-marked while the last computed is blue. The time duration is 10 seconds for each simulation.

Unlikely to ψ, there is no restriction on θ so that the trajectory of the point mass m_1 has an angular period of 2π whilst this is not valid for ψ which varies between $\arccos(\epsilon)$ and 2π in the first motion and between $\arccos(-|\epsilon|)$ and 2π in the second. Generally speaking the above periods of m_1 (angle θ) and m_2 (angle ψ) will not be commensurable each other: so that the global motion will not be periodic almost for a reasonable amount of time of simulation.

What we found in this paper is then resembling what happens in celestial mechanics. When to the basic law of attraction $1/r$ is added the bulge effect of the attractor, the original elliptic trajectory of the attracted body is forced to rotate around the attractor, so that its resulting orbit consists of nothing but a sequence of not overlapping ellipses shifted each other, which is called ”rosette”. But in our cases of motion we do not have regular micro-orbits such those described above, being the single, not overlapped loops quite irregular, so that our rosettes appear to be a bit deformed. Anyway, in the third case its aspect becomes more regular when ψ also is free of taking each possible value like θ.

8 Appendix: Short about Lauricella functions

We recall hereinafter an outline of the Lauricella hypergeometric functions.

The first hypergeometric historical series appeared in the Wallis's *Arithmetica infinitorum* (1656), the general expression of ${}_2F_1$ is:

$$ {}_2\mathrm{F}_1\left(\begin{array}{c|} a,b \\ c \end{array}\, x\right) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\frac{x^n}{n!}, \quad |z|<1 $$

A meaningful contribution on various ${}_2F_1$ topics is ascribed to Euler in three papers [(3)], [(4)], [(5)]; but he does not seem to have known the integral representation involving the Γ function too:

$$ {}_2\mathrm{F}_1\left(\begin{array}{c|} a,b \\ c \end{array}\, x\right) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \frac{u^{a-1}(1-u)^{c-a-1}}{(1-xu)^b}\,\mathrm{d}u $$

really due to Legendre [(9)]. The above integral relationship is true if $c>a>0$ and for $|x|<1$, even if this limitation can be discarded thanks to the analytic continuation.

Many functions have been introduced in 19$^{\text{th}}$ century for generalizing the hypergeometric functions to multiple variables such as the Appell F_1 two-variable hypergeometric series.

The functions introduced and investigated by G. Lauricella (1893) [(8)] and S. Saran (1954) [(11)], are those of our prevailing interest; and among them the hypergeometric function $F_D^{(n)}$ of $n\in\mathbb{N}^+$ variables (and $n+2$ parameters) defined as:

$$ \mathrm{F}_D^{(n)}\left(\begin{array}{c|} a,b \\ c \end{array}\, x\right) := \sum_{m_1,\ldots,m_n\in\mathbb{N}} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\cdots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}m_1!\cdots m_n!}\,x_1^{m_1}\cdots x_n^{m_m} $$

with $b=b_1,...,b_n$, $x=x_1,...,x_n$ and with the hypergeometric series usual convergence requirements $|x_1|<1,\ldots,|x_n|<1$. If $\mathrm{Re}\,(c)>\mathrm{Re}\,(a)>0$, the relevant Integral Representation Theorem provides:

$$ \mathrm{F}_D^{(n)}\left(\begin{array}{c|} a,b \\ c \end{array}\, x\right) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_0^1 \frac{u^{a-1}(1-u)^{c-a-1}}{(1-x_1u)^{b_1}\cdots(1-x_nu)^{b_n}}\,\mathrm{d}u \tag{8.1} $$

allowing the analytic continuation to $\mathbb{C}^n$ deprived of the cartesian n-dimensional product of the interval $]1,\infty[$ with itself.

Such functions are not much pervasive in Mathematical Physics [(10)], [(12)] and have their better employ in providing a valuable help in hyper-elliptic integrations, being the IRT used as a tool to transfer the integration task to a hypergeometric summation.

The Lauricella functions -*in lack of specific SW packages*-have been implemented in this work thanks to some reduction theorems which will form the object of a next paper. Here it will be enough to anticipate the following. All the Lauricella series are multi-valued power series whose coefficients are other series nested each other: so that e.g. $F_D^{(3)}$ consists of a triple infinite power series. But-by a proper work in re-scaling indices and other manipulations- it can be reduced to a single series whose coefficients are depending on a convenient ${}_2F_1$ function *for which SW packages are by long time available*. Furthermore $F_D^{(5)}$ has been led to a simple power series whose coefficients depend on 3 computable blocks each of them holds again the a.m. Gauss hypergeometric ${}_2F_1$. All above really helps such functions management increasing the series convergence speed.

References

[1] Bocci A. and Mingari Scarpello G. (2021). Analytic Inversion of Closed form Solutions of the Satellite's J_2 Problem. Asian Research Journal of Mathematics, 17(5), 50–68.

[2] Enolskii V. et al. (2003). Double Pendulum and θ-Divisor. J. Nonlinear Science, 13.

[3] Euler L. (1738). De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt. Commentarii academiae scientiarum Petropolitanae, 36–57.

[4] Euler L. (1769). De curva hypergeometrica hac aequatione expressa $y = 1.2.3.....x$. Novi Commentarii academiae scientiarum Petropolitanae, 3–66.

[5] Euler L. (1792). Institutionum calculi integralis. Impensis Academiae imperialis scientiarum Petropoli.

[6] Fay J. (2006). Theta functions on Riemann surfaces. Springer, 352.

[7] Gimeno E. and Beléndez A. (2009). Rational-harmonic balancing approach to nonlinear phenomena governed by pendulum-like differential equations. Zeitschrift für Naturforschung A, 64, 819–826.

[8] Lauricella G. (1893). Sulle funzioni ipergeometriche a piú variabili. Rendiconti del Circolo Matematico di Palermo, 7, 111–158.

[9] Legendre A. (1811). Exercices de calcul intégral sur divers ordres de transcendantes et sur les quadratures. Courcier.

[10] Mingari Scarpello G. and Ritelli D. (2011). Exact solutions of nonlinear equation of rod deflections involving the Lauricella hypergeometric functions. International Journal of Mathematics and Mathematical Sciences.

[11] Saran S. (1955), Transformations of certain hypergeometric functions of three variables. Acta Mathematica, 93, 293–312.

[12] Slater J. (1966). Generalized hypergeometric functions. Cambridge University Press.

[13] Talamucci F. (2018). Synchronization of a Double Pendulum with Moving Pivots: A Study of the Spectrum. J. Advances in Applied Mathematics, 3 .