Short Research Article

EVEN METRIC SPACE

Abstract. In this paper I have defined even metric spaces and established some conditions for a metric to be even in relation to translation invariant property of metric spaces and discuss completeness of even metric space.

1. Introduction

Definition 1.1. Metric space is an ordered pair (M,d) where M is a non empty set and d is metric on M

$d : MXM \to \mathbb{R}$ such that for any $x, y, z \in M$ the following holds

$$1) d \geq 0$$

$$2) d(x, y) = d(y, x)$$

$$3) d(x, z) \leq d(x, y) + d(y, z)$$

$$4) d(x, y) = 0 \iff x = y$$

Definition 1.2. A metric space (M,d) is said to be translation invariant if

$$d(x + a, y + a) = d(x, y)$$

$$for\ all\ x, y, a \in M$$

2. MAIN RESULT

Definition 2.1. A metric space (M,d) is said to be even if for all $x, y \in M$ the following holds

$$d(x, y) = d(x, -y)$$

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2.1. **Example.** $(\mathbb{R}, d)$ *be a metric space* such that

$$d(x,y) = |x| + |y| \textit{ when } x \neq y \textit{ and } 0 \textit{ when } x = y$$

clearly $d(x,y)$ *is metric on* $\mathbb{R}$

because for all x,y,z $\in \mathbb{R}$

$$d \geq 0$$

$$d(x,y) = d(y,x)$$

when $x \neq y \neq z$

$$|x| + |z| \leq |x| + |y| + |y| + |z|$$

implies

$$d(x,z) \leq d(x,y) + d(y,z)$$

when $x = z \neq y$ then

$$0 \leq |x| + |y| + |y| + |z|$$

$$d(x,z) \leq d(x,y) + d(y,z)$$

when $x \neq y = z$ then

$$|x| + |z| \leq |x| + |y|$$

implies

$$d(x,z) \leq d(x,y) + d(y,z)$$

$$d(x,x) = 0$$

by definition

$$d(x,y) = 0$$

$$|x| + |y| = 0$$

implies

$$x = y = 0$$

consider

$$d(x,-y) = |x| + |-y| = |x| + |y| = d(x,y) \ for \ all \ x, y \in \mathbb{R}$$

therefore $(M, \mathbb{R})$ is an even metric space

Proposition 2.2. *If a metric space* (M, d) *is even then it is not translation Invariant. Alternatively If a metric space is translation invariant then it cannot be even.*

Proof. Let (M, d) be an even metric space then for all x,y $\in M$

$$d(x,y) = d(x,-y)$$

Assume d is translation invariant then

$$d(x_1 + c, y_2 + c) = d(x_1, y_2)$$

for some $x_1, y_1 \in M$ and for all $c \in M$ also

$$d(x_1 + c, -y_2 + c) = d(x_1, -y_2)$$

since d is even

$$d(x_1, -y_2) = d(x_1, y_2)$$

implies

$$d(x_1 + c, -y_2 + c) = d(x_1 + c, +y_2 + c) \ for\ all\ c \in M$$

put $x_1 = y_2 = c$ implies

$$d(2c, 0) = d(2c, 2c)$$

$$d(2c, 0) = 0$$

using properties of metric space

$$2c = 0$$

implies

$$c = 0$$

a contradiction since c was arbitrary therefore (M, d) is not translation invariant

□

2.2. **Remark.** If a metric is not translation invariant is does not mean the metric space is even,translation invariant property can fail for other reasons also.

2.3. **Example.** Let $(\mathbb{R}, d) = |x^3 - y^3|$ clearly d is metric on $\mathbb{R}$ consider $d(1, 0) = 1\ and\ d(1 + 1, 0 + 1) = d(2, 1) = 7$
$1 \neq 7$ hence d is not translation invariant also consider $d(1, 1) = 0$ and $d(1, -1) = 2$
$d(1, 1) \neq d(1, -1)$ d is not an even metric space

2.4. **Example.** Let $(\mathbb{R}, d)$ be a discrete metric space
i.e

$$d = 1\ if\ x \neq y\ and\ 0\ if\ x = y$$

Let $x \neq y$ implies $x + c \neq y + c$ for all $c \in M$ implies $d(x + c, y + c) = 1 = d(x, y)$ for all $c \in M$ and when $x = y$ implies $x + c = y + c$ for all $c \in M$ which implies $d(x + c, x + c) = 0 = d(x, y) =$ for all $c \in M$ therefore d is translation invariant but $d(1, -1) = 1$ and $d(1, 1) = 0$ $d(1, -1) \neq d(1, 1)$ hence it is not even metric space

2.5. **Example.** $(\mathbb{R}, d)$ *be a metric space* such that

$$d(x, y) = |x| + |y| \textit{ when } x \neq y \textit{ and } 0 \textit{ when } x = y$$

we have already seen that it is an even metric space.
consider$d(1 + 2, 2 + 2) = d(3, 4) = |3| + |4| = 7$
and $d(1, 2) = |1| + |2| = 3$
$d(1 + 2, 2 + 2) \neq d(1, 2)$ therefore d is not translation invariant

2.6. **Example.** $(\mathbb{R}, d)$ *be a metric space* such that

$$d(x, y) = |x| + |y| + |x||y| \textit{ when } x \neq y \textit{ and } 0 \textit{ when } x = y$$

clearly it a metric space on $\mathbb{R}$ and $d(x, -y) = d(x, y)$ for x,y $\in M$ therefore it is an even space hence it is not translation invariant.

Proposition 2.3. *If an even metric space is complete then every cauchy sequence in that space must converge to zero.*

Proof. Let (M, d) be an even space and let it be complete,that means every cauchy sequence in that space converge to limit that is inside the space. let x_n be a cauchy sequence in M then

$$\lim_{x_n \to \infty} d(x_n, x) = 0$$

since M is complete and even, $x \in M$ then

$$\lim_{x_n \to \infty} d(x_n, -x) = 0$$

implies $x_n \to x$ as $n \to \infty$ and $x_n \to -x$ as $n \to \infty$ therefore $x = 0$

2.7. **Example.** $(\mathbb{R}, d)$ *be a metric space* such that

$$d(x, y) = |x| + |y| \textit{ when } x \neq y \textit{ and } 0 \textit{ when } x = y$$

Let it be complete and x_n be a cauchy sequence in $\mathbb{R}$
i.e

$$\lim_{n,m \to \infty} d(x_n, x_m) = 0$$

implies

$$\lim_{n,m \to \infty} (|x_n| + |x_m) = 0|)$$

implies

$$x_n \to 0, x_m \to 0$$

as n,m $\to \infty$

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□

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