Original Research Article

The Influence of Measurement Errors on Generalized Estimator of Population Mean

Abstract

This paper proposed a generalized estimator of population mean in the presence of correlated and uncorrelated measurement errors under simple random strategy. Some known estimators belong to this class of proposed estimator. Under the large sample approximation, the properties of the proposed estimator were obtained. Theoretical comparison was carried out on the members of the proposed class of estimators when measurement errors are correlated and when they are uncorrelated.

Keywords: Measurement errors, generalized estimator, bias, mean squared error, correlation coefficient.

1. Introduction

Notwithstanding the mathematical development in sampling survey, the general assumption in sampling survey is that data used in estimating population parameters are free of observational errors or measurement errors at data collection stage. In practice, this is not always the case as observed values in most case differ from the true value hence the data available for statistical analysis are subject to error.This may be due to the fact that all phases of sampling survey are possible sources of error. This can also be attributed to the bias on the part of the respondents or enumerators or both and due to natural variation in the subject, variation in the measurement process, or both (see Cochran 1977, Biemer et al 1991, and Tabasum 2012).

The difference between the individual observed values and their corresponding true values is termed measurement error. Measurement errors form a significant element of errors in anysurvey data and their presence may not be noticeable unless the responses are compared with some known standard values or the measurement processes are replicated a good number of times.

When measurement errors are present in both the study and auxiliary variables or present in either study variable or auxiliary variable, the influence drawn from the sample about the population parameter may be bias and inconsistent subjectto the level of the measurement errors. Thus to a large extent, the efficiency of an estimator is a function of the magnitude of the measurement errors.

In sampling, measurement errors have been studied by Sukhatme and Sukhatme (1970) and Cochran (1977), focusing on mathematical models to study the influence of measurement errors on estimators of population parameters. Shalabh (1997) developed a methodology for studying the influence of measurement errors on estimators like the ratio estimator in sampling survey. Following Shalabh (1997) methodology, Manish and Singh (2001) proposed class ofratio estimator when measurement errors are present in the variables. This estimator is a linear combination of ratio estimator and sample mean per unit estimator. Erum and Javid (2019) studied the effect of measurement errors on estimation of finite population mean for a sensitive variable using dual auxiliary information. The results which theyobtained from theoretical and empirical analysis shows that their proposed estimator performsbetter than some existing estimators under study.

Qi et al. (2021) studied the estimation ofpopulation mean in the presence of measurement errors and non-response error under stratifiedrandom sample and presented a comparison of the proposed estimator with some existing estimators which they found to be uniformly better than some existing estimators. Gajendra et al (2020) proposed ratio and regression type calibration estimators for the population mean under both correlated and uncorrelated measurement errors. The variances of the proposed calibrated estimators to the first order approximation were obtained and their efficiencies comparison withthe usual unbiased estimator carried out. The result shows their proposed calibrated estimatorto be uniformly better than the usual unbiased estimator. Using Monte Carlo simulation theystudied the effect of measurement errors on the proposed calibrated estimators. They calculated the percentage contribution of measurement errors (PCME) and found that PCME of theproposed calibrated estimators increases with the increase in variability of the measurementerrors present in both study and auxiliary variables. Decrease in PCME was recorded by themwhen measurement errors are positively correlated, the reverse is the case when they are negatively correlated.

The influence of measurement errors and randomized response technique on mean estimation under stratified double sampling was studied by Ronald et al. (2021). The numerical analysiscarried out on the efficiency of the proposed estimator using simulated and real dataset revealed that the use of the Randomized Response Technique (RRT) in a survey contaminatedwith measurement errors increases the variances and mean squared errors of estimators of the finite population mean. Study has shown that the properties of ratio estimator is distortedby the presence of measurement error on the auxiliary variate. It is on this premise thatthe statistical properties of three common ratio estimators was studied by Gregoire and Salas (2009) whenmeasurement error is present in the auxiliary variable. Under the effect of systematic measurement error, the bias is irregular around zero and precision may be enhanced or vitiated subject to the extent of the error. When the measurement error is stochastic in nature, the bias of classical ratio-of-means estimator is muchaffected so also is the mean square error when compared with the other estimators which theyconsidered. In summary, they concluded that ratio-of-means estimator appears to be less affected by the measurement error in the auxiliary variants.

Neha and Gajendra (2019) proposed a generalised class of estimators for mean when both studyvariable and auxiliary variable are contaminated with measurement errors under simple randomsampling. To evaluate the performance of the proposed class of estimators, they carried out simulation and empirical analysis and their result shows that the proposed class of estimators is more efficient than the ratio and product estimators for any value of correlation coefficient. Azeem and Hanif (2015) studied mean estimation when measurement errors and non-response exist simultaneously and presented theoretical and empirical analysis of the efficiency of the proposed estimator. Their findings show that the proposed estimator is moreefficient than three other estimators they considered and also less bias than the two of the three estimators considered. Shalabh and Tsai (2017) have proposed ratio and product method of estimation in the presence of correlated measurement error.

In this paper, we propose a generalized class of estimators of the population mean of study variable under the influence of measurement errors. Some existing estimators are a member of this class of estimators. The effect of measurement error on the mean square error of the proposed class of estimators be explored. Many authors have studied the effects of measurement error on ratio, product and regression estimator. The main objective is to obtain simultaneously the properties of particular members of the proposed class of estimators under the influence of measurement errors. Also will carried theoretical comparison of the performance of the members of proposed class estimators when the measurement errors are correlated when the measurement errors are uncorrelated.

2. Notations

In obtaining the properties of the proposed estimator, we will assume large sample approximation. Let the population mean and variance of X and Y be defined as

$$\bar{X} = \frac{1}{N}\sum_{i=1}^{N} X_i\,, \bar{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i\,, \sigma_X = \frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})^2\,, \sigma_Y = \frac{1}{N}\sum_{i=1}^{N}(Y_i - \bar{Y})^2$$

Further we define the coefficient of variation of X and Y as

$C_X = \frac{\sigma_X}{\bar{X}}$ and $C_Y = \frac{\sigma_Y}{\bar{Y}}$ respectively. Also Covariance of Y and X, Correlation Coefficient between Y and X, and Correlation Coefficient between u and v are defined as

$\sigma_{XY} = \frac{1}{N}\sum_{i=1}^{N}(X_i - \bar{X})\,(Y_i - \bar{Y}), \rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$ and $\rho^* = \frac{\sigma_{uv}}{\sigma_v \sigma_u}$ respectively. Using delta notation, we define the following:

$$\delta_0 = \frac{\bar{y}}{\bar{Y}} - 1 \implies \bar{y} = \bar{Y}(1 + \delta_o) \qquad (1)$$

$$\delta_1 = \frac{\bar{x}}{\bar{X}} - 1 \implies \bar{x} = \bar{X}(1 + \delta_1) \qquad (2)$$

Such that,

$$E(\delta_0) = E(\delta_1) = 0$$

(3)

$$E(\delta_0^2) = \frac{\sigma_Y^2}{n\bar{Y}^2}\left(\frac{\sigma_Y^2 + \sigma_v^2}{\sigma_Y^2}\right) = \frac{C_Y^2}{n}\left(\frac{\sigma_Y^2 + \sigma_v^2}{\sigma_Y^2}\right)(4)$$

$$E(\delta_1^2) = \frac{\sigma_X^2}{n\bar{X}^2}\left(\frac{\sigma_X^2 + \sigma_v^2}{\sigma_X^2}\right) = \frac{C_X^2}{n}\left(\frac{\sigma_X^2 + \sigma_v^2}{\sigma_X^2}\right)(5)$$

$$E(\delta_0\delta_1) = \frac{1}{n\bar{Y}\bar{X}}(\sigma_Y\sigma_X\rho + \sigma_u\sigma_v\rho^*)(6)$$

3. Measurement Error Model Definition

For a population $\mathrm{U} = (\mathrm{U}_1, \mathrm{U}_2, \ldots, \mathrm{U}_N)$ of size N. Let y and $x$denote study and auxiliary variables taking values y_i and x_i respectively on the i^{th} unit of U_i, $(i = 1, 2, \ldots, N)$. Assume SRSWOR of size n is drawn from population U. Let $\bar{y}$ and $\bar{x}$ be the sample meansof y and x respectively. Thus, for a simple random sampling scheme, let (y_i, x_i) be observedvalues instead of the true values $(y_i^{'}, x_i^{'})$ on the two characteristics (y, x) respectively for thei^{th} unit $(i = 1, 2, \ldots, n)$ in a sample of size n. Then the measurement errors in the study variable and auxiliary variableare respectively defined as:

i. When the Measurement Errors are correlated

$u_i = y_i - y_i^{'}$

$v_i = x_i - x_i^{'}$

$E(u) = E(v) = 0$

$Var(u) = \sigma_u^2 \qquad Var(v) = \sigma_v^2$

$Cov(u, v) = \rho^*\sigma_u\sigma_v$

ii. When the Measurement Errors are uncorrelated

$u_i = y_i - y_i^{'}$

$v_i = x_i - x_i^{'}$

$E(u) = E(v) = 0$

$Var(u) = \sigma_u^2 \qquad Var(v) = \sigma_v^2$

$Cov(u, v) = 0$

Expressing observed value as a function of true value and measurement error we have,

$$y_i = y_i' + u_i \tag{7}$$

$$x_i = x_i' + v_i \tag{8}$$

The measurement errors u and $v$are also assumed to be independent.

4. Adapted Estimators

In the presence of correlated measurement errors, the traditional sample mean per unit estimator for estimating population mean is given by:

$$t_0 = \bar{y} \tag{9}$$

The variance is given as

$$V(t_0) = \frac{C_Y^2}{n}\left(\frac{\sigma_Y^2 + \sigma_v^2}{\sigma_Y^2}\right) \tag{10}$$

Shalabh and Jia-Ren (2016) proposed ratio and product estimator in the presence of correlated measurement error as

$$t_r = \bar{y}\frac{\bar{X}}{\bar{x}} \tag{11}$$

$$t_p = \bar{y}\frac{\bar{x}}{\bar{X}} \tag{12}$$

They obtained the mean square error of ratio and product estimators as

$$MSE(t_r) = \frac{\bar{Y}^2}{n}\left(C_Y^2 + C_X^2 - 2C_Y C_X \rho + C_Y^2 \frac{\sigma_u^2}{\sigma_Y^2} + C_X^2 \frac{\sigma_v^2}{\sigma_X^2} - 2\frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right) \tag{13}$$

$$MSE(t_p) = \frac{\bar{Y}^2}{n}\left(C_Y^2 + C_X^2 + 2C_Y C_X \rho + C_Y^2 \frac{\sigma_u^2}{\sigma_Y^2} + C_X^2 \frac{\sigma_v^2}{\sigma_X^2} + 2\frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right) \tag{14}$$

In the presence of uncorrelated measurement error, the mean square errors of t_r and t_p were given as

$$MSE(t_r) = \frac{\bar{Y}^2}{n}\left(C_Y^2 + C_X^2 - 2C_Y C_X \rho + C_Y^2 \frac{\sigma_u^2}{\sigma_Y^2} + C_X^2 \frac{\sigma_v^2}{\sigma_X^2}\right) \tag{13}$$

$$MSE(t_p) = \frac{\bar{Y}^2}{n}\left(C_Y^2 + C_X^2 + 2C_Y C_X \rho + C_Y^2 \frac{\sigma_u^2}{\sigma_Y^2} + C_X^2 \frac{\sigma_v^2}{\sigma_X^2}\right) \tag{14}$$

5. Proposed Estimator

Motivated by the work of Shalabh and Jia-Ren (2016), we proposed the following generalized estimator of population mean in the presence of correlated and uncorrelated measurement errors.

$$t_{pro} = \bar{y}\left[\frac{\alpha\bar{X}+\psi}{\beta(\alpha\bar{x}+\psi)+(1-\beta)(\alpha\bar{X}+\psi)}\right]^{\kappa} \quad (15)$$

Where $\alpha > 0, \psi$ can either be a function of known population parameter of auxiliary variable x or a real number, $\kappa$is any real number chosen so as to minimize the mean squared error of$t_{pro}$and $\beta = 0$ or1

6. Properties of Proposed Estimator

The properties of the proposed estimator up to first order approximation are obtained using notations defined in section 2 thus:

Expressing (15) in terms of δ_i $(i = 0, 1)$ we have

$$t_{pro} = \bar{Y}(1+\delta_0)\left[\frac{\alpha\bar{X}+\psi}{\beta(\alpha\bar{X}(1+\delta_1)+\psi)+(1-\beta)(\alpha\bar{X}+\psi)}\right]^{\kappa} \quad (16)$$

After simplification, (16) can be written as

$$t_{pro} = \bar{Y}(1+\delta_0)[1+\beta\lambda\delta_1]^{-\kappa} \quad (17)$$

Where,

$$\lambda = \frac{\alpha\bar{X}}{\alpha\bar{X}+\psi}$$

Assuming that $|\delta_1| < 1$, the expression$(1+\beta\lambda\delta_1)^{-\kappa}$ can be expanded to a convergent infinite series using binomial expansion. Hence,

$$t_{pro} = \bar{Y}(1+\delta_0)\left(1 - \kappa\beta\lambda\delta_1 + \frac{\kappa(\kappa-1)}{2!}(\beta\lambda\delta_1)^2 - O(\delta_1)\right) \tag{18}$$

Ignoring high order of δ_1 and simplifying (18) we have

$$t_{pro} = \bar{Y}\left(1 + \delta_0 - \kappa\beta\lambda\delta_1 + \frac{\kappa(\kappa-1)}{2}\beta^2\lambda^2\delta_1^2 - \beta\lambda\delta_0\delta_1\right) \tag{19}$$

$$t_{pro} - \bar{Y} = \bar{Y}\left(\delta_0 - \kappa\beta\lambda\delta_1 + \frac{\kappa(\kappa-1)}{2}\beta^2\lambda^2\delta_1^2 - \beta\lambda\delta_0\delta_1\right) \tag{20}$$

Taking expectation of (20) and made necessary substitutions using (3) – (6), we obtained the bias of theproposed estimator(t_{pro})

i. **When the measurement errors are correlated as**

$$Bias^*(t_{pro}) = \frac{\bar{Y}}{2n}\left[\kappa(\kappa-1)\beta^2\lambda^2\frac{C_X^2}{\theta_X} - 2\kappa\beta\lambda\left(C_Y C_X\rho - \frac{\sigma_u}{\bar{Y}}\frac{\sigma_v}{\bar{X}}\rho^*\right)\right] \tag{21}$$

ii. **When the measurement error is uncorrelated ($\rho^* = 0$), as**

$$Bias(t_{pro}) = \frac{\bar{Y}}{2n}\left[\kappa(\kappa-1)\beta^2\lambda^2\frac{C_X^2}{\theta_X} - 2\kappa\beta\lambda C_Y C_X\rho\right] \tag{22}$$

Squaring and taking expectation of (20) and made necessary substitutions using (3) – (6), we obtained the mean square error of the proposed estimator (t_{pro})

i. **When the measurement errors are correlated as**

$$MSE^*(t_{pro}) = \frac{\bar{Y}^2}{n}\left[\frac{C_Y^2}{\theta_Y} + \kappa^2\beta^2\lambda^2\frac{C_X^2}{\theta_X} - 2\kappa\beta\lambda\left(C_Y C_X\rho + \frac{\sigma_u}{\bar{Y}}\frac{\sigma_v}{\bar{X}}\rho^*\right)\right] \quad (23)$$

The mean square error will be minimized when

$$\kappa = \frac{\theta_X\left(C_Y C_X\rho + \frac{\sigma_u}{\bar{Y}}\frac{\sigma_v}{\bar{X}}\rho^*\right)}{\beta\lambda C_X^2} \quad (24)$$

Thus the minimum mean square error of t_{pro} is obtained as

$$MSE_{min}^*(t_{pro}) = \frac{\bar{Y}^2}{n}\left[\frac{C_Y^2}{\theta_Y} - \frac{\theta_X\left(C_Y C_X\rho + \frac{\sigma_u}{\bar{Y}}\frac{\sigma_v}{\bar{X}}\rho^*\right)^2}{C_X^2}\right] \quad (25)$$

ii. When the measurement error is uncorrelated ($\rho^* = 0$), as

$$MSE(t_{pro}) = \frac{\bar{Y}^2}{2n}\left[\frac{C_Y^2}{\theta_Y} + \kappa^2\beta^2\lambda^2\frac{C_X^2}{\theta_X} - 2\kappa\beta\lambda C_Y C_X\rho\right] \quad (26)$$

The mean square error will be minimized when

$$\kappa = \frac{\theta_X C_Y C_X\rho}{\beta\lambda C_X^2} \quad (27)$$

Thus the minimum mean square error of t_{pro} is obtained as

$$MSE_{min}(t_{pro}) = \frac{\bar{Y}^2}{n}\left[\frac{C_Y^2}{\theta_Y} - \frac{\theta_X {C_Y}^2 {C_X}^2 \rho^2}{C_X^2}\right] = \frac{\bar{Y}^2}{n}\left[\frac{C_Y^2}{\theta_Y} - \theta_X {C_Y}^2 \rho^2\right] \qquad (28)$$

Some particular members of the proposed estimator of population mean and their mean square error can be obtained by chosen suitable values of β, α, ψ and κ (see table below).

Table 1: Estimators and Their Mean Square errors at Different Value of β, α, ψ and κ

	Chosen Values					**Mean Square Error In the Presence of**	
i	$\boldsymbol{\beta}$	$\boldsymbol{\alpha}$	ψ	$\boldsymbol{\kappa}$	**Estimator**	**Correlated Measurement Error**	**Uncorrelated Measurement Error**
1	0	0	0	0	$t_{pro1} = \bar{y}$	$\frac{\bar{Y}^2 C_Y^2}{n\theta_Y}$	$\frac{\bar{Y}^2 C_Y^2}{n\theta_Y}$
2	1	1	0	1	$t_{pro2} = \bar{y}\left(\frac{\bar{X}}{\bar{x}}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{C_X^2}{\theta_X} - 2\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{C_X^2}{\theta_X} - 2C_Y C_X \rho\right)$
3	1	1	0	-1	$t_{pro3} = \bar{y}\left(\frac{\bar{x}}{\bar{X}}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{C_X^2}{\theta_X} + 2\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{C_X^2}{\theta_X} + 2C_Y C_X \rho\right)$

4	1	1	C_X	1	$t_{pro4}=\bar{y}\left(\frac{\bar{X}+C_X}{\bar{x}+C_X}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_4^2 C_X^2}{\theta_X} - 2\lambda_4\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_4^2 C_X^2}{\theta_X} - 2\lambda_4 C_Y C_X \rho\right)$
5	1	1	C_X	-1	$t_{pro5}=\bar{y}\left(\frac{\bar{x}+C_X}{\bar{X}+C_X}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_5^2 C_X^2}{\theta_X} + 2\lambda_5\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_5^2 C_X^2}{\theta_X} + 2\lambda_5 C_Y C_X \rho\right)$

6	1	1	$\beta_1(x)$	1	$t_{pro6} = \bar{y}\left(\frac{\bar{X}+\beta_1(x)}{\bar{x}+\beta_1(x)}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_6^2 C_X^2}{\theta_X} - 2\lambda_6\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_6^2 C_X^2}{\theta_X} - 2\lambda_6 C_Y C_X \rho\right)$
7	1	1	$\beta_1(x)$	-1	$t_{pro7} = \bar{y}\left(\frac{\bar{x}+\beta_1(x)}{\bar{X}+\beta_1(x)}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_7^2 C_X^2}{\theta_X} + 2\lambda_7\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_7^2 C_X^2}{\theta_X} + 2\lambda_7 C_Y C_X \rho\right)$

8	1	$\beta_1(x)$	C_X	1	$t_{pro8} = \bar{y}\left(\frac{\beta_1(x)\bar{X}+C_X}{\beta_1(x)\bar{x}+C_X}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_8^2C_X^2}{\theta_X} - 2\lambda_8\left(C_YC_X\rho + \frac{\sigma_u\sigma_v\rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_8^2C_X^2}{\theta_X} - 2\lambda_8C_YC_X\rho\right)$
9	1	$\beta_1(x)$	C_X	-1	$t_{pro9} = \bar{y}\left(\frac{\beta_1(x)\bar{x}+C_X}{\beta_1(x)\bar{X}+C_X}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_9^2C_X^2}{\theta_X} + 2\lambda_9\left(C_YC_X\rho + \frac{\sigma_u\sigma_v\rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_9^2C_X^2}{\theta_X} + 2\lambda_9C_YC_X\rho\right)$

10	1	C_X	$\beta_1(x)$	1	$t_{pro10} = \bar{y}\left(\frac{C_X\bar{X}+\beta_1(x)}{C_X\bar{x}+\beta_1(x)}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{10}^2 C_X^2}{\theta_X} - 2\lambda_{10}\left(C_Y C_X\rho + \frac{\sigma_u\sigma_v\rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{10}^2 C_X^2}{\theta_X} - 2\lambda_{10}C_Y C_X\rho\right)$
11	1	C_X	$\beta_1(x)$	−1	$t_{pro11} = \bar{y}\left(\frac{C_X\bar{x}+\beta_1(x)}{C_X\bar{X}+\beta_1(x)}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{11}^2 C_X^2}{\theta_X} + 2\lambda_{11}\left(C_Y C_X\rho + \frac{\sigma_u\sigma_v\rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{11}^2 C_X^2}{\theta_X} + 2\lambda_{11}C_Y C_X\rho\right)$

12	1	1	$\beta_2(x)$	1	$t_{pro12} = \bar{y}\left(\frac{\bar{X}+\beta_2(x)}{\bar{x}+\beta_2(x)}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{12}^2 C_X^2}{\theta_X} - 2\lambda_{12}\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{12}^2 C_X^2}{\theta_X} - 2\lambda_{12} C_Y C_X \rho\right)$
13	1	1	$\beta_2(x)$	-1	$t_{pro13} = \bar{y}\left(\frac{\bar{x}+\beta_2(x)}{\bar{X}+\beta_2(x)}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{13}^2 C_X^2}{\theta_X} + 2\lambda_{13}\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{13}^2 C_X^2}{\theta_X} + 2\lambda_{13} C_Y C_X \rho\right)$

14	1	$\beta_2(x)$	C_X	1	$t_{pro14} = \bar{y}\left(\frac{\beta_2(x)\bar{X}+C_X}{\beta_2(x)\bar{x}+C_X}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{14}^2 C_X^2}{\theta_X} - 2\lambda_{14}\left(C_Y C_X \rho + \frac{\sigma_u\sigma_v\rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{14}^2 C_X^2}{\theta_X} - 2\lambda_{14} C_Y C_X \rho\right)$
15	1	$\beta_2(x)$	C_X	-1	$t_{pro15} = \bar{y}\left(\frac{\beta_2(x)\bar{x}+C_X}{\beta_2(x)\bar{X}+C_X}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{15}^2 C_X^2}{\theta_X} + 2\lambda_{15}\left(C_Y C_X \rho + \frac{\sigma_u\sigma_v\rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{15}^2 C_X^2}{\theta_X} + 2\lambda_{15} C_Y C_X \rho\right)$

16	1	C_X	$\beta_2(x)$	1	$t_{pro16} = \bar{y}\left(\frac{C_X\bar{X}+\beta_2(x)}{C_X\bar{x}+\beta_2(x)}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{16}^2C_X^2}{\theta_X} - 2\lambda_{16}\left(C_YC_X\rho + \frac{\sigma_u\sigma_v\rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{16}^2C_X^2}{\theta_X} - 2\lambda_{16}C_YC_X\rho\right)$
17	1	C_X	$\beta_2(x)$	-1	$t_{pro17} = \bar{y}\left(\frac{C_X\bar{x}+\beta_2(x)}{C_X\bar{X}+\beta_2(x)}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{17}^2C_X^2}{\theta_X} + 2\lambda_{17}\left(C_YC_X\rho + \frac{\sigma_u\sigma_v\rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{17}^2C_X^2}{\theta_X} + 2\lambda_{17}C_YC_X\rho\right)$

18	1	1	σ_x	1	$t_{pro18} = \bar{y}\left(\frac{\bar{X} + \sigma_x}{\bar{x} + \sigma_x}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{\lambda_{18}^2 C_X^2}{\theta_X} - 2\lambda_{18}\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{\lambda_{18}^2 C_X^2}{\theta_X} - 2\lambda_{18} C_Y C_X \rho\right)$
19	1	1	σ_x	−1	$t_{pro19} = \bar{y}\left(\frac{\bar{x} + \sigma_x}{\bar{X} + \sigma_x}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{\lambda_{19}^2 C_X^2}{\theta_X} + 2\lambda_{19}\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{\lambda_{19}^2 C_X^2}{\theta_X} + 2\lambda_{19} C_Y C_X \rho\right)$

20	1	$\beta_1(x)$	σ_x	1	$t_{pro20} = \bar{y}\left(\frac{\beta_1(x)\bar{X}+\sigma_x}{\beta_1(x)\bar{x}+\sigma_x}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{20}^2 C_X^2}{\theta_X} - 2\lambda_{20}\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{20}^2 C_X^2}{\theta_X} - 2\lambda_{20} C_Y C_X \rho\right)$
21	1	$\beta_1(x)$	σ_x	-1	$t_{pro21} = \bar{y}\left(\frac{\beta_1(x)\bar{x}+\sigma_x}{\beta_1(x)\bar{X}+\sigma_x}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{21}^2 C_X^2}{\theta_X} + 2\lambda_{21}\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{21}^2 C_X^2}{\theta_X} + 2\lambda_{21} C_Y C_X \rho\right)$

22	1	$\beta_2(x)$	σ_x	1	$t_{pro22} = \bar{y}\left(\frac{\beta_2(x)\bar{X}+\sigma_x}{\beta_2(x)\bar{x}+\sigma_x}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{22}^2 C_X^2}{\theta_X} - 2\lambda_{22}\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{22}^2 C_X^2}{\theta_X} - 2\lambda_{22} C_Y C_X \rho\right)$
23	1	$\beta_2(x)$	σ_x	-1	$t_{pro23} = \bar{y}\left(\frac{\beta_2(x)\bar{x}+\sigma_x}{\beta_2(x)\bar{X}+\sigma_x}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{23}^2 C_X^2}{\theta_X} + 2\lambda_{23}\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y}+\frac{\lambda_{23}^2 C_X^2}{\theta_X} + 2\lambda_{23} C_Y C_X \rho\right)$

24	1	1	ρ	1	$t_{pro24} = \bar{y}\left(\frac{\bar{X}+\rho}{\bar{x}+\rho}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{\lambda_{24}^2 C_X^2}{\theta_X} - 2\lambda_{24}\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{\lambda_{24}^2 C_X^2}{\theta_X} - 2\lambda_{24} C_Y C_X \rho\right)$
25	1	1	ρ	-1	$t_{pro25} = \bar{y}\left(\frac{\bar{x}+\rho}{\bar{X}+\rho}\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{\lambda_{25}^2 C_X^2}{\theta_X} + 2\lambda_{25}\left(C_Y C_X \rho + \frac{\sigma_u \sigma_v \rho^*}{\bar{Y}\bar{X}}\right)\right)$	$\frac{\bar{Y}^2}{n}\left(\frac{C_Y^2}{\theta_Y} + \frac{\lambda_{25}^2 C_X^2}{\theta_X} + 2\lambda_{25} C_Y C_X \rho\right)$

Where,

$$\lambda_i = \frac{\alpha_i \bar{X}}{\alpha_i \bar{X} + \psi_i}, \qquad (i = 4, 5, \ldots, 25)$$

7. Theoretical Efficiency Comparison

The proposed estimator t_{pro} was compare with some particular members of the proposed estimators shown in table 1. The results obtained are as follows

i. In the presence of correlated measurement error

1. $MSE^*_{min}(t_{pro}) < V(t_{pro1})$ if

$$V(t_{pro1}) - MSE^*_{min}(t_{pro}) = \frac{\theta_X}{C_X^2}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)^2 > 0$$

2. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro2})$ if

$$MSE^*(t_{pro2}) - MSE^*_{min}(t_{pro}) = \left(C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

3. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro3})$ if

$$MSE^*(t_{pro3}) - MSE^*_{min}(t_{pro}) = \left(C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

4. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro4})$ if

$$MSE^*(t_{pro4}) - MSE^*_{min}(t_{pro}) = \left(\lambda_4 C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

5. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro5})$ if

$$MSE^*(t_{pro5}) - MSE^*_{min}(t_{pro}) = \left(\lambda_5 C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

6. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro6})$ if

$$MSE^*(t_{pro6}) - MSE^*_{min}(t_{pro}) = \left(\lambda_6 C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

7. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro7})$ if

$$MSE^*(t_{pro7}) - MSE^*_{min}(t_{pro}) = \left(\lambda_7 C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

8. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro8})$ if

$$MSE^*(t_{pro8}) - MSE^*_{min}(t_{pro}) = \left(\lambda_8 C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

9. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro9})$ if

$$MSE^*(t_{pro9}) - MSE^*_{min}(t_{pro}) = \left(\lambda_9 C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

10. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro10})$ if

$$MSE^*(t_{pro10}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{10} C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

11. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro11})$ if

$$MSE^*(t_{pro11}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{11} C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u \sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

12. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro12})$ if

$$MSE^*(t_{pro12}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{12}C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

13. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro13})$ if

$$MSE^*(t_{pro13}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{13}C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

14. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro14})$ if

$$MSE^*(t_{pro14}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{14}C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

15. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro15})$ if

$$MSE^*(t_{pro15}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{15}C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

16. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro16})$ if

$$MSE^*(t_{pro16}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{16}C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

17. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro17})$ if

$$MSE^*(t_{pro17}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{17}C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

18. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro18})$ if

$$MSE^*(t_{pro18}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{18}C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

19. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro19})$ if

$$MSE^*(t_{pro19}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{19}C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

20. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro20})$ if

$$MSE^*(t_{pro20}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{20}C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

21. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro21})$ if

$$MSE^*(t_{pro21}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{21}C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

22. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro22})$ if

$$MSE^*(t_{pro22}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{22}C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

23. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro23})$ if

$$MSE^*(t_{pro23}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{23}C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

24. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro24})$ if

$$MSE^*(t_{pro24}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{24}C_X - \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

25. $MSE^*_{min}(t_{pro}) < MSE^*(t_{pro25})$ if

$$MSE^*(t_{pro25}) - MSE^*_{min}(t_{pro}) = \left(\lambda_{25}C_X + \frac{\theta_X}{C_X}\left(\rho C_Y C_X + \frac{\sigma_u\sigma_v}{\bar{Y}\bar{X}}\rho^*\right)\right)^2 > 0$$

ii. In the presence of uncorrelated measurement errors

1. $MSE_{min}(t_{pro}) < V(t_{pro1})$ if

   $$V(t_{pro1}) - MSE_{min}(t_{pro}) = \theta_X\rho^2C_Y^2 > 0$$

2. $MSE_{min}(t_{pro}) < MSE(t_{pro2})$ if

   $$MSE(t_{pro2}) - MSE_{min}(t_{pro}) = (C_X - \theta_X\rho C_Y)^2 > 0$$

3. $MSE_{min}(t_{pro}) < MSE(t_{pro3})$ if

   $$MSE(t_{pro3}) - MSE_{min}(t_{pro}) = (C_X + \theta_X\rho C_Y)^2 > 0$$

4. $MSE_{min}(t_{pro}) < MSE(t_{pro4})$ if

   $$MSE(t_{pro4}) - MSE_{min}(t_{pro}) = (\lambda_4C_X - \theta_X\rho C_Y)^2 > 0$$

5. $MSE_{min}(t_{pro}) < MSE(t_{pro5})$ if

   $$MSE(t_{pro5}) - MSE_{min}(t_{pro}) = (\lambda_5C_X + \theta_X\rho C_Y)^2 > 0$$

6. $MSE_{min}(t_{pro}) < MSE(t_{pro6})$ if

$$MSE(t_{pro6}) - MSE_{min}(t_{pro}) = (\lambda_6 C_X - \theta_X \rho C_Y)^2 > 0$$

7. $MSE_{min}(t_{pro}) < MSE(t_{pro7})$ if

$$MSE(t_{pro7}) - MSE_{min}(t_{pro}) = (\lambda_7 C_X + \theta_X \rho C_Y)^2 > 0$$

8. $MSE_{min}(t_{pro}) < MSE(t_{pro8})$ if

$$MSE(t_{pro8}) - MSE_{min}(t_{pro}) = (\lambda_8 C_X - \theta_X \rho C_Y)^2 > 0$$

9. $MSE_{min}(t_{pro}) < MSE(t_{pro9})$ if

$$MSE(t_{pro9}) - MSE_{min}(t_{pro}) = (\lambda_9 C_X + \theta_X \rho C_Y)^2 > 0$$

10. $MSE_{min}(t_{pro}) < MSE(t_{pro10})$ if

$$MSE(t_{pro10}) - MSE_{min}(t_{pro}) = (\lambda_{10} C_X - \theta_X \rho C_Y)^2 > 0$$

11. $MSE_{min}(t_{pro}) < MSE(t_{pro11})$ if

$$MSE(t_{pro11}) - MSE_{min}(t_{pro}) = (\lambda_{11} C_X + \theta_X \rho C_Y)^2 > 0$$

12. $MSE_{min}(t_{pro}) < MSE(t_{pro12})$ if

$$MSE(t_{pro12}) - MSE_{min}(t_{pro}) = (\lambda_{12} C_X - \theta_X \rho C_Y)^2 > 0$$

13. $MSE_{min}(t_{pro}) < MSE(t_{pro13})$ if

$$MSE(t_{pro13}) - MSE_{min}(t_{pro}) = (\lambda_{13} C_X + \theta_X \rho C_Y)^2 > 0$$

14. $MSE_{min}(t_{pro}) < MSE(t_{pro14})$ if

$$MSE(t_{pro14}) - MSE_{min}(t_{pro}) = (\lambda_{14}C_X - \theta_X\rho C_Y)^2 > 0$$

15. $MSE_{min}(t_{pro}) < MSE(t_{pro15})$ if

$$MSE(t_{pro15}) - MSE_{min}(t_{pro}) = (\lambda_{15}C_X + \theta_X\rho C_Y)^2 > 0$$

16. $MSE_{min}(t_{pro}) < MSE(t_{pro16})$ if

$$MSE(t_{pro16}) - MSE_{min}(t_{pro}) = (\lambda_{16}C_X - \theta_X\rho C_Y)^2 > 0$$

17. $MSE_{min}(t_{pro}) < MSE(t_{pro17})$ if

$$MSE(t_{pro17}) - MSE_{min}(t_{pro}) = (\lambda_{17}C_X + \theta_X\rho C_Y)^2 > 0$$

18. $MSE_{min}(t_{pro}) < MSE(t_{pro18})$ if

$$MSE(t_{pro18}) - MSE_{min}(t_{pro}) = (\lambda_{18}C_X - \theta_X\rho C_Y)^2 > 0$$

19. $MSE_{min}(t_{pro}) < MSE(t_{pro19})$ if

$$MSE(t_{pro19}) - MSE_{min}(t_{pro}) = (\lambda_{19}C_X + \theta_X\rho C_Y)^2 > 0$$

20. $MSE_{min}(t_{pro}) < MSE(t_{pro20})$ if

$$MSE(t_{pro20}) - MSE_{min}(t_{pro}) = (\lambda_{20}C_X - \theta_X\rho C_Y)^2 > 0$$

21. $MSE_{min}(t_{pro}) < MSE(t_{pro21})$ if

$$MSE(t_{pro21}) - MSE_{min}(t_{pro}) = (\lambda_{21}C_X + \theta_X\rho C_Y)^2 > 0$$

22. $MSE_{min}(t_{pro}) < MSE(t_{pro22})$ if

$$MSE(t_{pro22}) - MSE_{min}(t_{pro}) = (\lambda_{22}C_X - \theta_X \rho C_Y)^2 > 0$$

23. $MSE_{min}(t_{pro}) < MSE(t_{pro23})$ if

$$MSE(t_{pro23}) - MSE_{min}(t_{pro}) = (\lambda_{23}C_X + \theta_X \rho C_Y)^2 > 0$$

24. $MSE_{min}(t_{pro}) < MSE(t_{pro24})$ if

$$MSE(t_{pro24}) - MSE_{min}(t_{pro}) = (\lambda_{24}C_X - \theta_X \rho C_Y)^2 > 0$$

25. $MSE_{min}(t_{pro}) < MSE(t_{pro25})$ if

$$MSE(t_{pro25}) - MSE_{min}(t_{pro}) = (\lambda_{25}C_X + \theta_X \rho C_Y)^2 > 0$$

8. Empirical Efficiency Comparison

An empirical efficiency comparison was carried out by comparing the mean square error of the proposed estimator with the mean square error of some estimators that belong to the proposed generalized estimator. The percentage relative efficiency of the proposed estimator and some estimators that belong to the proposed generalized estimator over sample mean per unit was obtained. The dataset for the empirical analysis was from Okafor (2002) and the following parameters were computed:

$$N = 20, \bar{X} = 829.1635, \bar{Y} = 530.0755, \ \sigma_Y^2 = 61824.97, \ \ \sigma_X^2 = 190361.30,$$

$$\sigma_u^2 = 9.5705, \sigma_v^2 = 9.3084, \sigma_Y = 248.6463, \sigma_X = 436.3041, \ \ \sigma_u = 3.0937,$$

$$\sigma_v = 3.0510, \quad C_Y = 0.4691, C_X = 0.5262, \quad \rho = 0.8139, \quad \rho^* = 0.9980,$$

$$\beta_1(X) = 0.5735, \quad \beta_2(X) = -0.5858, \quad \theta_Y = 0.99985, \quad \theta_X = 0.99995$$

The results were shown in table 2.

ESTIMATORS	MEAN SQUARE ERROR		PERCENTAGE RELATIVE EFFICIENCY	
	CORRELATED $MSE^*(*)$	UNCORRELATED $MSE(*)$	CORRELATED	UNCORRELATED
t_{pro}	1043.7465	1044.1835	296.2433	296.1193
t_{pro1}	3092.0290	3092.0290	100.0000	100.0000
t_{pro2}	1336.6101	1337.2123	231.3337	231.2295
t_{pro3}	12627.7845	12627.1823	24.4859	24.4871
t_{pro4}	1340.1907	1340.7925	230.7156	230.6120
t_{pro5}	12624.2039	12623.6021	24.4929	24.4940
t_{pro6}	1340.5123	1341.1141	230.6602	230.5568
t_{pro7}	12623.8823	12623.2805	24.4935	24.4947
t_{pro8}	1342.85045	1343.4520	230.2586	230.1555
t_{pro9}	12621.5442	12620.9426	24.4980	24.4992
t_{pro10}	1344.0212	1344.6226	230.0581	229.9552

t_{pro11}	12620.3734	12619.7720	24.5003	24.5015
t_{pro12}	1332.6187	1333.2214	232.0265	231.9216
t_{pro13}	12631.7759	12631.1732	24.4781	24.4794
t_{pro14}	1330.4875	1331.0903	232.3982	232.2930
t_{pro15}	12633.9072	12633.3043	24.4741	24.4752
t_{pro16}	1329.0200	1329.6230	232.6548	232.5493
t_{pro17}	12635.3746	12634.7716	24.4712	24.4724
t_{pro18}	3283.0787	3283.4732	94.1808	94.1695
t_{pro19}	10681.3160	10680.9214	28.9480	28.9491
t_{pro20}	4037.9849	4038.2990	76.5736	76.5676
t_{pro21}	9926.4097	9926.0957	31.1495	31.1505
t_{pro22}	-48505.5366	-48499.6178	-6.3745	-6.3754
t_{pro23}	62469.9312	62464.0124	4.9496	4.9501
t_{pro24}	1342.1464	1342.7480	230.3794	230.2762
t_{pro25}	12622.2482	12621.6466	24.4967	24.4978

Table 2: Mean Square Error and Percentage Relative Efficiency

9. Discussion of Results

From the theoretical analysis, we observed that the bias of the proposed estimator $t_{pro}$is affected by the presence of measurement error on the auxiliary variable only. While the mean square error is affected by the presence of measurement errors both on the study and auxiliary variables.It is also our observation from both theoretical and empirical analysis that the proposed estimator t_{pro} at its optimum value yields the least mean square error when compared with the usual unbiased estimator $\bar{y}$ and other members of the proposed class of estimator. The high positive correlation coefficient between study variable and auxiliary variable accounted for efficient performance of the ratio type estimators against product type estimators. There is no much difference on the effect of the presence of measurement errors on the mean square error of the proposed estimator when the measurement errors are correlated as when it is uncorrelated.

Unexpected result was observed for the estimator t_{pro22} which recorded a negative mean square error. The implication of this is that the combination of kurtosis and standard deviation of auxiliary variable for the purpose improving the performance of the ratio type estimator cannot yield an efficient estimator at least for the dataset used.

10. Conclusion

The stated inequalities in section 7 provide the necessary conditions under which the proposed estimator at its optimum value is expected to be more efficient than the existing estimators of finite population mean. The empirical analysis buttresses these conditions, therefore the proposed estimator at its optimum value is recommended for use in practice.

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