Short Research Article

On an Erlang(2) risk model with dependence between interclaim arrivals and claim sizes

Abstract

In this paper, we consider an extension to the classical compound Poisson risk model by introducing a dependence structure between the claim sizes and interclaim times, which the claim inter-arrival distribution is Erlang(2). By studying the roots of the generalised Lundberg equation, the Laplace transform(LT) of the expected discounted penalty function is derived. We also show that the Gerber–Shiu functions satisfy some defective renewal equations. The ruin probability is an important case of the Laplace transform of the time to ruin. Some explicit expressions are obtained to measure the impact of the various dependence structures in the risk model on the ruin probability.

Keywords: dependence; Gerber-Shiu penalty function; Laplace Transform; defective renewal equation; ruin probability

1 Introduction

In the actuarial literature, many authors focus their research intertsts to two well-known risk models, namely the classial compound Poisson risk model and the risk model based on the renewal or the Sparre Andersen risk model. Ruin probabilities and many other ruin measures such as the marginal and the joint(defective or not) distributions of the time to ruin, the deficit at ruin and the surplus prior to ruin have been extensively studied(see Dickson and Hipp(1998)[1], Rolski et al.(1999)[2] and references therein). A unified approach to study these ruin measures with the discounted penalty function for the classical risk model has be introduced in the Gerber and Shiu(1998)[3].

Note that, for these two risk models, it is explicitly assumed that the interarrival times between two successive claims and the claim amounts are independent. This assumption is appropriate in certain practical circumstances and has the advantage of simplifying the models. However, this assumptions is inappropriate in the real world. For example, in modeling natural earthquake events, more considerable damages are expected with a longer period between claims. See Albrecher and Boxma(2004)[4] and Nikoloulopoulos & Karlis(2008)[5] for an example of this type of structure. M. Boudreault et al.(2006)[6] studied the dependence structure among the interclaim time and the subsequent size. Stathis et al.(2012)[7] considered an extension to the renewal process by introducing a dependence structure between the claim sizes and interclaim times through a Farlie-Gumbel-Morgenstern copula.and we can also see that in the H. Cossette et al(2008)[8].

Since then, several renewal risk models with different interclaim times have been studied by many authors. The Erlang distribution is one of the most commonly used distributions in risk and queueing theroy. See the paper writing by Dickson and Hipp(1998,2001)[1] [9], Willmot and Lin(1999)[10], Cheng and Tang(2003)[11], Gerber and Shiu(2005)[12] and the references therein.

In this paper, we consider that the interclaim times are distributed according to an Erlang(2).And we consider a dependence structure between the claim amount and the interclaim time.Therefore, our risk model is an extension of the classical Poisson.

The paper is organsized as follows. In Section 2, we briefly introduce the risk model and the dependence structure of the proposed model. We analyse the generalised Lundberg equation and its roots in Section 3. The Laplace transform (LT) of the Gerber-Shiu expected discount penalty function is given in Section 4. In Section 5, the defective renewal function is given. Finally, explicit expressions and numerical examples are given in Section 6.

2 The risk model and the dependence structure

In this section, we consider the surplus process $\{U(t), t \geq 0\}$ defined by $U(t) = u + ct - S(t)$, where $u = U(0) \geq 0$ is the initial surplus and c is the premium rate which is assumed to be a positive constant. $S(t), t \geq 0$ is the total claim amount process defined by $S(t) = \sum_{i=1}^{N(t)} X_i$ and $\sum_a^b = 0$ if $a > b$. The claim number process $\{N(t), t \geq 0\}$ is a renewal process defined via a sequence of i.i.d. interclaim times $\{W_i\}_{i=1}^{\infty}$. In this paper, we consider that the r.v. W has an Erlang(2) distribution with expectation $2/\beta, \beta > 0$ with p.d.f. given by

$$f_W(t) = \beta^2 t e^{-\beta t}, t \geq 0. \tag{1}$$

The individual claim amount r.v.'s $X_j, j \in N^+$ are assumed to be a sequence of strictly positive i.i.d. r.v.'s with cumulative distribution function(c.d.f.) $F_X(x) = 1 - \bar{F}_X(x)$ and Laplace transform $\hat{f}_X$. We assume that the claim amount and the interclaim time r.v.'s X_k and W_k is a dependence structure. We defined that the density of $X_k|W_k$ as a mixture of two arbitrary density function f_1 and f_2 (with respective means μ_1 and μ_2), i.e.

$$f_{X_k|W_k}(x) = e^{-\lambda W_k} f_1(x) + (1 - e^{-\lambda W_k}) f_2(x), \quad x \geq 0, \tag{2}$$

for $k = 1, 2, \ldots$.

We let $\tau = inf_{t \geq 0}\{t, U_t < 0\}$ be the time of ruin with $\tau = \infty$ if $U_t \geq 0$(i.e. ruin does not occur). The deficit at ruin is denoted by $|U_\tau|$ and the surplus just prior to ruin is $U_{\tau-}$. To ensure that ruin does not almost surely occur, the premium rate c is such that

$$E[cW_j - X_j] > 0, j = 1, 2, \ldots \tag{3}$$

providing a positive safety loading.

The Gerber-Shiu discounted penalty function $m_\delta(u)$ is defined as

$$m_\delta(u) = E[e^{-\delta\tau} w(U_{\tau-}, |U_\tau|) 1_{\tau<\infty} | U_0 = u], \tag{4}$$

where $\delta > 0, w : R^+ \times R^+ \to R^+$ is the penalty function. Especially, a special case of the Gerber-Shiu discounted penalty function is when $w(x, y) = 1$, for all $x, y \geq 0$. Then $m_\delta(u)$ becomes the LT of the time of ruin, denoted by $m_\tau(u)$. If $\delta = 0$ the $m_\delta(u)$ becomes the ruin probability $\psi(u) = E[1_{\tau<\infty} | U(0) = u]$.

3 Analysis of a Lundberg's generalised equation

In this section, we want to derive a generalised version of the Lundberg equation for the Erlang(2) risk process, and analyse the number of its roots, i.e. with $Re(s) > 0$. These roots are required to find the defective renewal equation for the Gerber-Shiu function $m_\delta(u)$ as well as to evaluate several ruin quantities.

To derive Lundberg's generalised equation, we consider the discrete-time process embedded in the continuous-time surplus process $\{U(t); t \geq 0\}$. Define the discrete-time process by $U_0 = u$ and for $k = 1, 2, \ldots,$

$$U_k = u + \sum_{i=1}^{k} (cW_i - X_i),$$

to be the surplus immediately after the kth claim. We seek a number such that the process $\{e^{-\delta \sum_{i=1}^{k} W_i + sU_k}, \quad k = 0, 1, 2, \ldots\}$ for $s > 0$ is a martingale if and only if

$$E[e^{-\delta W} e^{s(cW-X)}] = E[e^{(cs-\delta)W} e^{-sX}] = 1, \tag{5}$$

which is called the *generalised Lundberg equation* associated with our risk model. Given in Equation (1) and (2), the left-hand side of Equation (5) can be written as

$$\begin{aligned} E[e^{-\delta W} e^{s(cW-X)}] &= \int_0^\infty \int_0^\infty e^{-(\delta-cs)t} f_W(t)(e^{-\lambda t} f_1(x) + (1 - e^{-\lambda t} f_2(x)))e^{-sx} dx dt \\ &= \beta^2 \hat{f}_1(s) \frac{1}{(\delta+\lambda+\beta-cs)^2} + \beta^2 \hat{f}_2(s) \frac{1}{(\delta+\beta-cs)^2} \\ &- \beta^2 \hat{f}_2(s) \frac{1}{(\delta+\lambda+\beta-cs)^2} \end{aligned} \tag{6}$$

Then, Lundberg's generalised equation in (5) reduces to

$$\frac{\beta^2}{c^2} \frac{(\frac{\delta+\beta}{c} - s)^2 \hat{f}_1(s) + (\frac{\delta+\lambda+\beta}{c} - s)^2 \hat{f}_2(s) - (\frac{\delta+\beta}{c} - s)^2 \hat{f}_2(s)}{(\frac{\delta+\lambda+\beta}{c} - s)^2 (\frac{\delta+\beta}{c} - s)^2} = 1. \tag{7}$$

We use Rouche´'s theorem to show the numbers of roots of the generalized Lundberg equation in the following proposition.

Proposition 1. For $\delta > 0$, Lundberg's generalised equation in (7) has exactly 4 roots, say $\rho_1(\delta)$, $\rho_2(\delta)$, $\rho_3(\delta)$, $\rho_4(\delta)$, with $Re(\rho_i(\delta)) > 0$, $i = 1, 2, 3, 4$.

Proof. The generalised Lundberg Equation (7) also becomes

$$\begin{aligned} \beta^2(\delta+\beta-cs)^2 \hat{f}_1(s) + \beta^2 \hat{f}_2(s)[(\delta+\lambda+\beta-cs)^2 - (\delta+\beta-cs)^2] \\ = (\delta+\lambda+\beta-cs)^2(\delta+\beta-cs)^2, \end{aligned} \tag{8}$$

it can be seen that the above Equation (8) has exactly 4 roots with positive real parts. We let $r > 0$ and denote by C_r the contour containing the imaginary axis running from -ir to ir and a semicircle with radius r running clockwise fron -ir to ir, that is, $C_r = \{s \in C : |s| = r, Re(s) \geq 0, r > 0 \text{ is fixed } \}$. Also let $r \to \infty$ and denote by C the limiting contour. We apply Rouche's theorem on the closed contour C to prove the result.

(1) For $Re(s) > 0$, that is, for s on the semicircle, we have $|\delta + \beta - cs| \to \infty, \quad |\delta + \lambda + \beta - cs| \to \infty$ as $r \to \infty$, and thus

$$\begin{aligned} &\left| \frac{\beta^2}{(\delta+\lambda+\beta-cs)^2} \hat{f}_1(s) + \left[\frac{1}{(\delta+\beta-cs)^2} - \frac{1}{(\delta+\lambda+\beta-cs)^2} \right] \beta^2 \hat{f}_2(s) \right| \\ &\leq |\hat{f}_1(s)| \left| \frac{\beta^2}{(\delta+\lambda+\beta-cs)^2} \right| + |\hat{f}_2(s)| \left| \frac{\beta^2}{(\delta+\beta-cs)^2} + \frac{\beta^2}{(\delta+\lambda+\beta-cs)^2} \right| \to 0 \end{aligned}$$

on C. For $r \to \infty$, and hence it holds that

$$\left| \frac{\beta^2}{(\delta+\lambda+\beta-cs)^2}\hat{f}_1(s) + \left[\frac{1}{(\delta+\beta-cs)^2} - \frac{1}{(\delta+\lambda+\beta-cs)^2}\right]\beta^2\hat{f}_2(s)\right| < 1 \tag{9}$$

on C.

(2) For $Re(s) = 0$, that is, for s on the imaginary axis and for $\delta > 0$, similar to Cossette et al.(2008)[8], we let

$$\hat{d}_\delta(s) = \frac{\beta^2}{(\delta+\beta-cs)^2} - \frac{\beta^2}{(\delta+\lambda+\beta-cs)^2}$$

then we have

$$\begin{aligned} |\hat{d}_\delta(s)| &= \beta^2\left|\frac{(2\delta+2\beta-2cs)\lambda+\lambda^2}{(\delta+\beta-cs)^2(\delta+\lambda+\beta-cs)^2}\right| \\ &\le \beta^2\lambda\left|\frac{(2\delta+2\beta+\lambda)^2+(2cs)^2}{(\delta+\beta)^2(\delta+\beta+\lambda)^2(2\delta+2\beta+\lambda)}\right| \\ &\le \beta^2\lambda\left|\frac{(2\delta+2\beta+\lambda)^2}{(\delta+\beta)^2(\delta+\beta+\lambda)^2(2\delta+2\beta+\lambda)}\right| = |\hat{d}_\delta(0)| \end{aligned}$$

and

$$\begin{aligned} &\left| \frac{\beta^2}{(\delta+\lambda+\beta-cs)^2}\hat{f}_1(s) + \left[\frac{1}{(\delta+\beta-cs)^2} - \frac{1}{(\delta+\lambda+\beta-cs)^2}\right]\beta^2\hat{f}_2(s)\right| \\ &= \left|\frac{\beta^2}{(\delta+\lambda+\beta-cs)^2}\hat{f}_1(s) + \hat{f}_2(s)\hat{d}_\delta(s)\right| \\ &\le \left|\frac{\beta^2}{(\delta+\lambda+\beta-cs)^2}\right| + |\hat{d}_\delta(s)| \\ &\le \left|\frac{\beta^2}{(\delta+\lambda+\beta)^2}\right| + |\hat{d}_\delta(0)| \end{aligned} \tag{10}$$

For $\delta > 0$, it holds $\hat{d}_\delta(0) > 0$. Indeed,

$$\hat{d}_\delta(0) = \frac{\beta^2}{(\delta+\beta)^2} - \frac{\beta^2}{(\delta+\lambda+\beta)^2} > 0$$

Therefore, for s on the imaginary axis and for $\delta > 0$, Equation (10) becomes

$$\begin{aligned} &\left| \frac{\beta^2}{(\delta+\lambda+\beta-cs)^2}\hat{f}_1(s) + \left[\frac{1}{(\delta+\beta-cs)^2} - \frac{1}{(\delta+\lambda+\beta-cs)^2}\right]\beta^2\hat{f}_2(s)\right| \\ &\le \left|\frac{\beta^2}{(\delta+\lambda+\beta)^2}\right| + |\hat{d}_\delta(0)| < 1 \end{aligned}$$

Above all, we proved that

$$\begin{aligned} &|\beta^2(\delta+\beta-cs)^2\hat{f}_1(s) + \beta^2\hat{f}_2(s)[(\delta+\lambda+\beta-cs)^2 - (\delta+\beta-cs)^2]| \\ &< |(\delta+\lambda+\beta-cs)^2(\delta+\beta-cs)^2| \end{aligned}$$

in two case, and thus by Rouche's theorem, it follows that Equation(8) has the same number of roots as the equation $(\delta+\lambda+\beta-cs)^2(\delta+\beta-cs)^2 = 0$ inside C_r. Since the latter equation has exactly 4 positive roots inside C_r, we deduce that Equation (8), that is, Equation (7) has exactly 4 roots, say $\rho_1(\delta), \ldots, \rho_4(\delta)$ with positive real parts. Finally, we complete the proof by letting $r \to \infty$.

In the following, for simiplicity we write ρ_j for $\rho_j(\delta), j = 1, 2, 3, 4.$ when $\delta > 0$.

Remark. For $\delta = 0$, the conditions to Rouche´'s theorem are not satisfied, since

$$\left|\frac{\beta^2}{(\delta+\lambda+\beta-cs)^2}\hat{f}_1(s) + \left[\frac{1}{(\delta+\beta-cs)^2} - \frac{1}{(\delta+\lambda+\beta-cs)^2}\right]\beta^2\hat{f}_2(s)\right| = \left|\frac{\beta^2}{(\lambda+\beta)^2} + \left[1 - \frac{\beta^2}{(\lambda+\beta)^2}\right]\right| = 1$$

for $s = 0$. The case of $\delta = 0$ is important to evaluate several ruin related quantities, such as the ruin probablity, being special cases of the Gerber-Shiu penalty function at $\delta = 0$. We apply the Klimenok(2001)[13] to derive the number of roots to the generalized Lundberg's equation with $\delta = 0$.

Proposition 2. For $\delta = 0$, Lundberg's generalised Equation(7) has exaclty 3 roots, say $\rho_1(0)$, $\rho_2(0)$, $\rho_3(0)$, with positive real parts and one root equals zero.

Proof. Define the contour $D_k = s : |z| = 1$ and let $z = 1 - \frac{s}{k}$. In terms of s, the contour D_k is a circle with origin at k and radius k. Similarly as in Proposition 1, we let $k \to \infty$ and denote by D the limiting contour.Using the same arguments as in the proof of Proposition 1, one can show that Equation(8) also holds on D (excluding s=0 or equivalently z=1) for $\delta = 0$. besides, the functions $\beta^2(\delta+\beta-cs)^2\hat{f}_1(s) + \beta^2\hat{f}_2(s)[(\delta+\lambda+\beta-cs)^2 - (\delta+\beta-cs)^2]$ and $(\delta+\lambda+\beta-cs)^2(\delta+\beta-cs)^2$ are continuous on D. As Theorem 1 of Klimenok(2001), we need prove that

$$\frac{d}{dz}\left\{1 - \frac{\beta^2}{(\lambda+\beta-ck(1-z))^2}\hat{f}_1(k-kz) - \left[\frac{\beta^2}{(\beta-ck(1-z))^2} - \frac{\beta^2}{(\lambda+\beta-ck(1-z))^2}\right]\hat{f}_2(k-kz)\right\}\bigg|_{z=1} > 0$$

The left-hand side of this relation is equal to

$$\frac{d}{dz}\left\{1 - E\left[e^{(k-kz)(cW-X)}\right]\right\}\bigg|_{z=1} = kE\left[cW - X\right]$$

where $E\left[cW - X\right] > 0$ given the solvability condition in equation (3).

Based on Klimenok(2001), we conclude that inside D, the number of roots of Equation (8) is equal to 3, that is, the number of roots of $(\delta+\lambda+\beta-cs)^2(\delta+\beta-cs)^2$ inside D minus 1. Moreover, a trival root to Lundberg's generalised equation (7) equals zero.

4 LT of $m_\delta(u)$

In this section, we want to derive the LT $\hat{m}_\delta(s) = \int_0^\infty e^{-su}m_\delta(u)du$ of the Gerber-Shiu expected discount penalty function $m_\delta(u)$ defined by Equation (4). For $u \geq 0$, we define the following functions

$$\sigma_{1,\delta}(u) = \int_0^u m_\delta(u-x)f_1(x)dx + \gamma_1(u), \quad \gamma_1(u) = \int_u^\infty f_1(x)dx$$
$$\sigma_{2,\delta}(u) = \int_0^u m_\delta(u-x)f_2(x)dx + \gamma_2(u), \quad \gamma_2(u) = \int_u^\infty f_2(x)dx \tag{11}$$

By conditioning on the time and the amount of the first claim, we have

$$\begin{aligned} m_\delta(u) = &\int_0^\infty e^{-\delta t} f_W(t) \int_0^{u+ct} (m_\delta(u+ct-x)) \left(e^{-\lambda t} f_1(x) + \left(1-e^{-\lambda t}\right) f_2(x)\right) dxdt \\ &+ \int_0^\infty e^{-\delta t} f_W(t) \int_{u+ct}^\infty \left[e^{-\lambda t} f_1(x) + \left(1-e^{-\lambda t}\right) f_2(x)\right] dxdt \end{aligned} \tag{12}$$

Setting $y = u + ct$, and using Equation (11) and $f_W\left(\frac{y-u}{c}\right) = \beta^2\left(\frac{y-u}{c}\right)e^{-\beta(\frac{y-u}{c})}$, then Equation (12) yields

$$\begin{aligned} c^2 m_\delta(u) = \beta^2 &\int_u^\infty e^{-(\delta+\lambda+\beta)\frac{y-u}{c}}(y-u)(\sigma_{1,\delta}(y) - \sigma_{2,\delta}(y))dy + \\ &\beta^2 \int_u^\infty e^{-(\delta+\beta)\frac{y-u}{c}}(y-u)\sigma_{2,\delta}(y)dy \end{aligned} \tag{13}$$

Taking LTs gives

$$\begin{aligned} c^2\hat{m}_\delta(s) = \beta^2 &\int_0^\infty e^{-(\delta+\lambda+\beta)\frac{y}{c}}(\sigma_{1,\delta}(y) - \sigma_{2,\delta}(y)) \int_0^y (y-u)e^{-(s-\frac{\delta+\lambda+\beta}{c})u}dudy \\ &+ \beta^2 \int_0^\infty e^{-(\delta+\beta)\frac{y}{c}}\sigma_{2,\delta}(y) \int_0^y (y-u)e^{-(s-\frac{\delta+\beta}{c})u}dudy \end{aligned} \tag{14}$$

It can be easily proved that for $a > 0$

$$\int_0^y (y-u)e^{-au}du = \frac{y}{a} - \frac{1}{a^2} + \frac{e^{-ay}}{a^2} \tag{15}$$

Therefore, using Equation (15), Equation (14) can be written in the form

$$\begin{aligned} c^2\hat{m}_\delta(s) = &\frac{\beta^2}{(s-\frac{\delta+\lambda+\beta}{c})^2}(\hat{\sigma}_{1,\delta}(s) - \hat{\sigma}_{2,\delta}(s)) \\ &+ \beta^2 \int_0^\infty e^{-(\delta+\lambda+\beta)\frac{y}{c}}(\sigma_{1,\delta}(y) - \sigma_{2,\delta}(y)) \left(\frac{y}{s-\frac{\delta+\lambda+\beta}{c}} - \frac{1}{(s-\frac{\delta+\lambda+\beta}{c})^2}\right) dy \\ &+ \frac{\beta^2}{(s-\frac{\delta+\beta}{c})^2}\hat{\sigma}_{2,\delta}(s) + \beta^2 \int_0^\infty e^{-\frac{(\delta+\beta)y}{c}}\sigma_{2,\delta}(y) \left(\frac{y}{s-\frac{\delta+\beta}{c}} - \frac{1}{(s-\frac{\delta+\beta}{c})^2}\right) dy \\ = &\frac{\beta^2}{(s-\frac{\delta+\lambda+\beta}{c})^2}(\hat{\sigma}_{1,\delta}(s) - \hat{\sigma}_{2,\delta}(s)) + \frac{\beta^2}{(s-\frac{\delta+\beta}{c})^2}\hat{\sigma}_{2,\delta}(s) + \hat{B}_\delta(s) \end{aligned} \tag{16}$$

where

$$\hat{\sigma}_{i,\delta}(s) = \int_0^\infty e^{-su}\sigma_{i,\delta}(u)du \quad i = 1, 2.$$

and

$$\begin{aligned} \hat{B}_\delta(s) = \beta^2 &\int_0^\infty e^{-(\delta+\lambda+\beta)\frac{y}{c}}(\sigma_1(y) - \sigma_2(y)) \left(\frac{y}{s-\frac{\delta+\lambda+\beta}{c}} - \frac{1}{(s-\frac{\delta+\lambda+\beta}{c})^2}\right) dy \\ &+ \beta^2 \int_0^\infty e^{-\frac{(\delta+\beta)y}{c}}\sigma_2(y) \left(\frac{y}{s-\frac{\delta+\beta}{c}} - \frac{1}{(s-\frac{\delta+\beta}{c})^2}\right) dy. \end{aligned}$$

Let $\hat{\gamma}_i(s) = \int_0^\infty e^{-su}\gamma_i(u)du, \quad i = 1,2.$ Since from Equation (11) it holds $\hat{\sigma}_{i,\delta}(s) = \hat{m}_\delta(s)\hat{f}_i(s) + \hat{\gamma}_i(s) \quad i = 1,2$. The above Equation (16) reduces to

$$\hat{m}_\delta(s)\left\{c^2 - \frac{\beta^2}{(s-\frac{\delta+\lambda+\beta}{c})^2}(\hat{f}_1(s)-\hat{f}_2(s)) - \frac{\beta^2}{(s-\frac{\delta+\beta}{c})^2}\hat{f}_2(s)\right\}$$
$$= \frac{\beta^2}{(s-\frac{\delta+\lambda+\beta}{c})^2}(\hat{\gamma}_1(s)-\hat{\gamma}_2(s)) + \frac{\beta^2}{(s-\frac{\delta+\beta}{c})^2}\hat{\gamma}_2(s) + \hat{B}_\delta(s) \tag{17}$$

Now using Equation (17), we give the following theorem about the expression for $\hat{m}_\delta(s)$.

Theorem 1. In the Erlang(2) risk process with a dependence structure, the LT $\hat{m}_\delta(s)$ of the $m_\delta(u)$ is given by

$$\hat{m}_\delta(s) = \frac{\hat{\beta}_{1,\delta}(s) + \hat{\beta}_{2,\delta}(s)}{\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s)}, \tag{18}$$

where

$$\hat{h}_{1,\delta}(s) = \left(s - \frac{\delta+\lambda+\beta}{c}\right)^2\left(s - \frac{\delta+\beta}{c}\right)^2 \tag{19}$$

$$\hat{h}_{2,\delta}(s) = \frac{\beta^2}{c^2}\left(s - \frac{\delta+\beta}{c}\right)^2(\hat{f}_1(s)-\hat{f}_2(s)) + \frac{\beta^2}{c^2}\left(s - \frac{\delta+\lambda+\beta}{c}\right)^2\hat{f}_2(s) \tag{20}$$

$$\hat{\beta}_{1,\delta}(s) = \frac{\beta^2}{c^2}\left(s - \frac{\delta+\beta}{c}\right)^2(\hat{\gamma}_1(s)-\hat{\gamma}_2(s)) + \frac{\beta^2}{c^2}\left(s - \frac{\delta+\lambda+\beta}{c}\right)^2\hat{\gamma}_2(s) \tag{21}$$

and $\hat{\beta}_{2,\delta(s)}$ is a polynomial in s of degree 3 or less, given by

$$\hat{\beta}_{2,\delta}(s) = -\sum_{j=1}^{4}\hat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^{4}\frac{s-\rho_k}{\rho_j-\rho_k}.$$

Proof. Multiplying both sides of Equation (17) by $\left(s-\frac{\delta+\lambda+\beta}{c}\right)^2\left(s-\frac{\delta+\beta}{c}\right)^2/c^2$ and then solving the resulting equation for $\hat{m}_\delta(s)$ we get immediately the equation (18), with

$$\begin{aligned}
\hat{\beta}_{2,\delta}(s) &= \frac{1}{c^2}\hat{h}_{1,\delta}(s)\hat{B}_\delta(s)\\
&= \frac{1}{c^2}\left(s-\frac{\delta+\lambda+\beta}{c}\right)^2\left(s-\frac{\delta+\beta}{c}\right)^2\left[\frac{\beta^2}{s-\frac{\delta+\lambda+\beta}{c}}\int_0^\infty e^{-(\delta+\lambda+\beta)y/c}(\sigma_{1,\delta}(y)-\sigma_{2,\delta}(y))ydy\right.\\
&\quad\left.-\frac{\beta^2}{\left(s-\frac{\delta+\lambda+\beta}{c}\right)^2}\int_0^\infty e^{-(\delta+\lambda+\beta)y/c}(\sigma_{1,\delta}(y)-\sigma_{2,\delta}(y))dy\right]\\
&\quad+\frac{1}{c^2}\left(s-\frac{\delta+\lambda+\beta}{c}\right)^2\left(s-\frac{\delta+\beta}{c}\right)^2\left[\frac{\beta^2}{s-\frac{\delta+\beta}{c}}\int_0^\infty e^{-(\delta+\beta)y/c}\sigma_{2,\delta}(y)ydy\right.\\
&\quad\left.-\frac{\beta^2}{(s-\frac{\delta+\beta}{c})^2}\int_0^\infty e^{-(\delta+\beta)y/c}\sigma_{2,\delta}(y)dy\right]\\
&= \frac{\beta^2}{c^2}\left(s-\frac{\delta+\beta}{c}\right)^2\left(s-\frac{\delta+\lambda+\beta}{c}\right)^j\hat{\mu}_j\left(\frac{\delta+\lambda+\beta}{c}\right)\\
&\quad+\frac{\beta^2}{c^2}\left(s-\frac{\delta+\lambda+\beta}{c}\right)^2\left(s-\frac{\delta+\beta}{c}\right)^j\hat{\delta}_j\left(\frac{\delta+\beta}{c}\right)\quad (j=0,1).
\end{aligned}$$

which is a polynomial in s of degree 3 or less, where

$$\hat{\mu}_j\left(\frac{\delta+\lambda+\beta}{c}\right) = \int_0^\infty e^{-(\delta+\lambda+\beta)y/c}(\sigma_{1,\delta}(y)-\sigma_{2,\delta}(y))y^j dy$$
$$\hat{\delta}_j\left(\frac{\delta+\beta}{c}\right) = \int_0^\infty e^{-(\delta+\beta)y/c}\sigma_{2,\delta}(y)y^j dy \quad (j=0,1)$$

It is easy to see that the Lundberg's generalised equation (5) can be written as $\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) = 0$, which means that $\rho_i's, i = 1,\ldots,4$ are roots of the denominator in Equation (18). Since $\hat{m}_\delta(s)$ is analytic for $Re(s) \geq 0$, this implies that $\rho_i's, i = 1,\ldots,4$ are also roots of the numerator in Equation (18), and thus $\hat{\beta}_{2,\delta}(\rho_i) = -\hat{\beta}_{1,\delta}(\rho_i), i = 1,\ldots,4$. Since $\hat{\beta}_{2,\delta}(s)$ is a polynomial in s of degree 3, by the Lagrange interpolation formula at the 4 points $\rho_1,\rho_2,\rho_3,\rho_4$, we have

$$\hat{\beta}_{2,\delta}(s) = \sum_{j=1}^4 \hat{\beta}_{2,\delta}(\rho_j)\prod_{k=1,k\neq j}^4 \frac{s-\rho_k}{\rho_j-\rho_k} = -\sum_{j=1}^4 \hat{\beta}_{1,\delta}(\rho_j)\prod_{k=1,k\neq j}^4 \frac{s-\rho_k}{\rho_j-\rho_k},$$

and then the proof is completed.

5 Defection renewal function

Proposition 3. The LT of $m_\delta(u)$ is given by

$$\hat{m}_\delta(s) = \frac{T_sT_{\rho_1}\ldots T_{\rho_4}\beta_{1,\delta}(0)}{1-T_sT_{\rho_1}\ldots T_{\rho_4}h_{2,\delta}(0)} \tag{22}$$

Proof. By the Lagrange interpolating formula and using the Property of the Dickson-Hipp operator of Li and Garrido(2004)[14], we have

$$\hat{\beta}_{1,\delta}(s) + \hat{\beta}_{2,\delta}(s) = \pi_4(s)\left\{\frac{\hat{\beta}_{1,\delta}(s)}{\pi_4(s)} - \sum_{j=1}^4 \frac{\hat{\beta}_{1,\delta}(\rho_j)}{(s-\rho_j)\tau'(\rho_j)}\right\} = \pi_4(s)T_sT_{\rho_1}\ldots T_{\rho_4}\beta_{1,\delta}(0), \tag{23}$$

where $\pi_4(s) = \prod_{i=1}^4(s-\rho_j)$, and

$$\hat{h}_{1,\delta}(s) = \hat{h}_{1,\delta}(0)\prod_{k=1}^4\frac{s-\rho_k}{(-\rho_k)} + s\sum_{j=1}^4\frac{\hat{h}_{1,\delta}(\rho_j)}{\rho_j}\prod_{k=1,k\neq j}^4\frac{s-\rho_k}{\rho_j-\rho_k}.$$

Similar arguments as the Cossette et al.(2010)[15], the aforementioned relation implies that

$$\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) = \pi_4(s)\Bigg[\frac{\hat{h}_{1,\delta}(0)}{\pi_4(0)} - \sum_{j=1}^4\frac{\hat{h}_{2,\delta}(\rho_j)}{(-\rho_j)\pi_4'(\rho_j)} + \sum_{j=1}^4\frac{\hat{h}_{2,\delta}(\rho_j)}{(s-\rho_j)\pi_4'(\rho_j)} - \frac{\hat{h}_{2,\delta}(s)}{\pi_4(s)}\Bigg], \tag{24}$$

Since $\hat{h}_{2,\delta}(\rho_j) = \hat{h}_{1,\delta}(\rho_j)$, $j = 1, \ldots, 4$, for s=0, we obtain

$$\begin{aligned}
\frac{\hat{h}_{1,\delta}(0)}{\pi(0)} + \sum_{j=1}^{4} \frac{\hat{h}_{2,\delta}(\rho_j)}{\rho_j \pi'(\rho_j)} &= \frac{\left(\frac{\delta+\lambda+\beta}{c}\right)^2 \left(\frac{\delta+\beta}{c}\right)^2}{\prod_{i=1}^{4}(-\rho_i)} + \sum_{j=1}^{4} \frac{\left(\frac{\delta+\lambda+\beta}{c} - \rho_j\right)^2 \left(\frac{\delta+\beta}{c} - \rho_j\right)^2}{\rho_j \prod_{k=1,k\neq j}^{4} (\rho_j - \rho_k)} \\
&= \frac{(\delta+\lambda+\beta)^2(\delta+\beta)^2}{c^4 \prod_{i=1}^{4}(-\rho_j)} + \sum_{j=1}^{4} \frac{(\delta+\lambda+\beta - c\rho_j)^2(\delta+\beta-c\rho_j)^2}{c^4 \rho_j \prod_{k=1,k\neq j}^{4}(\rho_j - \rho_k)} \\
&= \frac{(\delta+\lambda+\beta)^2(\delta+\beta)^2}{c^4 \prod_{i=1}^{4}(-\rho_j)} + \left[1 - \frac{(\delta+\lambda+\beta)^2(\delta+\beta)^2}{c^4 \prod_{i=1}^{4}(\rho_j)}\right] \\
&= 1.
\end{aligned}$$

Then Equation (24) becomes

$$\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) = \pi_4(s)\left[1 - T_s T_{\rho_1} \ldots T_{\rho_4} h_{2,\delta}(0)\right]. \tag{25}$$

Finally, replacing Equation (23) and (25), we obtain Equation (22).

Proposition 4. The Gerber-Shiu discounted penalty function $m_\delta(u)$ admits a defective renewal equation

$$m_\delta(u) = \int_0^u m_\delta(u-y)\zeta_\delta(y)dy + G_\delta(u), \quad u \geq 0. \tag{26}$$

where

$$\zeta_\delta(y) = T_{\rho_1} \ldots T_{\rho_4} h_{2,\delta}(y),$$

$$G_\delta(u) = T_{\rho_1} \ldots T_{\rho_4} \beta_{2,\delta}(u).$$

Furthermore, Equation (26) admits the following alternative representation

$$m_\delta(u) = \frac{1}{1+\kappa_\delta} \int_0^u m_\delta(u-y)\theta_\delta(y)dy + \frac{1}{1+\kappa_\delta}\Lambda_\delta(u), \quad u \geq 0$$

where κ_δ is defined as

$$\frac{1}{1+\kappa_\delta} = T_0 T_{\rho_1} \ldots T_{\rho_4} h_{2,\delta}(0) = m_\delta(0).$$

Besides, we have

$$\Lambda_\delta(u) = (1+\kappa_\delta)G_\delta(u),$$

and

$$\theta_\delta(y) = (1+\kappa_\delta)\zeta_\delta(y),$$

which is a proper density function. From this Proposition, we can get that the LT of the time to ruin $m_\tau(u)$ is the tail of a compound geometric distribution.

Proposition 5. The LT of the time to ruin $m_\tau(u)$ satisfies the defective renewal equation

$$\begin{aligned}
m_\tau(u) &= \int_0^u m_\tau(u-y)\zeta_\delta(y)dy + \int_u^\infty \zeta_\delta(y)dy \\
&= \frac{1}{1+\kappa_\delta} \int_0^u m_\delta(u-y)\theta_\delta(y)dy + \frac{1}{1+\kappa_\delta} \int_u^\infty \zeta_\delta(y)dy, \quad u \geq 0.
\end{aligned}$$

6 Numberical illustration and impact of the dependence structure

In this section, we start with some example. We assume that the r.v. X representing the individual claim amount follows a mixed exponential distribution with parameter λ_1, λ_2, that is, $f_X(t) = e^{-\lambda x} f_1(x) + (1 - e^{\lambda x}) f_2(x)$,x>0, with $f_1(x) = \lambda_1 e^{-\lambda_1 x}$, $f_2(x) = \lambda_2 e^{-\lambda_2 x}$, $\hat{f}_1(s) = \frac{\lambda_1}{\lambda_1+s}$, $\hat{f}_2(s) = \frac{\lambda_2}{\lambda_2+s}$. At first, We find an explicit expression for Taking LTs in both sides of the first equation in Proposition 5, we obtain that

$$\hat{m}_\tau(s) = \frac{m_\tau(0) - \hat{\zeta}_\delta(s)}{s\left[1 - \hat{\zeta}_\delta(s)\right]} = \frac{1 - \hat{\zeta}_\delta(s) - [1 - m_\tau(0)]}{s\left[1 - \hat{\zeta}_\delta(s)\right]}. \tag{27}$$

From Equation (25) we get

$$\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) = \left[1 - \hat{\zeta}_\delta(s)\right] \prod_{i=1}^{4} (\rho_i - s),$$

and then Equation (27) becomes

$$\hat{m}_\tau(s) = \frac{\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) - [1 - m_\tau(0)] \prod_{i=1}^{4} (\rho_i - s)}{s\left[\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s)\right]}. \tag{28}$$

Bt the Equation (19),(20) we can get that

$$\hat{h}_{1,\delta}(s) - \hat{h}_{2,\delta}(s) = \frac{Q_{4,\delta}(s)}{c^4(\lambda_1 + s)(\lambda_2 + s)}, \tag{29}$$

where

$$\begin{aligned} Q_{4,\delta}(s) = {} & (\lambda_1 + s)(\lambda_2 + s)(\delta + \beta - cs)^2(\delta + \lambda + \beta - cs)^2 \\ & - \beta^2\lambda_1(\lambda_2 + s)(\delta + \beta - cs)^2 - \beta^2\lambda_2(\lambda_1 + s)(-2cs\lambda + \lambda^2 + 2\lambda(\lambda + \beta)). \end{aligned}$$

Since $Q_{4,\delta}(s)$ is a polynomial of degree 4 and then we have that $Q_{4,\delta}(s) = 0$ has 4 roots in the complex plane, and from Proposition 1 and Equation (29) that the equation $Q_{4,\delta}(s) = 0$ has 4 roots ρ_1, ρ_2, ρ_3, ρ_4 with positive real part and two roots say $-R_i = -R_i(\delta)$, with $Re(R_i) > 0, i = 1, 2$. Thus, we can rewrite $Q_{4,\delta}(s)$ as

$$Q_{4,\delta}(s) = c^4(s + R_1)(s + R_2) \prod_{i=1}^{4} (\rho_i - s). \tag{30}$$

From Equation (29)and (30), Equation (28) yields

$$\hat{m}_\tau(s) = \frac{\prod_{j=1}^{2} (s + R_j) - [1 - m_\tau(0)] (\lambda_1 + s)(\lambda_2 + s)}{s \prod_{j=1}^{2} (s + R_j)}. \tag{31}$$

Now that $\hat{m}_\tau(s) < \infty$ for $s \geq 0$, the numerator in Equation (31) is zero for $s = 0$, that is

$$1 - m_\tau(0) = \frac{R_1 R_2}{\lambda_1 \lambda_2}$$

and then Equation (31) yields

$$\hat{m}_\tau(s) = \frac{\left(1 - \frac{R_1 R_2}{\lambda_1 \lambda_2}\right) s + R_1 + R_2 - \frac{R_1 R_2(\lambda_1 + \lambda_2)}{\lambda_1 \lambda_2}}{(s + R_1)(s + R_2)}.$$

We assume that R_1, R_2 are distinct and we can get that

$$\hat{m}_\tau(s) = \sum_{j=1}^{2} \frac{\xi_{i,\delta}}{s + R_j},$$

where

$$\xi_{1,\delta} = \frac{R_2}{R_2 - R_1}\left(1 - \frac{R_1(\lambda_1 + \lambda_2)}{\lambda_1\lambda_2} + \frac{R_1^2}{\lambda_1\lambda_2}\right),$$

$$\xi_{2,\delta} = \frac{R_1}{R_2 - R_1}\left(1 - \frac{R_2(\lambda_1 + \lambda_2)}{\lambda_1\lambda_2} + \frac{R_2^2}{\lambda_1\lambda_2}\right).$$

Inverting $\hat{m}_\tau(s)$ is

$$m_\tau(u) = \xi_{1,\delta}e^{-R_1 u} + \xi_{2,\delta}e^{-R_2 u}, u \geq 0, \tag{32}$$

and by letting $\delta \to 0$, the ruin probability $\Psi(u)$ can be obtained.

6.1 Numberical examples of two compared models

In this subsection, we start with a numberical example. We consider the extension from the Poisson arrival process to Erlang(2) interarrival claim times as well as we indicate the impact of the dependence parameter λ on the ruin probability and the LT of the ruin time, where $\delta = 0$.

Here we compare the ruin probabilities calculated in an Erlang(2) risk model with those calculated by the exponential compound Poisson risk. Other settings for the two compared models are identical.

We assume for the claim amount r.v. that $f_{X|W} = e^{-\lambda t}f_1(x) + \left(1 - e^{-\lambda t}\right)f_2(x)$, where $f_1(x) = \lambda_1 e^{-\lambda_1 x}$, $f_2(x) = \lambda_2 e^{-\lambda_2 x}$ (the expectation is μ_1 and μ_2) for both risk model and also we assume that the interclaim r.v. is $f_w(t) = \beta^2 t e^{-\beta t}$ for Erlang(2) model and $f_w(t) = \beta e^{-\beta t}$ for exponential model.

The ruin probability $\psi_p(u)$ for the Exponential Poisson risk model are taken from Cossette et al.(2010)[16] and for the Erlang(2) risk model, using $\delta = 0$ from Equation (32). We give expressions for the ruin probability $\psi(u)$ as function of $u \geq 0$ and for different values of the dependence parameter λ, both can see in Figure 1.

Let $\lambda_1 = 3$, $\lambda_2 = 1$, $c = 1.5$, $\beta = 2$, and then we have

with $\lambda = 0.5$

$$\psi(u) = -0.0711385440307139e^{-2.72611056853693u} + 0.1583937580081028e^{-0.8478757687088427u}$$

$$\psi_p(u) = -0.15951259519348063e^{-1.931774360594839u} + 0.4366144680653996e^{-0.6272051410032553u}$$

with $\lambda = 0.75$

$$\psi(u) = -0.05905603558214539 4e^{-2.74918048198971u} + 0.21623621102415286e^{-0.7908259477411941u}$$

$$\psi_p(u) = -0.1128431290755753e^{-2.010106953258826u} + 0.5464987972543871e^{-0.5084178328686777u}$$

with $\lambda = 1$

$$\psi(u) = -0.0499755210860354e^{-2.7690937726761637u} + 0.26379296752529546e^{-0.7434542500799464u}$$

$$\psi_p(u) = -0.08233338133410702e^{-2.0733044134625125u} + 0.6326231557343817e^{-0.41244806273596246u}$$

with $\lambda = 2$

$$\psi(u) = -0.029493060910398744e^{-2.827448729560705u} + 0.3866120955900488e^{-0.6195283091024653u}$$

$$\psi_p(u) = -0.024656369294917924e^{-2.2469643956889267u} + 0.852452192972948 2e^{-0.16407661550122832u}$$

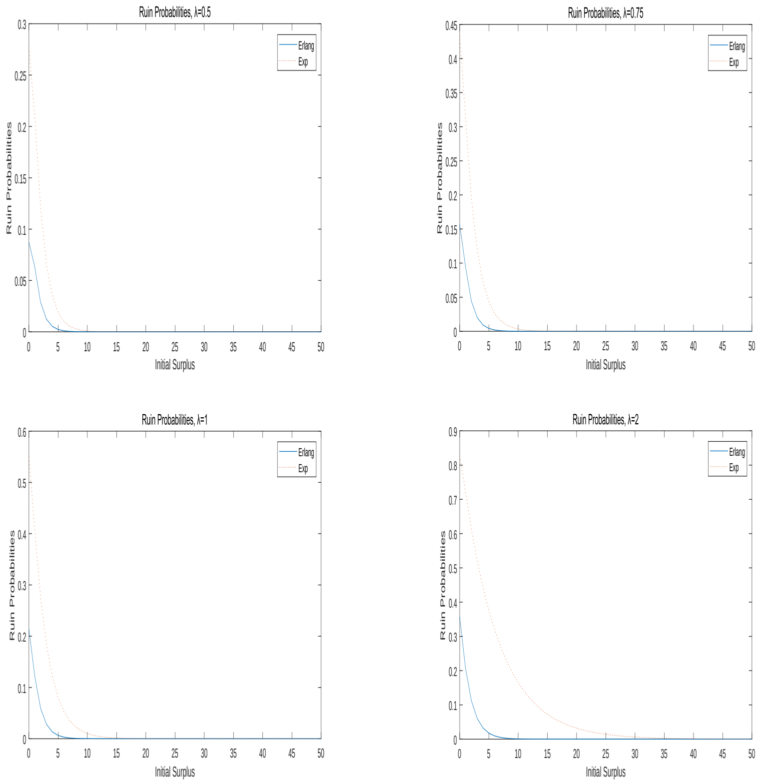

Figure 1: Ruin Probabilities in corresponding risk models

Table 1 is the numerical values of these ruin probabilities in corresponding risk models.

Figure 1 and Table 1 both show that the ruin probabilities $\psi(u)$ for the Erlang(2) risk model are much smaller than the exponential risk model for different initial surplus u and for all $\lambda > 0$, so we see that it is worthwhile to consider Erlang(2) risk models.

6.2 Impact of the dependence parameter λ

We plot the values $\psi(u)$ calculated in Figure 2, and we easily get that the dependence parameter λ has an impact on the ruin probabilities. It is clear that the lower the dependence parameter the lower the ruin probability is.

We may interpret the impact of the dependence relation λ as follows. When the dependence relation λ is low, the probability of having an important claim increases as the time elapsed since the last claim increases. Thus the ruin probability will be lower since the probability that the insurance company will have enough premium income to pay the claim will be higher.

Furthermore using $\delta = 0.05$ and for different values of the dependence parameter λ, we arrive the analytic expressions for the LT of the time of ruin $m_\delta(u)$ as function of the initial surplus u, $(u \geq 0)$, where $\lambda_1 = 3$, $\lambda_2 = 1$, $c = 1.5$, $\beta = 2$,

with $\lambda = 0.5$

$$m_\tau(u) = -0.07128752934999309e^{-2.7307092800651613u} + 0.14690015980947344e^{-0.8589112541726275u}$$

with $\lambda = 0.75$

$$m_\tau(u) = -0.059772554449779 79e^{-2.7531272965384335u} + 0.20007700234772152e^{-0.8065196739719653u}$$

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Table 1: Ruin probabilities in two risk models.

u	$\lambda = 0.5$		$\lambda = 0.75$		$\lambda = 1$		$\lambda = 2$	
	$\psi(u)$	$\psi_p(u)$	$\psi(u)$	$\psi_p(u)$	$\psi(u)$	$\psi_p(u)$	$\psi(u)$	$\psi_p(u)$
0	8.7255e-02	2.7710e-01	1.5718e-01	4.3366e-01	2.1382e-01	5.5029e-01	3.5712e-01	8.2780e-01
5	2.2834e-03	1.8963e-02	4.1463e-03	4.3006e-02	6.4102e-03	8.0447e-02	1.7458e-02	3.7530e-01
10	3.2920e-05	8.2448e-04	7.9509e-05	3.3850e-03	1.5577e-04	1.0231e-02	7.8831e-04	1.6523e-01
15	4.7459e-07	3.5828e-05	1.5246e-06	2.6641e-04	3.7852e-06	1.3010e-03	3.5597e-05	7.2746e-02
20	6.8420e-09	1.5569e-06	2.9235e-08	2.0967e-05	9.1982e-08	1.6545e-04	1.6074e-06	3.2027e-02
25	9.8638e-11	6.7656e-08	5.6059e-10	1.6501e-06	2.2352e-09	2.1040e-05	7.2583e-08	1.4100e-02
30	1.4220e-12	2.9400e-09	1.0749e-11	1.2987e-07	5.4315e-11	2.6756e-06	3.2775e-09	6.2079e-03

with $\lambda = 1$

$$m_\tau(u) = -0.05105583625093204e^{-2.7725204232005805u} + 0.24356338303999145e^{-0.7632558174934445u}$$

with $\lambda = 2$

$$m_\tau(u) = -0.031068233976860926e^{-2.829560489332212u} + 0.3547980400374833e^{-0.651126273816391u}$$

From Figure 3, we can see that the lower the dependence parameter λ, the lower the value of the LT of time to ruin is.

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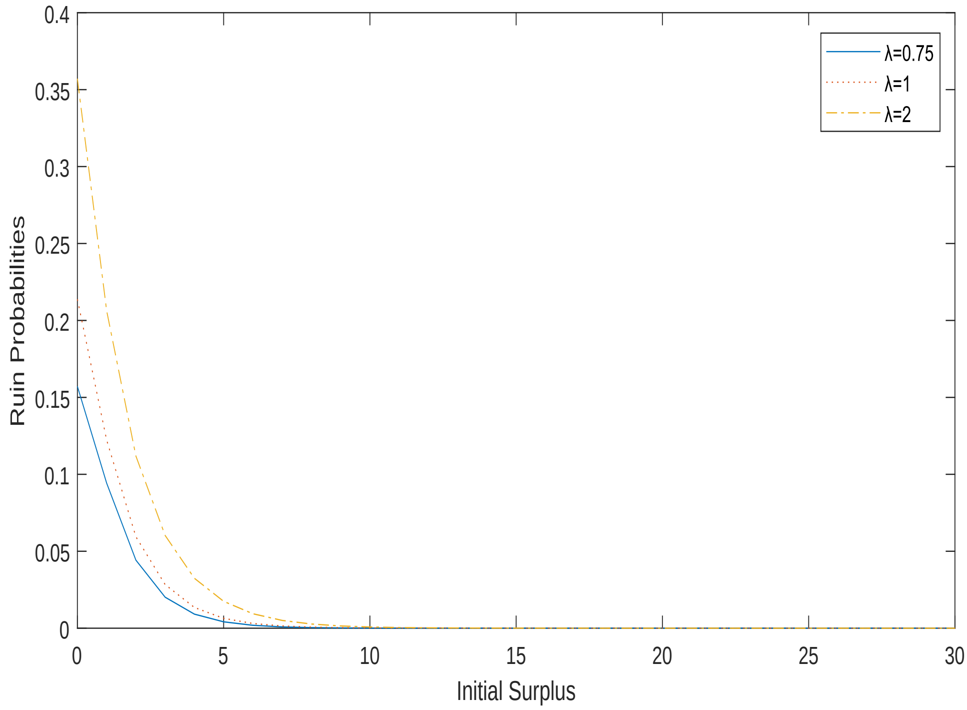

Figure 2: Ruin probabilities

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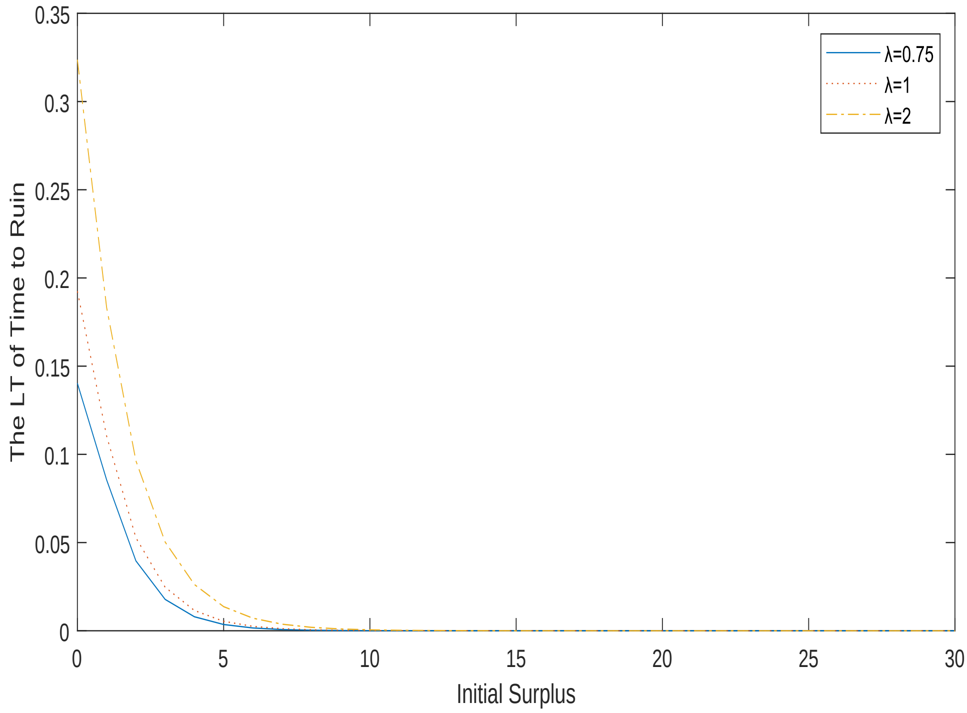

Figure 3: Laplace transform of time to ruin

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