Original Research Article

New Zero -Truncated Distribution: Properties and Applications

Abstract

A new zero-truncated distribution called zero-truncated Poisson-Pseudo Lindley distribution is introduced. Its statistical properties including general expression of probabilities, moments, cumulative function and the quantile function were examined. The parameters of zero-truncated Poisson Pseudo Lindley distribution have been estimated by using a moment's method. An application of the model to a real data set is presented and compared with the fit attained by some other distributions. A simulation study is carried out to examine the bias and mean square error of the maximum likelihood estimators of the parameters.

Keywords: Pseudo Lindley distribution, zero-truncated Poisson -Lindley distribution, Poisson Pseudo Lindley distribution, Maximum likelihood estimation.

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Introduction

In statistics sciences, truncated distributions are more commonly used when a random variable is restricted to be observed over a given range.

Zero-truncated distributions, in probability theory, are positive discrete distributions witch modeling data excluding counts. There are many works study the zero-truncated distributions: Ghitany, Al-Mutairi, and Nadarajah (2008), Shanker et al. (2015), Shanker and Fesshaye (2016), ….

Recently, Zeghdoudi and Nedjar (2016) introduced a new distribution, called pseudo Lindley distribution, and study the mathematical and statistical properties, estimation of parameters of pseudo Lindley distribution and its applications to model count data. This distribution is based on mixtures of gamma (2, θ) and exponential (θ) distributions, where the density function of this distribution is given by

$$f_{PsLD}(x;\theta,\beta) = \frac{\theta(\beta - 1 + \theta x)e^{-\theta x}}{\beta}, \quad x > 0, \theta > 0, \beta \geq 1 \qquad (1)$$

The Lindley distribution, introduced by Lindley (1958) and studied in detail by Ghitany, Atieh and Nadarjah (2008) is a particular case of (4) for $(\beta = \theta + 1)$. Zeghdoudi and Nedjar (2016) have discussed various interesting properties including cumulative, characteristic function, failure, rate function and stochastic ordering.

Also, Zeghdoudi and Nedjar (2017) and Grine and Zeghdoudi (2017) introduced another new distributions which Poisson-Lindley distribution (see Sankaran (1970)) is a particular case, by compounding Poisson, pseudo Lindley and quasi Lindley distributions which will add some value to the existing literature on modeling lifetime data and biological sciences.

The p.m.f of Poisson pseudo Lindley distribution given by

$$P_0(x;\theta,\beta) = \frac{\theta(\theta\beta + \beta - 1 + \theta x)}{\beta(\theta + 1)^{x+2}}, \quad x > 0, \theta > 0, \beta \geq 1. \qquad (2)$$

In this paper, a zero-truncated Poisson Pseudo Lindley distribution (ZTPPsLD), of which zero truncated Poisson-Lindley distribution (ZTPLD) is a particular case, has been obtained by compounding size-biased Poisson distribution (SBPD) with a continuous distribution.

The paper is organized as follows. In Section 2, we introduce the ZTPPsL distribution, and give immediate properties. In Section 3 the moment estimation has been obtained and the maximum likelihood estimation has been discussed to estimate the parameters of ZTPPsLD. In this last section, simulation studies are reported and the goodness of fit of this new distribution has been compared with ZTPLD and ZTPD using real data sets.

Main properties

Zero-Truncated Poisson pseudo Lindley Distribution

Using $P(x;\theta,\beta)=\frac{P_0(x;\theta,\beta)}{1-P_0(0;\theta,\beta)}$, $x=1,2,\ldots$, and the p.m.f of Poisson pseudo Lindley (2), we obtained the p.m.f of zero-truncated Poisson pseudo Lindley distribution **(ZTPPsLD)**

$$P(x;\theta,\beta)=\frac{\theta(\theta\beta+\beta-1+\theta x)}{(\theta\beta+\beta+\theta)(\theta+1)^x},\quad x=1,2,3,\ldots,\theta>0,\beta\geq\left(\frac{1}{\theta+1}\right)\quad (3)$$

Remark 1.

It can be easily verified that at $\beta=\theta+1$, (3) reduces to the p.m.f of zero-truncated Poisson-Lindley distribution (ZTPLD) introduced by Ghitany, Al-Mutairi, and Nadarajah (2008). where the p.m.f is

$$P(x;\theta)=\frac{\theta^2}{\theta^2+3\theta+1}\frac{x+\theta+2}{(\theta+1)^x},\quad x=1,2,3,\ldots,\theta>0\,.\qquad (4)$$

The nature and behavior of ZTPPsLD for varying values of parameters θ and β are explained graphically in fig.1. It is obvious from the graphs of the p.m.f of ZTPPsLD that it is monotonically decreasing for increasing values of the parameters θ and . If θ is constant and β increases, the graph of the p.m.f of ZTPPsLD shift upward slowly and decreasing slowly for increasing value of x. If β is constant and θ increases, the graph of the p.m.f of ZTPPsLD starts from higher value and decreasing fast with increasing value of x . That is the change in the parameter θ makes a difference in the shape of the p.m.f of ZTPPsLD and thus the parameter θ is the dominating parameter. The change in the value of parameter β does not make much impact on the shape of the p.m.f of ZTPPsLD.

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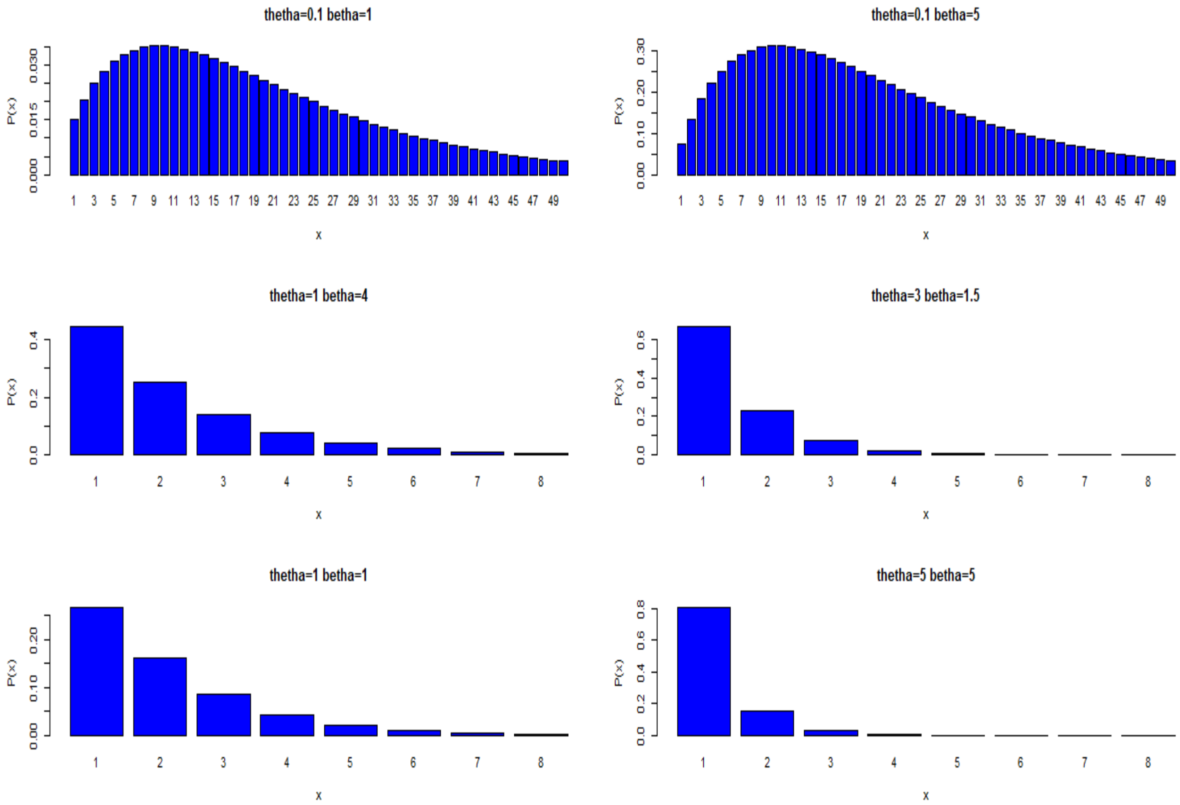

Fig 1. Plots of the p.m.f of ZTPPsLD for some parameter values

The cumulative distribution function (c.d.f) of ZTPPsLD

The cumulative distribution function of ZTPPsLD is given by

$$F(x;\theta,\beta) = 1 - \frac{(\theta\beta + \theta + \beta + \theta x)}{(\theta\beta + \beta + \theta)(\theta + 1)^x}, x = 1,2,3, \dots, \theta > 0, \beta \geq \left(\frac{1}{\theta + 1}\right) \quad (5)$$

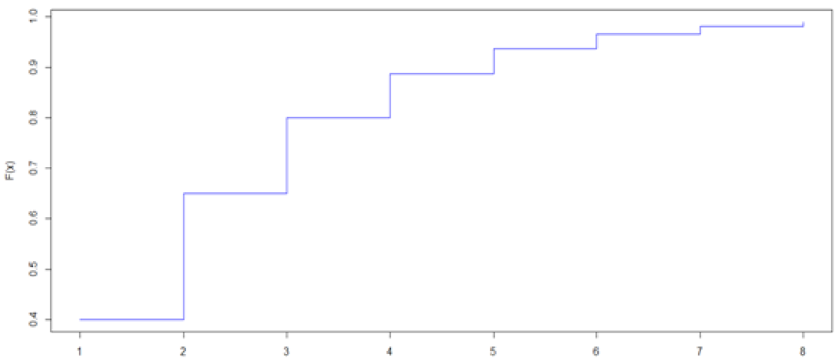

Fig 2. Plot of the c.d.f of the ZTPPsLD for θ=1 and β=2

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$\frac{P(x+1;\theta,\beta)}{P(x;\theta,\beta)} = \frac{1}{1+\theta}\left[1+\frac{\theta}{(\theta\beta+\beta-1+\theta x)}\right]$ is a decreasing function in x, the ZTPPsLD is unimodal. (Johnson, Kotz and Kemp (2005)).

It should be noted that it is very tedious and complicated to find the moments of ZTPPsLD (5) directly. To find moments of ZTPPsLD in an easy and interesting way, firstly we have obtained the p.m.f of ZTPPsLD as a size-biased Poisson mixture of an assumed continuous distribution (Ghitany, Al-Mutairi, and Nadarajah (2008),

The ZTPPsLD (5) can also be obtained from size-biased Poisson distribution (SBPD) having p.m.f

$$f(x/\lambda) = \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}, \quad x = 1,2,3,\dots,\lambda > 0$$

when the parameter λ follows a distribution having p.d.f

$$h(\lambda;\theta) = \frac{\theta}{(\theta\beta+\beta+\theta)}[(\theta\beta+\beta-1)+\theta(\theta+1)\lambda]e^{-\lambda\theta}, \quad \lambda > 0, \theta > 0.$$

Thus the p.m.f of ZTPPsLD is obtained as

$$P(X=x) = \int_0^\infty f(x/\lambda)\, h(\lambda,\theta) d\lambda$$

$$= \int_0^\infty \frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\frac{\theta}{(\theta\beta+\beta+\theta)}[(\theta\beta+\beta-1)+\theta(\theta+1)\lambda]e^{-\lambda\theta} d\lambda \qquad (6)$$

$$= \frac{\theta}{(\theta\beta+\beta+\theta)(x-1)!}\int_0^\infty e^{-(\theta+1)\lambda}[(\theta\beta+\beta-1)\lambda^{x-1}+\theta(\theta+1)\lambda^{x}] d\lambda$$

$$= \frac{\theta}{(\theta\beta+\beta+\theta)}\left[\frac{(\theta\beta+\beta-1)}{(\theta+1)^x}+\frac{\theta x}{(\theta+1)^x}\right]$$

$$P(x;\theta,\beta) = \frac{\theta(\theta\beta+\beta-1+\theta x)}{(\theta\beta+\beta+\theta)(\theta+1)^x}, \quad x = 1,2,3,\dots,\theta > 0, \beta \geq \frac{1}{\theta+1} \;.$$

Where is the p.m.f of ZTPPsLD.

Moments of ZTPPsLD

The $r\,th$ factorial moment of the ZTPPsLD can be obtained as

$$\acute{\mu}_{(r)} = E\left[X^{(r)}/\lambda\right], \text{ where } X^{(r)} = X(X-1)(X-2)\dots(X-r+1)$$

Using (5), we get

$$\acute{\mu}_{(r)} = \int_0^\infty \sum_{x=1}^{\infty}\left[x^{(r)}\frac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}\right]\frac{\theta}{(\theta\beta+\beta+\theta)}[(\theta\beta+\beta-1)+\theta(\theta+1)\lambda]e^{-\lambda\theta} d\lambda$$

$$= \int_0^\infty \left[\lambda^{r-1}\sum_{x=r}^{\infty} x\frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right]\frac{\theta}{(\theta\beta+\beta+\theta)}[(\theta\beta+\beta-1)+\theta(\theta+1)\lambda]e^{-\lambda\theta} d\lambda$$

Taking $x + r$ in place of x, we get

$$\acute{\mu}_{(r)} = \int_0^\infty \left[\lambda^{r-1} \sum_{x=0}^{\infty} (x + r) \frac{e^{-\lambda}\lambda^x}{x!}\right] \frac{\theta}{(\theta\beta + \beta + \theta)} [(\theta\beta + \beta - 1) + \theta(\theta + 1)\lambda] e^{-\lambda\theta} d\lambda$$

It is obvious that the expression within the bracket is $\lambda + r$ and hence, we have

$$\acute{\mu}_{(r)} = \frac{\theta}{(\theta\beta + \beta + \theta)} \int_0^\infty \lambda^{r-1}(\lambda + r)\,[(\theta\beta + \beta - 1) + \theta(\theta + 1)\lambda] e^{-\lambda\theta} d\lambda$$

Using gamma integral and little algebraic simplification, we get finally, a general expression for the factorial moment of the ZTPPsLD as

$$\acute{\mu}_{(r)} = \frac{r!\,(\theta + 1)^2 (r + \beta)}{\theta^r (\theta\beta + \theta + \beta)}\ ; r = 1,2,3, \ldots. \qquad (7)$$

Substituting $r = 1,2,3$, and 4 in (7), the first four factorial moment can be obtained and then using the relationship between factorial moments and moments about origin, the first four moments about origin of the ZTPPsLD are given by

$$\mu_1 = \frac{(\theta + 1)^2 (\beta + 1)}{\theta(\theta\beta + \theta + \beta)}$$

$$\mu_2 = \frac{(\theta + 1)^2 [\theta(\beta + 1) + 2(\beta + 2)]}{\theta^2 (\theta\beta + \theta + \beta)}$$

$$\mu_3 = \frac{(\theta + 1)^2 [\theta^2(\beta + 1) + 6\theta(\beta + 2) + 6(\beta + 3)]}{\theta^3 (\theta\beta + \theta + \beta)}$$

$$\mu_4 = \frac{(\theta + 1)^2 [\theta^3(\beta + 1) + 14\theta^2(\beta + 2) + 36(\beta + 3) + 24(\beta + 4)]}{\theta^4 (\theta\beta + \theta + \beta)}$$

Then the *n* th moments about origin of the ZTPPsLD are given by

$$\mu_n = \frac{(\theta + 1)^2 \sum_{r=1}^{n} p(n,r)(r + \beta)\theta^{n-r} r!}{\theta^r (\theta\beta + \theta + \beta)}$$

with

$$\mu_n = \sum_{r=1}^{n} p(n,r) \acute{\mu}_{(r)}$$

$$p(n,r) = p(n-1, r-1) + r(n-1, r)$$

and $$p(n,1) = 1; p(1,r) = 0;\ r > 1.$$

Again using the relationship between moments about origin and moments about mean, the mean and variance of ZTPPsL distribution are obtained as

$$E(X) = \frac{(\theta + 1)^2 (\beta + 1)}{\theta(\theta\beta + \theta + \beta)}$$

$$V(X) = \frac{(\theta + 1)^2 [\beta^2\theta^2 + \theta(\beta^2 + 3\beta + 2) + 2\beta]}{\theta^2 (\theta\beta + \theta + \beta)^2}$$

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Generating functions

The probability generating function of the ZTPPsLD is obtained as

$$P_X(t) = E(t^X) = \frac{\theta}{(\theta\beta+\theta+\beta)} \sum_{x=1}^{\infty} t^x \frac{(\theta\beta+\beta-1+\theta x)}{(\theta+1)^x}$$

$$= \frac{\theta}{(\theta\beta+\theta+\beta)} \left[(\theta\beta+\beta-1) \sum_{x=1}^{\infty} \left(\frac{t}{\theta+1}\right)^x + \theta \sum_{x=1}^{\infty} x \left(\frac{t}{\theta+1}\right)^x \right]$$

So, the probability generating function of the ZTPPsLD is

$$P_X(t) = \frac{\theta t}{(\theta\beta+\theta+\beta)} \left[\frac{(\theta\beta+\beta-1)}{\theta+1-t} + \frac{\theta(\theta+1)}{(\theta+1-t)^2} \right]$$

The moment generating function of the ZTPPsLD is thus given by

$$P_X(t) = \frac{\theta e^t}{(\theta\beta+\theta+\beta)} \left[\frac{(\theta\beta+\beta-1)}{\theta+1-e^t} + \frac{\theta(\theta+1)}{(\theta+1-e^t)^2} \right]$$

Table 1. Displays the mean, variance coefficients of variation, skewness, and kurtosis for ZTPPsLD for different choices of parameter θ and β

	$\theta = 0.5; \beta = 1.1$	$\theta = 1; \beta = 1.5$	$\theta = 1.5; \beta = 2$
Mean	*4.3953*	*2.5*	*1.9230*
Variance	*11.030*	*3.25*	*1.6437*
C.V	*0.7556*	*0.7211*	*0.6666*
Skewness	*57.856*	*10.5*	*4.1319*
Kurtosis	*802.35*	*79.563*	*22.555*

The Quantile Function of the Zero Truncated Poisson Pseudo Lindley Distribution

Theorem 1.

From any $\theta > 0, \beta \geq \frac{1}{\theta+1}$, *the quantile function of the Zero Truncated Poisson Pseudo Lindley distribution X is*

$$Q_X(u) = -\frac{\theta\beta+\theta+\beta}{\theta} - \frac{1}{log(\theta+1)} W_{-1}\left(\frac{(\theta\beta+\theta+\beta)log(\theta+1)}{\theta(\theta+1)^{\frac{\theta\beta+\theta+\beta}{\theta}}} (u-1) \right)$$

where $0 < u < 1$*and* W_{-1} *denote negative branch of lambert W function.*

Proof.

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For any fixed $\theta > 0, \beta \geq \frac{1}{\theta+1}$, let $u \in (0,1)$. We have to solve the equation $F_X(x) = u$, with respect to x, for $x > 0$. Then, by taking the ceiling of that solution we acheive the desired result. From the expressin of F_{ZTPPsL} given in formula (4), we have to solve the following:

$$\frac{(\theta\beta + \theta + \beta + \theta x)}{(\theta\beta + \beta + \theta)(\theta + 1)^x} = 1 - u\,, \quad x = 1,2,3,\dots, \theta > 0, \beta \geq (\frac{1}{\theta + 1}) \quad (8)$$

Equation (8) can be written as follows:

$$x + \frac{\theta\beta + \theta + \beta}{\theta}(u - 1)(\theta + 1)^x = -\frac{\theta\beta + \theta + \beta}{\theta} \quad (9)$$

Now we apply *Theorem 1* (see Jodrá, P. (2010)) to solve equation (9) with respect to x. As a consequence, the following equality is achieved:

$$W\left(\frac{(\theta\beta + \theta + \beta)log(\theta + 1)}{\theta(\theta + 1)^{\frac{\theta\beta+\theta+\beta}{\theta}}}(u - 1)\right) = -\left(\frac{\theta\beta + \theta + \beta}{\theta} + x\right)log(\theta + 1)\,, \quad (10)$$

where W denotes the Lambert W function.

Let us now consider (10). For any $\theta > 0, \beta \geq \frac{1}{\theta+1}$, $x \geq 1$ and $u \in (0,1)$ the following inequalities hold:

(i) $-\frac{1}{e} < \frac{(\theta\beta+\theta+\beta)log(\theta+1)}{\theta(\theta+1)^{\frac{\theta\beta+\theta+\beta}{\theta}}}(u - 1) < 0,$

(ii) $\left(\frac{\theta\beta+\theta+\beta}{\theta} + x\right)log(\theta + 1) > 1$

By virtue of inequalities (i) and (ii) above together with the properties of the Lambert W function, the real branch of W involved in (10) is precisely the negative branch W_{-1}, which leads to the desired result. This completes the proof of Theorem 1.

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Estimation and Simulation

Method of moment

Since ZTPPsLD (5) has two parameters θ, and β to be estimated, the methode of moment is based on equating the ratio of the first two probabilities and the mean. From (3) we have

$$p_1 = \frac{\theta(\theta\beta + \beta - 1 + \theta)}{(\theta + 1)(\theta\beta + \theta + \beta)} \quad (11)$$

and

$$p_2 = \frac{\theta(\theta\beta + \beta - 1 + 2\theta)}{(\theta + 1)(\theta\beta + \theta + \beta)} \quad (12)$$

Taking the ratio of these probabilities, we get

$$\frac{p_1}{p_1} = \frac{(\theta\beta + \beta - 1 + 2\theta)}{(\theta + 1)(\theta\beta + \beta - 1 + \theta)} \qquad (13)$$

After a little algebraic simplification, above equation reduces to

$$\beta(\theta + 1) = \frac{p_1(2\theta - 1) + p_2(\theta + 1)(1 - \theta)}{p_2(\theta + 1) - p_1} \qquad (14)$$

Again, simplifying the expression of the mean of ZTPPsLD (4), we get

$$\beta(\theta + 1) = \frac{(1 - \mu)\theta^2 + 2\theta + 1}{(\mu\theta - \theta - 1)} \qquad (15)$$

Now, equating equation (14) and (15), we get a quadratic equation in θ as

$$A\theta^2 + B\theta + c = 0 \qquad (16)$$

where

$A = (1 - \mu)p_1 + (2 - \mu)p_2$

$B = (\mu - 1)p_1 + (4 - \mu)p_2$

$A = 2(p_1 - p_2)$

From equation (16), MM $\hat{\theta}$ of θ can be obtained from

$$\hat{\theta} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \qquad (17)$$

Replacing the population mean μ by the corresponding sample mean $\bar{x}$ and p_1 *and* p_2 from the respective probabilities from the given data set, the values of *A*, *B*, and *C* in (16) can be obtained and substituting these values in (17) will give a MM estimate $\hat{\theta}$ of θ. After obtaining the MM $\hat{\theta}$, MOME $\widehat{\beta}$ of β can be obtained either from

$$\widehat{\beta} = \frac{1}{\theta + 1}\left[\frac{p_1(2\theta - 1) + p_2(\theta + 1)(1 - \theta)}{p_2(\theta + 1) - p_1}\right]$$

Or

$$\widehat{\beta} = \frac{1}{\theta + 1}\left[\frac{(1 - \mu)\theta^2 + 2\theta + 1}{(\mu\theta - \theta - 1)}\right]$$

Maximum Likelihood Estimate (MLE)

Given a random sample $x_1, x_2, \ldots, x_n$, of size n from the ZTPPsLD with the probability mass function (6). The likelihood function $\boldsymbol{L}$ of ZTPPLD is given by:

$$L(x; \theta, \beta) = \left(\frac{\theta}{\theta\beta + \theta + \beta}\right)^n \prod_{i=1}^{n} \left(\frac{\theta\beta + \beta - 1 + \theta x_i}{(\theta + 1)^{\sum_{i=1}^{n} x_i}}\right)$$

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The ln-likelihood function is

$$lnL(x;\theta,\beta) = n\,ln\left(\frac{\theta}{\theta\beta+\theta+\beta}\right) + \sum_{i=1}^{n} ln(\theta\beta+\beta-1+\theta x_i) - ln(\theta+1)\sum_{i=1}^{n} x_i$$

The maximum likelihood estimates $\hat{\theta}$ of θ and $\hat{\beta}$ of β of ZTPPsLD (6) is the solution of the following ln likelihood equations

$$\frac{\partial lnL(x;\theta,\beta)}{\partial\theta} = \frac{n\beta}{\theta(\theta\beta+\theta+\beta)} + \sum_{i=1}^{n}\left(\frac{\beta+x_i}{\theta\beta+\beta-1+\theta x_i}\right) - \frac{\sum_{i=1}^{n} x_i}{\theta+1} = 0$$

$$\frac{\partial lnL(x;\theta,\beta)}{\partial\beta} = -\frac{n(\theta+1)}{(\theta\beta+\theta+\beta)} + \sum_{i=1}^{n}\left(\frac{\theta+1}{\theta\beta+\beta-1+\theta x_i}\right) = 0$$

This non-linear system can be solved by any numerical iteration methods such as Newton-Raphson method, Regula-falsi method,…, etc.

Also, the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 lnL(x;\theta,\beta)}{\partial^2\theta} = -\frac{n(\beta+1)}{(\theta\beta+\theta+\beta)^2} + \sum_{i=1}^{n}\frac{(\beta+x_i)^2}{(\theta\beta+\beta-1+\theta x_i)^2} + \frac{\sum_{i=1}^{n} x_i}{(\theta+1)^2}$$

$$\frac{\partial^2 lnL(x;\theta,\beta)}{\partial^2\beta} = \frac{n(\theta+1)^2}{(\theta\beta+\theta+\beta)^2} + \sum_{i=1}^{n}\frac{(\theta+1)^2}{(\theta\beta+\beta-1+\theta x_i)^2}$$

$$\frac{\partial^2 lnL(x;\theta,\beta)}{\partial\theta\partial\beta} = \frac{n}{(\theta\beta+\theta+\beta)^2} + \sum_{i=1}^{n}\frac{x+1}{(\theta\beta+\beta-1+\theta x_i)^2}$$

For the maximum likelihood estimates $\hat{\theta}$ of θ and $\hat{\beta}$ of β, following equations can be solved

$$\begin{bmatrix} \frac{\partial^2 lnL(x;\theta,\beta)}{\partial^2\theta} & \frac{\partial^2 lnL(x;\theta,\beta)}{\partial\theta\partial\beta} \\ \frac{\partial^2 lnL(x;\theta,\beta)}{\partial\theta\partial\beta} & \frac{\partial^2 lnL(x;\theta,\beta)}{\partial^2\beta} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\beta}=\beta_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\beta}-\beta_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial lnL(x;\theta,\beta)}{\partial\theta} \\ \frac{\partial lnL(x;\theta,\beta)}{\partial\beta} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\beta}=\beta_0}}$$

where θ_0 and β_0 are the initial values of θ and β respectively.
In this paper, we will use the R package BB which has high capabilities for solving a nonlinear system of equations.

Algorithm for generating random data

Assume the conditional distribution of X given the size-biased Poisson distribution (SBPD).We proceed to our algorithm for generating random data from the ZTPPsL distribution; let us realize the following steps

Step 1. Generate λ with the p.d.f (4) as follows:

(i). Generate u~Uniform (0,1),

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(i). If $u \leq p = \frac{\theta\beta+\beta-1}{\theta\beta+\theta+\beta}$, generate $\lambda \sim exp(\theta)$; otherwise, generate $\lambda \sim gamma(2,\theta)$,

Step 2. Generate $x=1+z$, where $z \sim Poisson(\lambda)$.

Remark 2.

We can generate a variant x from the ZTPPsLD using the quantile function $Q_X(u)$.

Simulation study

Using the above algorithm to generate samples from ZTPPsL distribution, a simulation study is carried out *N=10.000* times for each triplet (θ,β,n) where (θ,β)=(0.5,1), (1,1.5), (2,2) and *n*=10,50,100. The study calculates the following measures:

- Choose the initial values of θ_0, β_0 for the corresponding elements of the parameter vector Θ= (θ,β) to specify ZTPPsL distribution;
- choose sample size *n*;
- generate N independent samples of size n from ZTPPsLD (θ,β);
- compute the ML estimate $\hat{\Theta}_n$ of Θ_0 for each of the *N* samples;
- compute the mean of the obtained estimators over all *N* samples,

$$average\ bias(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\Theta_i} - \Theta_0\right) ,$$

and the average square error

$$MSE(\theta) = \frac{1}{N}\sum_{i=1}^{N}\left(\widehat{\Theta_i} - \Theta_0\right)^2 .$$

We used the R 3.4.2 package BB which presents very high performances for nonlinear systems. The results are as follows:

Table 2. Average bias of the simulated estimators with $(\theta = 0.5, \beta = 1)$

	$bias(\theta)$	$bias(\beta)$	$MSE(\theta)$	$MSE(\beta)$
n=10	*-0.06554(0.43445)*	*-0.00381(0.99615)*	*0.02588*	*0.34143*
n=50	*-0.08044(0.41955)*	*-0.03504(0.96498)*	*0.01001*	*0.34045*
n=100	*-0.08001(0.41899)*	*-0.03762(0.96237)*	*0.00835*	*0.33976*

Table 3. Average bias of the simulated estimators with $(\theta = 1, \beta = 1.5)$

	$bias\theta(\hat{\theta})$	$bias\beta(\hat{\beta})$	$MSE(\theta)$	$MSE(\beta)$
n=10	*-0.02134(1.02134)*	*0.11419(1.61419)*	*0.25385*	*0.48250*
n=50	*-0.06983(0.93016)*	*0.01664(1.51664)*	*0.03200*	*0.35220*
n=100	*-0.07649(0.92350)*	*-0.00047(1.49952)*	*0.01879*	*0.34520*

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Table 4. Average bias and MSE of the simulated estimators with ($\theta = 2, \beta = 2$)

	$bias\theta(\hat{\theta})$	$bias\beta(\hat{\beta})$	$MSE(\theta)$	$MSE(\beta)$
n=10	*1.48105(3.48105)*	*3.92869(5.92869)*	*324.7766*	*324.7766*
n=50	*0.06615(2.06615)*	*0.13063(2.13063)*	*0.23953*	*0.23953*
n=100	*0.02846(2.02846)*	*0.08026(2.08026)*	*0.10551*	*0.10551*

Shown in Tables 2, 3, and 4, the bias (θ) → 0 for $\theta \to 0$, and the MSE (θ) → 0 where $\theta \to 0$ and $n \to \infty$.

Goodness of fit

The goodness of fit of ZTPPsL distribution has been discussed with a count data and the fit has been compared with ZTP and ZTPL distributions. In this section, the goodness of fit of ZTP, ZTPL, and ZTPPsL distributions have been compared to a data set due to Mathews and Appleton (1993) who gave counts of sites with 1,2,3,4 and 5 particles from Immunogold data. The values of the ML estimate of parameters, chi-square **(**χ^2), p-values are given in the **Table 5**, where the value of the chi-square of ZTPPsL distribution is smaller than that of ZTPD and ZTPL distribution but the p-value is otherwise. That is due to the difference in the number of parameters, the ZTP and ZTPL distributions are one parameter distributions while the ZTPPsLD is a two parameter distribution, and we confirm this goodness of fit in **Fig 3.** The negative log- likelihood (-logL), the Akaike Information Criterion (AIC), the Corrected Akaike Information Criterion (AICC) and the Bayesian Information Criterion (BIC) are given in the **Table 6** where the data is taken from Finney and Varley (1955) who gave counts of number of flower having number of fly eggs.

From the **Table 6**, the value of the negative log-likelihood is minimum of ZTPPsLD as compared to ZTPLD and ZTPD. According to the AIC, AICC, and BIC values, the best model for prediction and most plausibly generated the data is the ZTPPsLD, so our distribution can be considered as an important model for biological science data.

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Table 5. Immunogold data

Number of attached particles	*Observed Frequency*	*ZTPD*	*ZTPLD*	*ZTPPsLD*
1	*122*	*115.9*	*124.7*	*121.9*
2	*50*	*57.4*	*46.8*	*50.3*
3	*18*	*18.9*	*17.1*	*17.9*
4	*4*	*4.7*	*6.1*	*5.9*
5	*4*	*1.1*	*3.3*	*2*
Total	*198*	*198*	*198*	*198*

ML Estimate	$\hat{\theta}$ =0.9905	$\hat{\theta}$ =2.1830	$\hat{\theta}$ =2.899,$\hat{\beta}$ = 0.750
χ^2	2.140	0.5614	**0.0031**
d.l	2	2	1
P-value	0.343	0.7552	0.9550

Let the number of counts of flower heads as per the number of fly eggs reported by Finney and Varley (1955).

x_i	1	2	3	4	5	6	7	8	9
n_i	22	18	18	11	9	6	3	0	1

Table 6. Maximum likelihood estimates, AIC, AICC, BIC statistics values under considered models on real data

Distribution	ZTPD($\hat{\theta}$=2.8604)	ZTPLD($\hat{\theta}$=0.7185)	ZTPPsLD($\hat{\theta}$=1.0208, $\hat{\beta}$=0.4237)
LogL	166.5481	167.3841	165.2412
AIC	335.0962	336.7628	334.4824
AICC	335.1427	336.8093	334.6235
BIC	335.0406	336.7072	334.3712

The corresponding fitted p.m.f of ZTPD, ZTPLD, and ZTPPsLD along with the original data points in the data set in **Table 1** are given in **Fig 3**.

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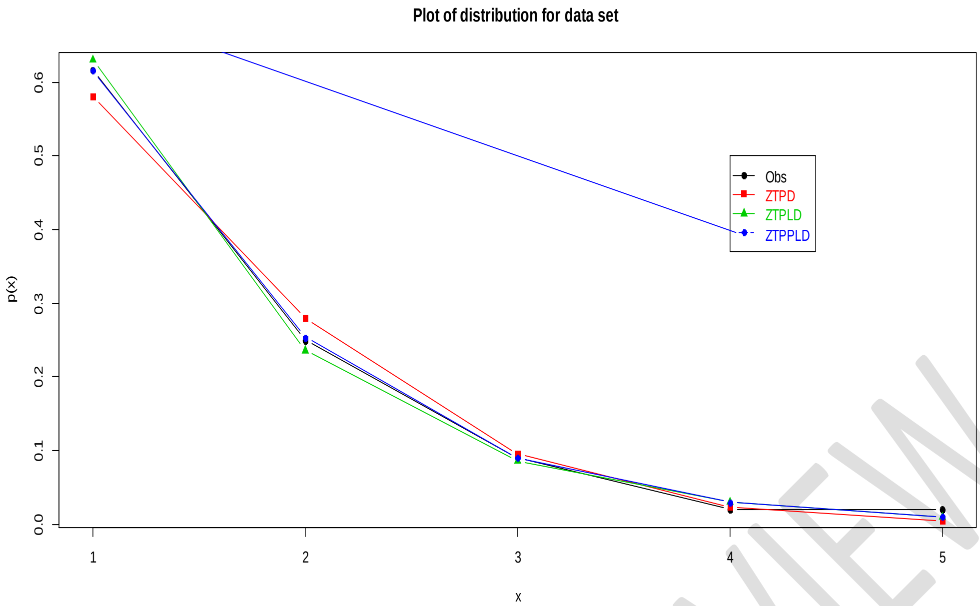

Fig 3: Corresponding fitted pmf of ZTPD, ZTPLD and ZTPPsLD for data set in Table1

Conclusion

The paper introduces a two-parameters Zero-Truncated Poisson Pseudo Lindley distribution (ZTPPsLD). The nature and behavior of ZTPPsLD has been studied by drawing different graphs of their probability mass function for the different values of their parameters. We discussed more statistical properties of the distribution, including the cumulative distribution function, moments and the quantile function.

Method of maximum likelihood and moment estimation has been discussed for estimating the parameters of ZTPPsLD and the goodness of fit has been discussed with a data from biological science and the fit shows a quite satisfactory fit over ZTPD and ZTPLD.

For future studies, we can make an actuarial application to risk classification models for the annual number of claims reported to the insurer and in Bonus-Malus Systems in Automobile Insurance.

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