Original Research Article

Inverted Power Ishita Distribution and Its Application to Lifetime Data

Abstract

In this paper, a new distribution, named 'the Inverted Power Ishita distribution', was introduced. It is an extension of the Ishita distribution and its capable of modelling real life data with upside down bathtub shape and heavy tails was introduced. Mathematical and statistical characteristics such as the quantile function, mode, moments and moment generating function, entropy measure, stochastic ordering and distribution of order statistics have been derived. Furthermore, reliability measures like survival function, hazard function and odds function have been derived. The method of maximum likelihood was used for estimating the parameters of the distribution. To demonstrate the applicability of the distribution, a numerical example was given. Based on the results, the proposed distribution performed better than the competing distributions.

Keyword: Ishita Distribution, Inverted Ishita distribution, Inverted power Ishita distribution, lifetime data, order statistics,

1 Introduction

Modelling life time data has been the interest of many statistical investigations, and this has led to the proposal of some statistical distributions. The behavior of the hazard rate is a strong determiner when modeling of lifetime data is considered. In real life, we can say we have some life time data with monotone (increasing and non-increasing) hazard rates while some have non-monotone (bathtub and upside-down bathtub or unimodal) hazard rates. Several statistical distributions exist for modeling lifetime data for any of these categories of data. The Ishita distribution introduced by [1] is a statistical distribution used in the modeling of lifetime data in biomedical science and engineering. It is distinctively used for lifetime data with monotone hazard rates, and cannot be used to appropriately model data with non-monotone hazard rates. As at the time of the conduct of this study, many research works have been done with the aim of developing better extensions of the Ishita distribution with the aim of developing a more flexible model and also a transformation of the distribution. For instance, [2] developed a transmuted Ishita distribution which is more flexible for modeling life time data. Also, [3] developed a transformed version of the Ishita distribution, using the inverse transformation in order to obtain a distribution that could model non monotone hazard rate data. Further, the work by [4] proposed a two-parameter power Ishita distribution and applied to modeling lifetime data.

Undoubtedly, the Ishita distribution and its extensions, asides the inverse Ishita distribution, reviewed in this paper do not provide a reasonable fit for lifetime data with non-monotone hazard rates, such as the upside-down bathtub hazard rates, which are common in many statistical investigations. For example, the lifetime models that present upside-down bathtub hazard rates curves can be observed when modeling a disease whose mortality reaches a peak after some finite period and then declines gradually. The Inverse Ishita Distribution proposed by [3] could model such life data, that is, the non-monotone hazard rates. The need for an extended form of the inverse Ishita distribution is to develop a more flexible model that can best capture lifetime data with non-monotone upside-down bathtub hazard rates in some applied areas. Other forms of statistical distributions that have been used by researchers to model lifetime data exhibiting upside-down bathtub hazard rate are those of [5,6,7,8,9,10] and [11] among others.

The aim of this article is to introduce an inverted power Ishita distribution that fits well lifetime data with upside-down bathtub hazard rate and, however, skewed data. The rest of the paper is organized as follows. In Section 2, some properties of the inverted power Ishita (IPI) distribution are derived. Section 3 deals with reliability analyses such as survival function, hazard rate function and odds function. Section 4 deals with the maximum likelihood estimation of the parameters of the IPI distribution, derivation of the Fisher Information matrix and construction of confidence intervals for the parameters of the distribution. The analysis of two real data sets is presented in Section 5. Finally, in Section 6, we conclude the paper.

To derive the inverted power Ishita distribution, we recall that for a random variable Y, Rama S. [1] defines the probability density function (PDF) of the Ishita distribution as

$$f(y,\theta)=\frac{\theta^3}{\theta^3+2}(\theta+y^2)e^{-\theta y}, y>0, \theta>0 \qquad (1)$$

Suppose X is related to Y by the inverse power function $X=h_2(y)=Y^{-\frac{1}{\alpha}}$, then, the pdf of X is

$$f_{IPID}(x)=f_{IPID}(h_2^{-1}(x))\left|\frac{dx}{dy}\right| \quad \text{Hogg and Craig(2013)} \qquad (2)$$

one obtains the PDF of the inverted Power Rama distributed random variable X as

$$f_{IPID}(x;\alpha,\theta)=\frac{\alpha\theta^3}{\theta^3+2}\left(\frac{\theta}{x^{\alpha+1}}+\frac{1}{x^{3\alpha+1}}\right)e^{-\frac{\theta}{x^{\alpha}}}; x,\alpha,\theta>0 \qquad (3)$$

The corresponding cumulative density function (CDF) of the inverted Ishita distributed random variable X is

$$F_{IPID}(x,\theta)=\frac{\theta^3}{\theta^3+2}\left(1+\frac{1}{x^{2\alpha}\theta}+\frac{2}{x^{\alpha}\theta^2}+\frac{2}{\theta^3}\right)e^{-\frac{\theta}{x^{\alpha}}}; x,\alpha,\theta>0 \qquad (4)$$

density function of Power Inverse Ishita distribution

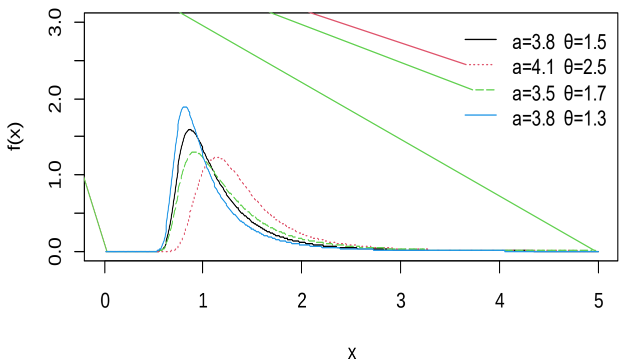

Fig 1a:pdf plot of IPI

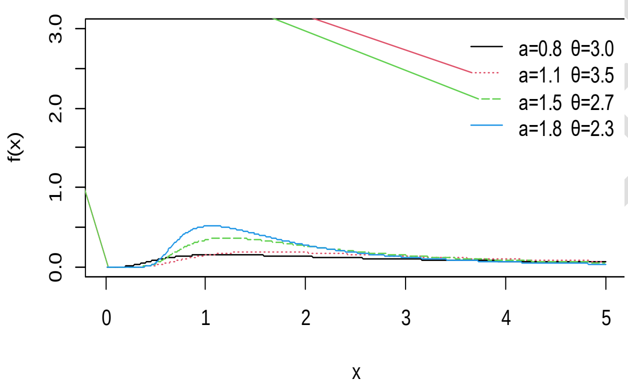

Fig 1b:pdf plot of PII

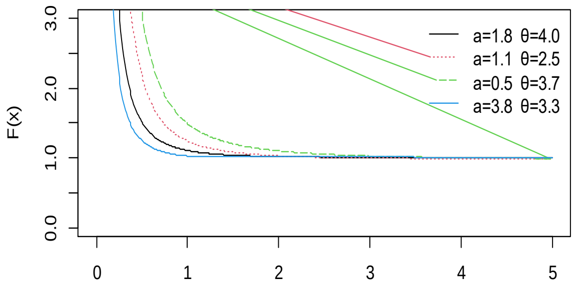

Fig 1a:cdf plot of IPI

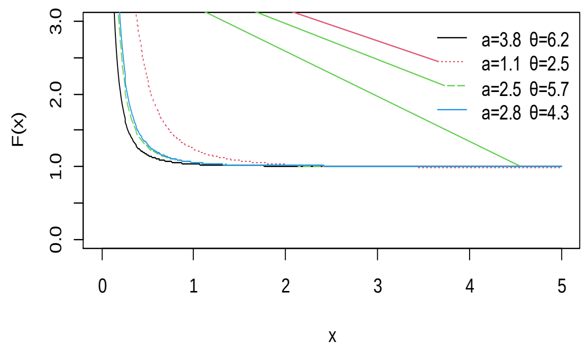

Fig 2b:cdf plot of IPI

Figs. 1a, 1b, 1c and 1d. Show the pdf and cdf plots of the inverted power Ishita distribution for varying values of θ and α

It may be noted that, when $\alpha = 1$, the proposed distribution reduces to the Inverted Ishita distribution, with PDF given by

$$f_{IID}(x;\theta) = \frac{\theta^3}{\theta^3+2}\left(\frac{\theta}{x^2}+\frac{1}{x^4}\right)e^{-\frac{\theta}{x}}; x,\theta > 0 \tag{5}$$

2 Mathematical Characteristics

2.1 Mode of the inverted power Ishita distribution

The mode is useful in determining the shape of the distribution. So, for the proposed distribution, the first derivative of $f(x)$ is obtained from (2) as follows

$$f'_{IPID}(x) = \frac{\alpha\theta^3}{\theta^3+2}e^{-\theta x^{-\alpha}}\begin{bmatrix}\alpha\theta^2 x^{-2\alpha-1} + (-\alpha-1)\theta x^{-\alpha-2} + \\ \alpha\theta x^{-4\alpha-2} + (-3\alpha-1)x^{-3\alpha-2}\end{bmatrix} \tag{6}$$

$$f'_{IPID}(x) = \frac{\alpha\theta^3}{\theta^3+2}e^{-\theta x^{-\alpha}}\ x^{-\alpha-2}\left(\theta^2\alpha x^{-\alpha+1} - \theta(\alpha+1) + \alpha\theta x^{-3\alpha} - x^{-2\alpha}(3\alpha+1)\right) \tag{7}$$

Let b= $x^{-\alpha}$ in (7), we have

$$f'_{IPID}(x) = \frac{\alpha\theta^3}{\theta^3+2}e^{-\theta x^{-\alpha}}\ x^{-\alpha-2}\eta[b] \tag{8}$$

Where;

$$\eta[b] = \left(\theta^2\alpha b^{1-1/\alpha} - \theta(\alpha+1) + \alpha\theta b^3 - (3\alpha+1)b^2\right)$$

If we let $\eta[b]=0$, and numerically solve the non linear equation, the positive root gives the mode of the IPID distribution. To observe the asymptotic behavior , the limit of $f_{IPID}^{'}(x)$ is evaluated at $x=0$, and $x=\infty$respectively.

$$\lim_{x\to 0}\left(f_{IPID}(x;\theta,\alpha)=\frac{\alpha\theta^3}{\theta^3+2}\left(\frac{\theta}{x^{\alpha+1}}+\frac{1}{x^{3\alpha+1}}\right)e^{-\theta x^{-\alpha}}\right)=0$$

$$\lim_{x\to \infty}\left(f_{IPID}(x;\theta,\alpha)=\frac{\alpha\theta^3}{\theta^3+2}\left(\frac{\theta}{x^{\alpha+1}}+\frac{1}{x^{3\alpha+1}}\right)e^{-\theta x^{-\alpha}}\right)=0$$

Since $\lim_{x\to 0} f(x;\theta,\alpha)=0$ and $\lim_{x\to\infty} f(x;\theta,\alpha)=0$, the inverted Power Ishita distribution is unimodal. To also further this claim it is also observed that $\lim_{x\to\infty} F_{IPID}(x;\theta,\alpha)=1$.

2.2 Quantile Function

The quantile function is used for the generation of random numbers. It can also be used to derive percentiles of a distribution. The quantile function is defined by;

$$u=F(x) \tag{9}$$

Where U is distributed as uniform distribution, $U \sim [0,1]$,and $F(x)$ is the CDF.

Proposition 2.1: Let X be a random variable having the PDF of IPID, then the quantile $Q(p)$ function is

Proof: To proof proposition 2.2, recall that the quantile function $Q(p)$ satisfies the equation

$$Q(p)=F_{IPID}(x,\alpha,\theta) \tag{10}$$

Where $Q(p)\sim(0,1)$ and $F_{IPID}(x,\alpha,\theta)$is the CDF of the Power inverse Ishita distribution. Thus,

$$Q(p)=\frac{\theta^3}{\theta^3+2}\left(1+\frac{1}{x^{2\alpha}\theta}+\frac{2}{x^{\alpha}\theta^2}+\frac{2}{\theta^3}\right)e^{-\frac{\theta}{x^{\alpha}}}$$

$$Q(p)=\frac{\theta^3}{\theta^3+2}\left(1+\frac{1}{x^{2\alpha}\theta}+\frac{2}{x^{\alpha}\theta^2}+\frac{2}{\theta^3}\right)e^{-\frac{\theta}{x^{\alpha}}}$$

$$e^{\frac{\theta}{x^{\alpha}}}=\frac{1}{Q(p)}\frac{\theta^3}{\theta^3+2}\left(1+\frac{1}{x^{2\alpha}\theta}+\frac{2}{x^{\alpha}\theta^2}+\frac{2}{\theta^3}\right)$$

$$x=\left\{\frac{1}{\theta}1n\left[\frac{1}{Q(p)}\frac{\theta^3}{\theta^3+2}\left(1+\frac{1}{x^{2\alpha}\theta}+\frac{2}{x^{\alpha}\theta^2}+\frac{2}{\theta^3}\right)\right]\right\}^{-\frac{1}{\alpha}} \tag{11}$$

Eq. (11) completes the proof of the quantile of the inverted power Ishita distribution

2.3 Moments

Several fascinating characteristics of a distribution can be studied via the moments. For instance, measure of central tendency, dispersion, coefficient of skewness and coefficient of kurtosis. Consequently, it is essential to derive the moments for any new distribution proposed.

Proposition 2.2: Given a random variable X from a continuous distribution, the r^{th} non-central moment $E\left(X^r\right)$ is given by;

$$E\left(X^k\right)=\theta^{\frac{k}{\alpha}}\frac{\Gamma 1-\frac{k}{\alpha}}{\theta^3+2}\left(\theta^3+\frac{2\alpha^2-3\alpha k+k^2}{\alpha^2}\right),\ \alpha>0$$

Proof: By the rth moment about the origin is given by,

$$E\left(X^k\right)=\int_0^\infty x^k f_{IPID}(x,\alpha,\theta)dx \qquad (12)$$

$$\frac{\alpha\theta^3}{\theta^3+2}\int_0^\infty x^k\left(\frac{\theta}{x^{\alpha+1}}+\frac{1}{x^{3\alpha+1}}\right)e^{-\frac{\theta}{x^\alpha}}dx$$

$$\frac{\alpha\theta^3}{\theta^3+2}\int_0^\infty x^k\left(\theta x^{-\alpha-1}+x^{-3\alpha-1}\right)e^{-\frac{\theta}{x^\alpha}}dx$$

$$\frac{\alpha\theta^3}{\theta^3+2}\int_0^\infty \theta x^k x^{-\alpha-1}e^{-\frac{\theta}{x^\alpha}}dx+\frac{\alpha\theta^3}{\theta^3+2}\int_0^\infty x^k x^{-3\alpha-1}e^{-\frac{\theta}{x^\alpha}}$$

$$\frac{\alpha\theta^4}{\theta^3+2}\int_0^\infty x^{-\alpha+k-1}e^{-\frac{\theta}{x^\alpha}}dx+\frac{\alpha\theta^3}{\theta^3+2}\int_0^\infty x^{-3\alpha+k-1}e^{-\frac{\theta}{x^\alpha}}$$

Letting $y=x^\alpha,\Rightarrow x=y^{\frac{1}{\alpha}}$, and applying little algebra gives

$$\frac{\alpha\theta^4}{\theta^3+2}\int_0^\infty y^{1-\frac{k}{\alpha}+\frac{1}{\alpha}} e^{-\frac{\theta}{y}}\left(\frac{1}{\alpha}\right) y^{-\frac{1}{\alpha}-1} dy + \frac{\alpha\theta^3}{\theta^3+2} y^{3-\frac{k}{\alpha}+\frac{1}{\alpha}} e^{-\frac{\theta}{y}}\left(\frac{1}{\alpha}\right) y^{-\frac{1}{\alpha}-1} dy$$

$$\frac{\alpha\theta^4}{\theta^3+2}\left(\frac{1}{\alpha}\right)\int_0^\infty y^{-\left(\frac{k}{\alpha}-1\right)-1} e^{-\frac{\theta}{y}} dy + \frac{\alpha\theta^3}{\theta^3+2}\int_0^\infty y^{-\left(\frac{k}{\alpha}-3\right)-1} e^{-\frac{\theta}{y}}\left(\frac{1}{\alpha}\right) dy \qquad (13)$$

Using the fact that $\int_0^\infty x^{-\alpha-1} e^{-\frac{\beta}{x}} dx = \frac{\Gamma(\alpha)}{\beta^{(\alpha)}}$ one manipulates the Eqn. (7) to obtain

Using the transformation technique,

Let w=x^α, x=$w^{1/\alpha}$, dw=$\alpha x^{\alpha-1}dx$, dx=$\frac{dw}{\alpha x^{\alpha-1}}$, $dx = \frac{dw}{\alpha w^{1-1/\alpha}}$

$$\frac{\theta^4}{\theta^3+2}\int_0^\infty w^{-1+\frac{k}{\alpha}-\frac{1}{\alpha}} w^{-1+1/\alpha} e^{-\frac{\theta}{w}} dw + \frac{\theta^3}{\theta^3+2}\int_0^\infty w^{-3+\frac{k}{\alpha}-\frac{1}{\alpha}} w^{-1+1/\alpha} e^{-\frac{\theta}{w}} dw$$

$$\frac{\theta^4}{\theta^3+2}\int_0^\infty w^{-\left[\left(1-\frac{k}{\alpha}\right)+1\right]} e^{-\frac{\theta}{w}} dw + \frac{\theta^3}{\theta^3+2}\int_0^\infty w^{-\left[\left(3-\frac{k}{\alpha}\right)+1\right]} e^{-\frac{\theta}{w}} dw$$

From the theorem of inverted gamma, $\int_0^\infty e^{-\frac{\beta}{x}} x^{-\alpha-1} dx = \frac{\Gamma(\alpha)}{\beta^\alpha}$

Hence,

$$= \frac{\theta^3}{\theta^3+2}\left[\frac{\left(\theta\Gamma 1-\frac{k}{\alpha}\right)}{\theta^{1-\frac{k}{\alpha}}} + \frac{\Gamma\left(3-\frac{k}{\alpha}\right)}{\theta^{3-\frac{k}{\alpha}}}\right]$$

Recall that$\Gamma\alpha = (\alpha-1)!$, hence, $\Gamma\left(3-\frac{k}{\alpha}\right) = \left(2-\frac{k}{\alpha}\right)\left(1-\frac{k}{\alpha}\right)\Gamma 1-\frac{k}{\alpha}$

Hence, the k^{th} moment is given by

$$E(X^k) = \frac{\theta^3\Gamma 1-\frac{k}{\alpha}}{(\theta^3+2)}\left[\frac{\theta}{\theta^{1-\frac{k}{\alpha}}} + \frac{\left(2-\frac{k}{\alpha}\right)\left(1-\frac{k}{\alpha}\right)}{\theta^{3-\frac{k}{\alpha}}}\right]$$

In a simplified form it can be written as

$$E\left(X^{k}\right)=\theta^{\frac{k}{\alpha}}\frac{\Gamma 1-\dfrac{k}{\alpha}}{\theta^{3}+2}\left(\theta^{3}+\frac{2\alpha^{2}-3\alpha k+k^{2}}{\alpha^{2}}\right) \tag{14}$$

Eq. (14) completes the computation of the rth crude moment of the IPI distribution.

The mean of the IPI distribution is obtained by setting r=1 in (14). Thus,

$$\mu_{1}^{1}=\theta^{\frac{1}{\alpha}}\frac{\Gamma 1-\dfrac{1}{\alpha}}{\theta^{3}+2}\left(\theta^{3}+\frac{2\alpha^{2}-3\alpha+1}{\alpha^{2}}\right) \tag{15}$$

For r=2 in Eq. (14), the second crude moment $E\left(X^{2}\right)$ of the IPI distribution becomes,

$$\mu_{2}^{1}=\theta^{\frac{2}{\alpha}}\frac{\Gamma 1-\dfrac{2}{\alpha}}{\theta^{3}+2}\left(\theta^{3}+\frac{2\alpha^{2}-6\alpha+4}{\alpha^{2}}\right) \tag{16}$$

The variance of the IPI distribution is obtained as follows;

$$\operatorname{var}(X)=E\left(X^{2}\right)-(E(X))^{2}$$

$$\operatorname{var}(X)=\theta^{\frac{2}{\alpha}}\frac{\Gamma 1-\dfrac{2}{\alpha}}{\theta^{3}+2}\left(\theta^{3}+\frac{2\alpha^{2}-6\alpha+4}{\alpha^{2}}\right)-\left(\theta^{\frac{1}{\alpha}}\frac{\Gamma 1-\dfrac{1}{\alpha}}{\theta^{3}+2}\left(\theta^{3}+\frac{2\alpha^{2}-3\alpha+1}{\alpha^{2}}\right)\right)^{2} \tag{17}$$

2.4 Moment generating function of the inverted power Ishita distribution

Proposition 2.3 Given a random variable X that follows Inverted Power Ishita distribution, the moment generating function is given by;

$$M_{x}(t)=\sum_{k=0}^{\infty}\frac{\left(t\theta^{\frac{1}{\alpha}}\right)^{k}}{k!}\frac{\Gamma 1-\dfrac{k}{\alpha}}{\theta^{3}+2}\left(\theta^{3}+\frac{2\alpha^{2}+3\alpha k+k^{2}}{\alpha^{2}}\right)$$

Proof. Let X be a random variable which has the pdf defined in equation (3). Then, the moment generating function is obtained as follows

$$M_x(t) = E\left(e^{tx}\right) = \int_0^\infty e^{tx} f(x) dx \tag{18}$$

$$= \int_0^\infty \left[1 + tx + \frac{(tx)^2}{2!} + \ldots\right] f(x) dx$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left(X^r\right)$$

Substituting for $E\left(X^r\right)$, we obtain an expression for the moment generating function as

$$M_x(t) = \sum_{r=0}^{\infty} \frac{\left(t\theta^{\frac{1}{\alpha}}\right)^r}{r!} \frac{\Gamma 1 - \frac{r}{\alpha}}{\theta^3 + 2} \left(\theta^3 + \frac{2\alpha^2 + 3\alpha r + r^2}{\alpha^2}\right) \tag{19}$$

2.5 Entropy measure

Entropy is a measure that can be very useful in determining the uncertainty of a distribution. It has applications in economics, probability and statistics, communication theory etc. Large value of entropy signifies large uncertainty in the data . In this section, we derive an expression for the Rényi entropy of the IPI distribution.

Proposition 2.4 Suppose X is a random variable having the PDF of IPI distribution . The Rényi entropy is given by

$$\operatorname{Re}(\gamma) = \frac{1}{1-\gamma} \log \sum_{j=0}^{\infty} \binom{\gamma}{j} \frac{\alpha^{\gamma-1} \theta^{3\gamma+j} (\gamma)^{-3\gamma+2j} \Gamma(3\gamma - 2j)}{\left(\theta^3 + 2\right)^{\gamma} \theta^{3\gamma-2j}}$$

Proof. The Ŕenyi entropy of a random variable X for a continuous distribution is given by

$$\operatorname{Re}(\gamma) = \frac{1}{1-\gamma} \log \left\{\int f^{\gamma}(x, \alpha, \theta) dx\right\}, \gamma > 0, \gamma \neq 1 \tag{20}$$

$$= \frac{1}{1-\gamma} \log \int_0^\infty \left(\frac{\alpha\theta^3}{\theta^3 + 2}\left(\frac{\theta}{x^{\alpha+1}} + \frac{1}{x^{3\alpha+1}}\right) e^{-\theta/x^{\alpha}}\right)^{\gamma} dx$$

$$=\frac{1}{1-\gamma}\log\left\{\int_0^\infty \frac{\alpha^{\gamma}\theta^{3\gamma}}{\left(\theta^3+2\right)^{\gamma}}\left(\frac{\theta}{x^{\alpha+1}}+\frac{1}{x^{3\alpha+1}}\right)^{\gamma} e^{-\theta\gamma/x^{\alpha}}\right\}$$

$$=\frac{1}{1-\gamma}\log\frac{\alpha^{\gamma}\theta^{3\gamma}}{\left(\theta^3+2\right)^{\gamma}}\left\{\int_0^\infty\left(\frac{\theta x^{2\alpha}+1}{x^{3\alpha+1}}\right)^{\gamma} e^{-\theta\gamma/x^{\alpha}}dx\right\}$$

Recall the following

- $(1+x)^r=\sum_{i=0}^{\infty}\binom{r}{i}x^i$
- $\int_0^\infty e^{-\frac{\beta}{x}}x^{-\alpha-1}dx=\frac{\Gamma(\alpha)}{\beta^{\alpha}}$

Applying the two relations above, we have

$$=\frac{1}{1-\gamma}\log\frac{\alpha^{\gamma}\theta^{3\gamma}}{\left(\theta^3+2\right)^{\gamma}}\int_0^\infty x^{-(3\alpha+1)\gamma}e^{-\theta\gamma/x^{\gamma}}\sum_{j=0}^{\infty}\binom{\gamma}{j}\theta^{j}x^{2\alpha j}dx$$

$$=\frac{1}{1-\gamma}\log\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\alpha^{\gamma}\theta^{3\gamma+j}}{\left(\theta^3+2\right)^{\gamma}}\int_0^\infty x^{-3\alpha\gamma-1+2\alpha j}e^{-\theta\gamma/x^{\alpha}}dx$$

$$=\frac{1}{1-\gamma}\log\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\alpha^{\gamma}\theta^{3\gamma+j}}{\left(\theta^3+2\right)^{\gamma}}\int_0^\infty x^{-(3\alpha\gamma-2\alpha j)-1}e^{-\theta\gamma/x^{\alpha}}dx$$

Let $\mathrm{w}=\frac{x^{\alpha}}{\gamma}$, $x^{\alpha}=w\gamma$, $x=(w\gamma)^{1/\alpha}$, $\frac{dx}{dw}=\frac{1}{\alpha}\gamma^{1/\alpha}\mathrm{w}^{1/\alpha-1}$

$$=\frac{1}{1-\gamma}\log\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\alpha^{\gamma}\theta^{3\gamma+j}}{\left(\theta^3+2\right)^{\gamma}}\int\left((w\gamma)^{1/\alpha}\right)^{-(3\alpha\gamma-2\alpha j)-1}e^{-\theta/\mathrm{w}}\frac{1}{\alpha}\gamma^{1/\alpha}\mathrm{w}^{1/\alpha-1}dw$$

$$=\frac{1}{1-\gamma}\log\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\alpha^{\gamma-1}\theta^{3\gamma+j}\gamma^{-3\gamma+2j}}{\left(\theta^3+2\right)^{\gamma}}\int_0^\infty (w)^{-(3\alpha\gamma-2\alpha j)-1}e^{-\theta/w}dw$$

Recall that, $\int_0^\infty x^{-\alpha-1}e^{-\theta/x}dx=\frac{\Gamma\alpha}{\theta^{\alpha}}$ hence we have

$$\operatorname{Re}(\gamma)=\frac{1}{1-\gamma}\log\sum_{j=0}^{\infty}\binom{\gamma}{j}\frac{\alpha^{\gamma-1}\theta^{3\gamma+j}\gamma^{-3\gamma+2j}}{\left(\theta^{3}+2\right)^{\gamma}}\frac{\Gamma(3\gamma-2j)}{\theta^{3\gamma-2j}} \tag{21}$$

Eq. (21) completes the computation of Renyi entropy

2.6 Order statistics

Suppose $x_1, x_2, ..., x_n$ are a random sample of size n from a continuous distribution with PDF and CDF, $f(x)$ and $F(x)$ respectively. If these random variables are arranged in ascending order, they are referred to as order statistics. That is, the order statistics is such that $x_1 \le x_2 \le x_3 \le ... \le x_n$. The PDF of the ω^{th} ordered statistics is

$$f_x(x)=\frac{n!}{(\omega-1)!(n-\omega)!}F^{\omega-1}(x)(1-F(x))^{n-\omega}f(x) \tag{22}$$

$$=\sum_{j=0}^{n-\omega}\frac{n!}{(\omega-1)!(n-\omega)!}\binom{n-\omega}{j}(-1)^{j}F^{\omega-1}(x)F^{j}(x)f(x)$$

$$=\frac{n!}{(\omega-1)!(n-\omega)!}\sum_{0}^{n-\omega}\binom{n-\omega}{j}(-1)^{j}F^{\omega+j-1}(x)f(x) \tag{23}$$

Substituting (3) and (4) into (23), we have

$$f_x(x)=\frac{n!}{(\omega-1)!(n-\omega)!}\sum_{0}^{n-\omega}\binom{n-\omega}{j}(-1)^{j}\left(\frac{\theta^3}{\theta^3+2}\left(1+\frac{1}{x^{2\alpha}\theta}+\frac{2}{x^{\alpha}\theta^2}+\frac{2}{\theta^3}\right)e^{-\frac{\theta}{x^{\alpha}}}\right)^{\omega+j-1}\left(\frac{\alpha\theta^3}{\theta^3+2}\left(\frac{\theta}{x^{\alpha+1}}+\frac{1}{x^{3\alpha+1}}\right)e^{-\frac{\theta}{x^{\alpha}}}\right)$$

$$=\frac{\theta^{3\omega+3j-3}}{\left(\theta^3+2\right)^{\omega+j-1}}\left(\frac{x^{2\alpha}\theta^3+\theta^2+2\theta x^{\alpha}+2x^{2\alpha}}{\theta^3x^{2\alpha}}\right)^{\omega+j-1}e^{-\theta(\omega+j-1)/x^{\alpha}}$$

$$=\frac{\theta^{3\omega+3j-3}}{\left(\theta^3+2\right)^{\omega+j-1}}\frac{\left(x^{2\alpha}\theta^3+\theta^2+2\theta x^{\alpha}+2x^{2\alpha}\right)^{\omega+j-1}}{\theta^{3\omega+3j-3}x^{2\alpha\omega+2\alpha j-2\alpha}}e^{-\theta(\omega+j-1)/x^{\alpha}}$$

where

$$F^{\omega+j-1}(x)=\frac{\left(x^{2\alpha}\theta^3+\theta^2+2\theta x^{\alpha}+2x^{2\alpha}\right)^{\omega+j-1}}{\left(\theta^3+2\right)^{\omega+j-1}x^{2\alpha\omega+2\alpha j-2\alpha}}e^{-\theta(\omega+j-1)/x^{\alpha}}$$

and

$$f(x)=\frac{\alpha\theta^3\left(\theta x^{2\alpha}+1\right)e^{-\theta/x^{\alpha}}}{\left(\theta^3+2\right)x^{3\alpha+1}}$$

Substituting the simplified $F^{\omega+j-1}(x)$ and $f(x)$ in (23), we have

$$f_x(x)=\frac{\alpha\theta^3\left(\theta x^{2\alpha}+1\right)e^{-\theta/x^{\alpha}}n!}{(\omega-1)!(n-\omega)!}\sum_{0}^{n-\omega}\binom{n-\omega}{j}(-1)^j\frac{\left(x^{2\alpha}\theta^3+\theta^2+2\theta x^{\alpha}+2x^{2\alpha}\right)^{\omega+j-1}}{\left(\theta^3+2\right)^{\omega+j}x^{2\alpha\omega+2\alpha j+\alpha+1}}e^{-\theta(\omega+j-1)/x^{\alpha}} \tag{24}$$

The corresponding CDF, $F_x(x)$ of the order statistics of the IPI distribution is obtained as follows;

$$F_x(x)=\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l F^{j+l}(x)$$

$$F^{j+l}(x)=\left[\frac{\theta^3}{\theta^3+2}\frac{\left(x^{2\alpha}\theta^3+\theta^2+2\theta x^{\alpha}+2x^{2\alpha}\right)}{\theta^3x^{2\alpha}}e^{-\theta/x^{\alpha}}\right]^{j+l}$$

$$F^{j+l}(x)=\left[\frac{\left(x^{2\alpha}\theta^3+\theta^2+2\theta x^{\alpha}+2x^{2\alpha}\right)}{\left(\theta^3+2\right)x^{2\alpha}}e^{-\theta/x^{\alpha}}\right]^{j+l}$$

Thus,

$$F_x(x)=\sum_{j=0}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l\left[\frac{\left(x^{2\alpha}\theta^3+\theta^2+2\theta x^{\alpha}+2x^{2\alpha}\right)}{\left(\theta^3+2\right)x^{2\alpha}}e^{-\theta/x^{\alpha}}\right]^{j+l} \tag{25}$$

2.7 Stochastic ordering

Stochastic ordering is an essential tool for quantifying the behavior of random variables in terms of their sizes. Given that X and Y are distributed according to Eq. (2). Let $f_x(x,\alpha,\theta)$, $f_y(x,\alpha,\theta)$, and $F_X(x,\alpha,\theta)$, $F_Y(x,\alpha,\theta)$ denote the probability density function and distribution function of X and Y, respectively. The random variable X is said to be smaller than the random variable Y, if the following holds;

a) Stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x);\ \forall x$
b) Hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x);\ \forall x$
c) Mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \geq m_Y(x);\ \forall x$
d) Likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(y)}$ decreases in x.

These results were established by M. Shaked and J. G Shanthikumar (1994). The order of the distributions is as follows

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \Rightarrow X \leq_{st} Y$$

Proof. The likelihood ratio is

$$\frac{f_X(x;\alpha_1,\theta_1)}{f_Y(x;\alpha_2,\theta_2)} = \frac{\frac{\alpha_1\theta_1^3}{\theta_1^3+2}\left(\theta_1 x^{-\alpha_1-1} + x^{-3\alpha_1-1}\right)e^{-\theta_1/x^{\alpha_1}}}{\frac{\alpha_2\theta_2^3}{\theta_2^3+2}\left(\theta_2 x^{-\alpha_2-1} + x^{-3\alpha_2-1}\right)e^{-\theta_2/x^{\alpha_2}}} \tag{26}$$

$$= \frac{\alpha_1\theta_1^3}{\theta_1^3+2}\left(\theta_1 x^{-\alpha_1-1} + x^{-3\alpha_1-1}\right)e^{-\theta_1/x^{\alpha_1}} * \frac{\left(\theta_2^3+2\right)e^{\theta_2/x^{\alpha_2}}}{\alpha_2\theta_2^3\left(\theta_2 x^{-\alpha_2-1} + x^{-3\alpha_2-1}\right)}$$

$$= \frac{\alpha_1\theta_1^3\left(\theta_2^3+2\right)}{\alpha_2\theta_2^3\left(\theta_1^3+2\right)} * \frac{\left(\theta_1 x^{-\alpha_1-1} + x^{-3\alpha_1-1}\right)}{\left(\theta_2 x^{-\alpha_2-1} + x^{-3\alpha_2-1}\right)} e^{-\theta_1/x^{\alpha_1} + \theta_2/x^{\alpha_2}} \tag{27}$$

Taking natural log of 27, we have

$$\ln\frac{f_X(x;\alpha_1,\theta_1)}{f_Y(x;\alpha_2,\theta_2)} = \ln\left[\frac{\alpha_1\theta_1^3\left(\theta_2^3+2\right)}{\alpha_2\theta_2^3\left(\theta_1^3+2\right)} * \frac{\left(\theta_1 x^{-\alpha_1-1} + x^{-3\alpha_1-1}\right)}{\left(\theta_2 x^{-\alpha_2-1} + x^{-3\alpha_2-1}\right)} e^{-\theta_1/x^{\alpha_1} + \theta_2/x^{\alpha_2}}\right]$$

$$= \ln\left[\frac{\alpha_1\theta_1^3\left(\theta_2^3+2\right)}{\alpha_2\theta_2^3\left(\theta_1^3+2\right)}\right] + \ln\left[\frac{\left(\theta_1 x^{-\alpha_1-1} + x^{-3\alpha_1-1}\right)}{\left(\theta_2 x^{-\alpha_2-1} + x^{-3\alpha_2-1}\right)}\right] - \left[\frac{\theta_1}{x^{\alpha_1}} - \frac{\theta_2}{x^{\alpha_2}}\right]$$

Taking the derivative of $ln\frac{f_X(x;\theta_1)}{f_Y(x;\theta_2)}$ gives

$$\frac{d}{dx}\ln\frac{f_X(x;\alpha_1;\theta_1)}{f_Y(x;\alpha_2;\theta_2)} = \frac{\left\{\begin{array}{l}\theta_1 x^{-3\alpha_2-\alpha_1-3}\left(3\alpha_2-\alpha_1\right) + \theta_1\theta_2 x^{-\alpha_2-\alpha_1-3}\left(\alpha_2-\alpha_1\right) \\ +\theta_2 x^{-3\alpha_1-\alpha_2-3}\left(3\alpha_1+\alpha_2+2\right) \\ +x^{-3\alpha_2-3\alpha_1-3}\left(3\alpha_2-3\alpha_1+2\right)\end{array}\right\}}{\left(\theta_1 x^{-\alpha_1-1} + x^{-3\alpha_1-1}\right)\left(\theta_2 x^{-\alpha_2-1} + x^{-3\alpha_2-1}\right)} + \frac{\theta_1\alpha_1 x^{\alpha_2+1} - \theta_2\alpha_2 x^{\alpha_1+1}}{x^{\alpha_1+\alpha_2+2}} = 0$$

Thus, for $\theta_2 \geq \theta_1$ and $\alpha_1 = \alpha_2$(or for $\alpha_2 \geq \alpha_1$ and $\theta_1 = \theta_2$, $\frac{d}{dx} ln\frac{f_X(x;\theta_1)}{f_Y(x;\theta_2)} \leq 0$, This implies that $X \leq_{lr} Y$and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

3 Reliability Analyses

3.1 Survival function

Survival function, $S(x)$ is the probability that the survival time is greater than or equal to x. In engineering, it is the probability that an item does not fail prior to some time, x. We use survival functions in reliability analysis to determine the survival time of items. Let X be a continuous random Variable with CDF, $F(x)$, the survival function of X is

$$S(x)=1-F(x) \tag{28}$$

Thus, inserting (4) in (28), the survival function of the inverted power Ishita distribution is

$$S(x)=1-\frac{\theta^3}{\theta^3+2}\left(1+\frac{1}{x^{2\alpha}\theta}+\frac{2}{x^{\alpha}\theta^2}+\frac{2}{\theta^3}\right)e^{-\frac{\theta}{x^{\alpha}}} \tag{29}$$

3.2 Hazard function

Hazard function, also known as failure rate is the probability that an individual dies at time x given that the individual survived to that time x. Hazard function is extensively used to express the risk of an event (example, death) occurring at some time t. Given a random variable X from a continuous distribution, the hazard rate $h(x)$is given by

$$h(x)=\frac{f(x)}{1-F(x)} \tag{30}$$

Inserting (3) and (4) in (30), we have

$$h(x)=\frac{\dfrac{\alpha\theta^3\left(\theta x^{2\alpha}+1\right)e^{-\theta/x^{\alpha}}}{\left(\theta^3+2\right)x^{3\alpha+1}}}{1-\dfrac{\theta^3}{\theta^3+2}\left(1+\dfrac{1}{x^{2\alpha}\theta}+\dfrac{2}{x^{\alpha}\theta^2}+\dfrac{2}{\theta^3}\right)e^{-\frac{\theta}{x^{\alpha}}}} \tag{31}$$

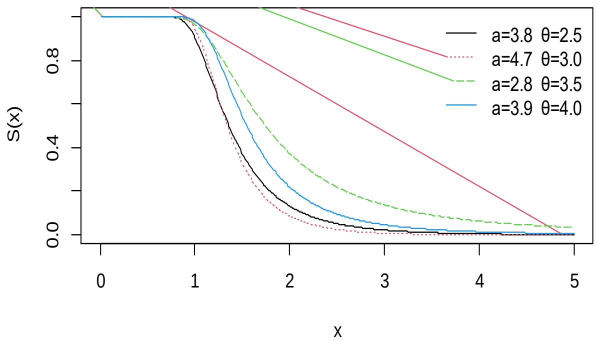

Fig 3A:Survival rate function plot of IPI distribution

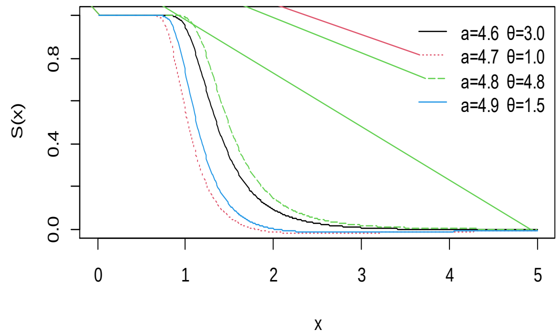

Fig 3B:Survival rate function plot of IPI distribution

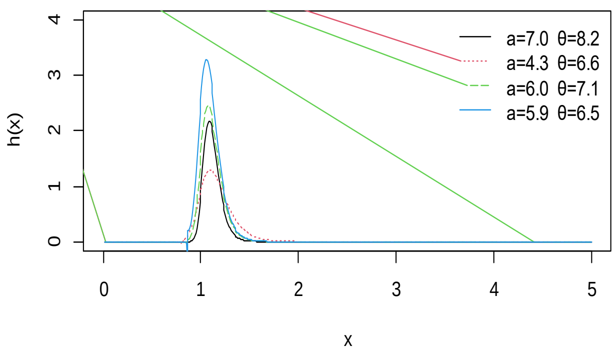

Fig 4a:Hazard rate plot of IPI

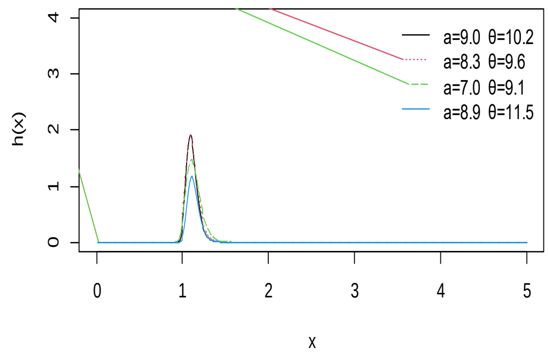

Fig 4a:Hazard rate plot of IPI

3.3 Odds function

In reliability analysis, the odd rate is an important tool used for modeling real life data set that shows non-monotone hazard rate. Let X be a random variable from a continuous distribution with distribution function $F(x)$ and reliability function $1-F(x)$, the odd function is

$$O(x)=\frac{F(x)}{1-F(x)} \tag{32}$$

Thus, the odds function of the inverted power Ishita distribution is given by

$$O(x)=\frac{\frac{\theta^3}{\theta^3+2}\left(1+\frac{1}{x^{2\alpha}\theta}+\frac{2}{x^\alpha\theta^2}+\frac{2}{\theta^3}\right)e^{-\frac{\theta}{x^\alpha}}}{1-\left(\frac{\theta^3}{\theta^3+2}\left(1+\frac{1}{x^{2\alpha}\theta}+\frac{2}{x^\alpha\theta^2}+\frac{2}{\theta^3}\right)e^{-\frac{\theta}{x^\alpha}}\right)} \tag{33}$$

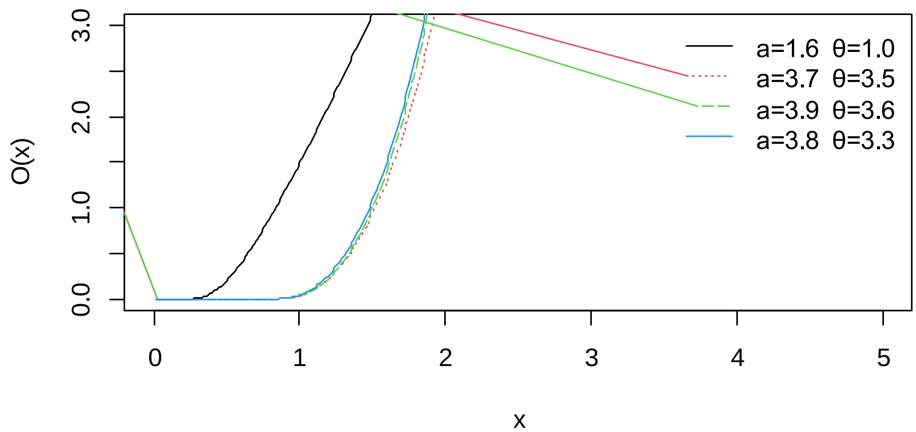

Fig 5:odd function plot of IPI distribution

4 Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a random sample of size n from an inverted power Ishita distribution. Then, the log-likelihood (LL) function is

$$LL(\alpha,\theta) = 1n\prod_{i=1}^{n} f(x_i;\alpha,\theta) \tag{34}$$

$$= 1n\prod_{i=1}^{n}\frac{\alpha\theta^3\left(\theta x_i^{\,2\alpha}+1\right)e^{-\theta/x_i^{\alpha}}}{\left(\theta^3+2\right)x_i^{\,3\alpha+1}}$$

$$= 1n\left[\left(\frac{\alpha\theta^3}{\theta^3+2}\right)^n * \sum_{i=1}^{n}\left(x_i^{2\alpha}\theta+1\right) * \sum_{i=1}^{n}\left(x_i^{3\alpha+1}\right)^{-1} * e^{-\theta\sum_{i=1}^{n} x^{-\alpha}}\right]1$$

$$= n1n\left(\alpha\theta^3\right) - n1n\left(\theta^3+2\right) + \sum_{i=1}^{n}1n\left(x_i^{2\alpha}\theta+1\right) - \sum_{i=1}^{n}1n\left(x_i^{3\alpha+1}\right) - \theta\sum_{i=1}^{n}x_i^{-\alpha}$$

The partial derivatives in terms of the parameter (α,θ), are given as follows

$$\frac{dLL}{d\theta} = \frac{3n}{\theta} - \frac{3n\theta^2}{\theta^3+2} + \sum_{i=1}^{n}\frac{x_i^{2\alpha}}{\left(x_i^{2\alpha}\theta+1\right)} - \sum_{i=1}^{n}x_i^{-\alpha} = 0 \tag{35}$$

$$\frac{dLL}{d\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\left(\frac{2\theta x_i^{2\alpha}\left(1nx_i\right)}{\left(x_i^{2\alpha}\theta+1\right)}\right)-\sum_{i=1}^{n}31nx_i+\theta\sum_{i=1}^{n}\frac{1nx_i}{x_i^{\alpha}}=0 \tag{36}$$

The simultaneous solutions of the nonlinear Equation (35) and (36), at $\frac{dLL}{d\theta}=0$ and $\frac{dLL}{d\alpha}=0$, yields the maximum likelihood of the parameter (α,θ).

To derive the confidence intervals for the parameters using maximum likelihood estimators $\left(\hat{\alpha},\hat{\theta}\right)$, the fisher information matrix will be used, which for a vector parameters ξ, and $n=1$ is given by the expression.

$$I(\xi)=I_{i,j}(\xi)=E\left[-\frac{\partial^2}{\partial\xi_i\partial\xi_j}1nf\left(X/\xi\right)\right] \tag{37}$$

As a result of the complexity involved in evaluating the information matrix given in (37), the inverse Hessian matrix is used in the maximum likelihood estimates. Consequently, the second-order derivatives of the loglikelihood function are given as follows

$$\frac{\partial^2 LL}{\partial\alpha^2}=-\frac{n}{\alpha^2}+\sum_{i=1}^{n}\left[\frac{4\theta x_i^{2\alpha}\left(1nx_i\right)^2}{\left(\theta x_i^{2\alpha}+1\right)^2}\right]-\sum_{i=1}^{n}\frac{\left(1nx_i\right)^2}{x_i^{\alpha}} \tag{38}$$

$$\frac{\partial^2 LL}{\partial\theta^2}=\frac{3n\left(\theta^4-4\theta\right)}{\left(\theta^3+2\right)^2}-\frac{3n}{\theta^2}-\sum_{i=1}^{n}\frac{x_i^{4\alpha}}{\left(x_i^{2\alpha}+1\right)^2} \tag{39}$$

$$\frac{\partial^2 LL}{\partial\alpha\partial\theta}=\sum_{i=1}^{n}\left[\frac{2x_i^{2\alpha}1nx_i}{\left(\theta x_i^{2\alpha}+1\right)^2}\right]+\sum_{i=1}^{n}\frac{1nx_i}{x_i^{\alpha}} \tag{40}$$

$$\frac{\partial^2 LL}{\partial\theta\partial\alpha}=\sum_{i=1}^{n}\left[\frac{2x_i^{2\alpha}1nx_i}{\left(\theta x_i^{2\alpha}+1\right)^2}\right]+\sum_{i=1}^{n}\frac{1nx_i}{x_i^{\alpha}} \tag{41}$$

In order to determine the Fisher information matrix for the IPI distribution, the expectations of (38), (39), (40) and (41) are taken, assuming $n=1$. Thus

$$I(\xi)=I_{i,j}(\xi)=E\begin{bmatrix}\frac{\partial^2 LL}{\partial\alpha^2} & \frac{\partial^2 LL}{\partial\theta\partial\alpha}\\ \frac{\partial^2 LL}{\partial\alpha\partial\theta} & \frac{\partial^2 LL}{\partial\theta^2}\end{bmatrix}=\begin{bmatrix}E\left[\frac{\partial^2 LL}{\partial\alpha^2}\right] & E\left[\frac{\partial^2 LL}{\partial\theta\partial\alpha}\right]\\ E\left[\frac{\partial^2 LL}{\partial\alpha\partial\theta}\right] & E\left[\frac{\partial^2 LL}{\partial\theta^2}\right]\end{bmatrix}=\begin{bmatrix}I_{11} I_{12}\\ I_{21} I_{22}\end{bmatrix}$$

The asymptotic distribution of the maximum likelihood estimator $\hat{\xi}$ for ξ under consistency state is given by:

$$\sqrt{n}\left(\hat{\xi}-\xi\right)\to N\left(0, I^{-1}(\xi)\right)$$

where $I^{-1}(\xi)$ is the inverse Fisher information matrix, defined as;

$$\frac{1}{n}I^{-1}(\xi)=\frac{1}{n}\begin{pmatrix}\mathrm{I}_{11} & \mathrm{I}_{12}\\ \mathrm{I}_{21} & \mathrm{I}_{22}\end{pmatrix}^{-1}=\begin{pmatrix}Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\theta})\\ Cov(\hat{\alpha},\hat{\theta}) & Var(\hat{\theta})\end{pmatrix} \tag{42}$$

Having obtained the expression in (42), we can now define the asymptotic $100(1-\tau)\%$ confidence intervals for α and θ as given below;

$$\hat{\alpha}\pm \mathrm{Z}_{\frac{\tau}{2}}\sqrt{Var(\hat{\alpha})} \text{ and } \hat{\theta}\pm \mathrm{Z}_{\frac{\tau}{2}}\sqrt{Var(\hat{\theta})} \tag{43}$$

where $Var(\hat{\alpha})$ and $Var(\hat{\theta})$ denote the elements of the main diagonal of the variance covariance matrix defined in (42).

5 Numerical Applications

In this section, we present two real life data sets to exhibit the practicality of the proposed model. The first data set is the monthly actual taxes revenue in Egypt from January 2006 to November 2010 used in [14], [15] and [16]. The data (in 1000 million Egyptian pounds) is provided below;

5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17.0, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36.0, 8.5, 8.0, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1,16.5, 11.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

The second dataset represents the relief times of twenty patients receiving an analgesic. It was used by [17], reported by [18]

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0

Tables 1 and 2 below show the estimates of the Inverted power Ishita (IPI) distribution and the competing distributions, namely; Inverse Power Rama (IPR) distribution, Ishita distribution (ID), Inverse Ishita distribution (IID) and the Sujatha distribution (SD), respectively obtained using the first and second datasets. Comparisons of the estimates, computed using maximum likelihood estimation method was made. To select the best distribution, three criteria were used. The criteria include:

- Akaike information criterion developed by [19] is defined by

$$AIC = 21nL + 2K$$

- Bayesian Information Criiterion proposed by [20] is defined by

$$BIC = 1n(n)k + 21n\left(\hat{L}\right)$$

- Corrected Akaike Information Criterion stated by [21] is defined by

$$AICc = AIC + \frac{2k(k+1)}{(n-k-1)}$$

where $\hat{L}$ denotes the log-likelihood at Maximum Likelihood Estimates (MLEs), k is the number of parameters in the distribution, and n is the sample size. The distribution with least AIC, BIC, AICc and log-likelihood is considered as best. Table 1 and 2 show that the Inverse Power Ishita distribution compared to Inverse Ishita distribution, Ishita distribution, sujatha distribution has the least values of AIC, BIC, and AICc. While compared to the Inverse Power Rama distribution, we observe that both distribution have their AIC, BIC, AICc, and log-likelihood approximately equal. Thus, the Inverse Power Ishita distribution is considered to provide best fit than the Inverse Ishita, Ishita and Sujatha distributions.

Table 1. MLEs, S.E, LL, AIC, BIC, and AICc (Data 1)

Model	**MLE**	**S.E**	**LL**	**AIC**	**BIC**	**AICc**
IPID	$\alpha = 2.24664$	0.22245	188.9396	381.8792	386.0343	382.3156

	$\theta = 144.62744$	67.49581				
IID	$\theta = 10.59798$	1.36746	212.2291	426.4581	428.5356	426.8945
ID	$\theta = 0.21869$	0.01635	194.1055	390.211	392.2885	390.6474
SD	$\theta = 0.21248$	0.01591	195.0663	392.1326	394.2101	392.5689
IPRD	$\alpha = 2.24665$	0.22245	188.9396	381.8792	386.0342	382.3155
	$\theta = 144.62895$	67.49593				

Table 2. MLEs, S.E, LL, AIC, BIC, and AICc (Data 2)

Model	MLE	S.E	LL	AIC	BIC	AICc
IPID	$\alpha = 4.03629$	0.68884	15.4073	34.8146	36.80608	36.3146
	$\theta = 6.16314$	1.90240				
IID	$\theta = 2.25893$	0.33081	33.7432	69.4864	70.48213	70.9864
ID	$\theta = 1.09485$	0.12169	30.0824	62.1647	63.16043	63.6647
SD	$\theta = 1.13675$	0.14984	28.7488	59.4975	60.49327	60.9975
IPRD	$\alpha = 4.11806$	0.66709	15.4089	34.8178	36.80929	36.3178
	$\theta = 6.61108$	1.83603				

Tables 3 and 4 show the 95% confidence interval constructed for the parameters of the IPR distribution, using the first and second datasets respectively.

Table 3. MLEs of the Parameters of IPI distribution and their C.I (Data 1)

Model	MLE	S.E	95% Confidence Interval	
			Lower limit	**Upper limit**
IPID	$\alpha = 2.24664$	0.22245	1.81064	2.68264
	$\theta = 144.62744$	67.49581	12.33565	276.91923
IID	$\theta = 10.59798$	1.36746	7.91776	13.27820
ID	$\theta = 0.21869$	0.01635	0.18664	0.25074
SD	$\theta = 0.21248$	0.01591	0.18130	0.24366
IPRD	$\alpha = 2.24665$	0.22245	1.81065	2.68265
	$\theta = 144.62895$	67.49593	12.33693	276.92097

Table 4. MLEs of the Parameters of IPI distribution and their C.I (Data 2)

Model	MLE	S.E	95% Confidence Interval	
			Lower limit	**Upper limit**
IPID	$\alpha = 4.03629$	0.68884	2.68616	5.38642
	$\theta = 6.16314$	1.90240	2.39947	9.92681
IID	$\theta = 2.25893$	0.33081	1.61054	2.90732
ID	$\theta = 1.09485$	0.12169	0.85634	1.33336
SD	$\theta = 1.13675$	0.14984	0.84306	1.43044
IPRD	$\alpha = 4.11806$	0.66709	2.81056	5.42556
	$\theta = 6.61108$	1.83603	3.01246	10.20970

6 Conclusions

Generalization in distribution theory is often made to improve the distribution under consideration and to make it more flexible so as to extend its application to other areas. The capability of the data to fit more appropriately into a given distribution shows the superiority of such distribution over others. In this paper, we have proposed a new distribution named the Inverted Power Ishita distribution. The mathematical characteristics and reliability measures such as survival function, hazard rate and odds function are derived. The method of maximum likelihood was used to estimate the parameters of the distribution and besides, the distribution was subjected to real life data to illustrate its application. Based on the empirical results obtained, the IPI distribution, asides the IPR whose MLE, S.E, LL, AIC, BIC, and AICc are approximately equal, outperforms the other competing models considered in the article. Hence, we recommend the use of the proposed model when modeling lifetime data that are heavy tailed and have upside down bathtub shape.

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