*Original Research Article*

# An A- Stable block integrator scheme for the solution of first order system of IVP of ordinary differential equations


## Abstract

This paper considers deriving four blocks numerical scheme of block backward differentiation formula for the approximate solutions of initial value problem of the first order system of ordinary differential equations. The block scheme at a single integration step produces four approximate solution values of $y_{n+1}$, $y_{n+2}$, $y_{n+3}$ and $y_{n+4}$ at point $x_{n+1}$, $x_{n+2}$, $x_{n+3}$ and $x_{n+4}$ respectively. The work investigated the order and stability property of the scheme, the method is zero, A–Stable and of order 6. Test problems are solved with the scheme and the comparison shows that the proposed block scheme has advantages over some existing methods in terms of accuracy, minimum error and less CPU time.


**Keyword:** Block, IVPs, Order, Ordinary Differential equation, Stiff

## Introduction

A number of real life issues that we encounter, especially in the field of engineering, sciences both physical, social and life sciences can be modeled in Mathematics as differential equations. Considering the vast application of differential equations, analytical and numerical methods are being developed to find solutions.

This study consider a method for solving system of first order initial value problem of ordinary differential equation of the form

$$y' = f(x,\ \hat{Y}), \qquad \hat{Y}(a) = \eta, \qquad a \leq x \leq b \qquad (1)$$
$$\hat{Y} = (y_1, y_2, y_3, \dots\dots\dots y_n), \qquad \hat{\eta} = (\eta_1, \eta_2, \eta_3, \dots\dots\dots \eta_n)$$

Ordinary differential equations can be solved by analytical and numerical methods. The solutions generated by the analytical method are generally exact values, whereas with the numerical method an approximation is given as a solution approaching the real value (Fatokun *et al* 2005). Implicit numerical schemes proved to be more efficient in solving problems than explicit ones. Most common implicit algorithms are based on Backward Differentiation Formula (BDF). The BDF first appeared in the work of (Curtiss and Hirschfelder 1952). Researchers continued to improve on the BDF methods. Such improvements include the Extended Backward Differential Formula by (Cash, 1980), modified extended backward differential formula by (Cash, 2000), 2 point diagonally implicit super class of backward differentiation formula (Musa *et al.*, 2016), an order five implicit 3-step block method for solving ordinary differential equation (Yahaya and Sagir, 2013), Implicit r-point block backward differentiation formula for solving first- order stiff ODEs (Ibrahim *et al*., 2007), a new variable step size block backward differentiation formula for solving stiff initial value problems (Suleiman *et al*., 2013), a new fifth order implicit block method for solving first order stiff ordinary differential equations by (Musa *et al* 2014), an accurate computation of block hybrid method for solving stiff ODEs (Sagir, 2012), One-leg Multistep Method for first Order Differential Equations (Fatunla, 1984), Sagir (2014), Numerical Treatment of Block Method for the Solution of Ordinary Differential Equations. All the schemes mentioned above developed by different scholars possesses various sort of accuracy, minimum error and less computation time at one step or the other. However, there is need of developing a numerical algorithm that will solve system of ODEs with minimal computational time and converge faster, hence the motivation for this research.

## Formulation of the Methods

Consider the general k-step linear multistep method of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h\beta_{k} f_{n+k} \qquad (2)$$

This study consider adding a future point in (2), with three step backward, to came-up with the formula of the form

$$\sum_{j=0}^{7} \alpha_j y_{n+j-3} = h\beta_{k} f_{n+k-3} \quad k = 1,2,3,4 \qquad (3)$$

The implicit four point method (3) is constructed using a linear operator $L_i$. To derive the four point, define the linear operator $L_i$ associated with (3) as

$$L_i[y(x_n), h]: \alpha_0\, y_{n-3} + \alpha_1\, y_{n-2} + \alpha_2\, y_{n-1} + \alpha_3\, y_n + \alpha_4\, y_{n+1} + \alpha_5\, y_{n+2} + \alpha_6\, y_{n+3} + \alpha_7\, y_{n+4} - h\beta_k f_{n+k-3} = 0 \quad k = i = 1,2,3,4 \qquad (4)$$

To derive the first, second, third, and fourthpoints as $y_{n+1}$, $y_{n+2}$, $y_{n+3}$ *and* $y_{n+4}$ respectively Using Taylor series expansion in (4) and normalizing $\alpha_3 = 1$, $\alpha_4 = 1$, $\alpha_5 = 1$ and $\alpha_6 = 1$ as coefficient's of the four points, $k = 1$, $k = 2$, $k = 3$ and $k = 4$ respectively. To obtain

$$y_{n+1} = -\frac{1298881}{341643939} y_{n-3} + \frac{341643939}{569406565} y_{n-2} - \frac{72003623}{113881313} y_{n-1} + \frac{426060731}{341643939} y_n + \frac{6274637}{16268759} y_{n+2} - \frac{143998979}{1708219695} y_{n+3} + \frac{1847955}{113881313} y_{n+4} - \frac{9603792}{113881313} f_{n-2}$$

$$y_{n+2} = -\frac{79696}{845265} y_{n-3} + \frac{41929759}{9861425} y_{n-2} - \frac{68414023}{3944570} y_{n-1} + \frac{189894686}{5916855} y_n - \frac{7210474}{394457} y_{n+1} + \frac{21582821}{59168550} y_{n+3} + \frac{14016}{1972285} y_{n+4} + \frac{19789614}{1972285} f_{n-1} \qquad (5)$$

$$y_{n+3} = \frac{70450}{1797393} y_{n-3} - \frac{1295843}{1198262} y_{n-2} + \frac{5593225}{599131} y_{n-1} + \frac{676840}{105729} y_n - \frac{11495780}{599131} y_{n+1} + \frac{6496015}{1198262} y_{n+2} + \frac{42690}{599131} y_{n+4} + \frac{845710}{46087} f_n$$

$$y_{n+4} = -\frac{338687}{348237} y_{n-3} - \frac{353855969}{77076456} y_{n-2} + \frac{2326014617}{19269114} y_{n-1} - \frac{4938738481}{115614684} y_n + \frac{1117145237}{19269114} y_{n+1} - \frac{11296250177}{77076456} y_{n+2} + \frac{495749336}{28903671} y_{n+3} + \frac{951570371}{3211519} f_{n+1}$$

## Order of the Method

In this section, we derive the order of the methods (5). It can be transform to a general matrix form as follows

$$\sum_{j=0}^{1} C_j^* Y_{m+j-1} = h \sum_{j=0}^{1} D_j^* Y_{m+j-1}, \qquad (6)$$

Let $C_0^*, C_1^*, D_0^*$ and $D_1^*$ be block matrices defined by

$$C_0^* = [C_{0,} C_1, C_2, C_3], C_1^* = [C_4, C_5, C_6, C_7], \quad D_0^* = [D_0, D_1, D_2, D_3], \mathrm{D}_1^* = [D_4, D_5, D_6, , D_7]$$

Where $C_0^*$, $C_1^*$, $D_0^*$ and $D_1^*$ are square matrices and $\mathrm{Y}_{\mathrm{m}-1}, \mathrm{Y}_{\mathrm{m},} \mathrm{F}_{\mathrm{m}-1}$ and $\mathrm{F}_{\mathrm{m}}$ are column vectors defined by

$$Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \\ y_{3m+4} \end{bmatrix}, \; Y_{m-1} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \\ y_{3(m-1)+4} \end{bmatrix}, \; F_{m-1} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_{3(m-1)+1} \\ f_{3(m-1)+2} \\ f_{3(m-1)+3} \\ f_{3(m-1)+4} \end{bmatrix}$$

$$F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} = \begin{bmatrix} f_{3m+1} \\ f_{3m+2} \\ f_{3m+3} \\ f_{3m+4} \end{bmatrix} \qquad (7)$$

Thus, equations (5) can be rewritten as

$$\begin{bmatrix} -\frac{1298881}{341643939} & \frac{341643939}{569406565} & -\frac{7200362}{113881313} & \frac{342606073}{34164393} \\ -\frac{79696}{845265} & \frac{41929759}{9861425} & -\frac{6841402}{3944570} & \frac{318989468}{5916855} \\ \frac{70450}{1797393} & -\frac{1295843}{1198262} & \frac{5593225}{599131} & \frac{676840}{105729} \\ -\frac{338687}{348237} & -\frac{353855969}{77076456} & \frac{2326014617}{19269114} & -\frac{4938848}{1156146} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} +$$

$$\begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{7129625017}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = h$$

$$\begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{84571}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \qquad (8)$$

From the (8) we have

$$C_0^* = \begin{bmatrix} -\frac{1298881}{341643939} & \frac{341643939}{569406565} & -\frac{72003623}{113881313} & \frac{42606073}{34164393} \\ -\frac{79696}{845265} & \frac{41929759}{9861425} & -\frac{68414023}{3944570} & \frac{18989468}{5916855} \\ \frac{70450}{1797393} & -\frac{1295843}{1198262} & \frac{5593225}{599131} & \frac{676840}{105729} \\ -\frac{338687}{348237} & -\frac{353855969}{77076456} & \frac{2326014617}{19269114} & -\frac{4938848}{1156146} \end{bmatrix}$$

$$C_1^* = \begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{7129625017}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix}$$

$$D_0^* = \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{84571}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} D_1^* = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix}$$

Where

$$C_0 = \begin{bmatrix} -\frac{1298881}{3416439} \\ -\frac{79696}{845265} \\ \frac{70450}{1797393} \\ -\frac{338687}{348237} \end{bmatrix} C_1 = \begin{bmatrix} \frac{34164393}{56940656} \\ -\frac{4192975}{9861425} \\ -\frac{1295843}{1198262} \\ -\frac{3538559}{348237} \end{bmatrix} C_2 = \begin{bmatrix} -\frac{7200362}{1138813} \\ -\frac{6841402}{3944570} \\ \frac{5593225}{599131} \\ \frac{23260146}{1926911} \end{bmatrix} C_3 = \begin{bmatrix} \frac{42606073}{34164393} \\ \frac{18988946}{5916855} \\ \frac{676840}{105729} \\ -\frac{4938848}{1156146} \end{bmatrix}$$

$$C_4 = \begin{bmatrix} 1 \\ \frac{7210474}{394457} \\ \frac{11495780}{599131} \\ -\frac{11171452}{1926911} \end{bmatrix} C_5 = \begin{bmatrix} -\frac{6274637}{1626875} \\ 1 \\ -\frac{6496015}{1198262} \\ \frac{112962507}{7707645} \end{bmatrix} C_6 = \begin{bmatrix} \frac{14399897}{17082196} \\ -\frac{2158282}{5916855} \\ 1 \\ -\frac{4957493}{2890367} \end{bmatrix} C_7 = \begin{bmatrix} -\frac{9603792}{1138813} \\ -\frac{14016}{1972285} \\ -\frac{42690}{599131} \\ 1 \end{bmatrix}$$

$$D_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_1 = \begin{bmatrix} \frac{9603792}{1138813} \\ 0 \\ 0 \\ 0 \end{bmatrix} D_2 = \begin{bmatrix} 0 \\ \frac{1978961}{197228} \\ 0 \\ 0 \end{bmatrix} D_3 = \begin{bmatrix} 0 \\ 0 \\ \frac{84571}{46087} \\ 0 \end{bmatrix} D_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{9515703}{3211519} \end{bmatrix}$$

$$D_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{9515703}{3211519} \end{bmatrix} D_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} D_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

**Definition:** (Order of the method) the order of the block method (5) and its associated linear operator are given by

$$L[y(x);h] = \sum_{j=0}^{7}\left[C_j y(x+jh)\right] - h\sum_{j=0}^{7}\left[D_j y'(x+jh)\right] \quad (9)$$

where p is unique integer such that

$E_q = 0, \; q = 0,1, \ldots \; p$ and $E_{p+1} \neq 0$, where the $E_q$ are constant Matrix.
With

$$E_0 = \sum_{J=0}^{7} C_J = 0,$$

$$E_1 = \sum_{J=0}^{7} \left[ jC_j - 2D_j \right] = 0,$$

$$E_2 = \sum_{J=0}^{7} \left[ \frac{1}{2!} j^2 C_j - 2jD_j \right] = 0,$$

$$E_3 = \sum_{J=0}^{7} \left[ \frac{1}{3!} j^3 C_j - 2\frac{1}{2!} j^2 D_j \right] = 0,$$

$$E_4 = \sum_{J=0}^{7} \left[ \frac{1}{4!} j^4 C_j - 2\frac{1}{3!} j^3 D_j \right] = 0,$$

$$E_5 = \sum_{J=0}^{7} \left[ \frac{1}{5!} j^5 C_j - 2\frac{1}{4!} j^4 D_j \right] = 0,$$

$$E_6 = \sum_{J=0}^{7} \left[ \frac{1}{6!} j^6 C_j - 2\frac{1}{5!} j^5 D_j \right] = 0$$

$$E_7 = \sum_{J=0}^{7} \left[ \frac{1}{7!} j^7 C_j - 2\frac{1}{6!} j^6 D_j \right] \neq 0$$

Therefore, the method is of order 6, with error constant as: $$E_7 = \begin{bmatrix} -\frac{210}{829358} \\ \frac{324}{3184255} \\ -\frac{981}{692640} \\ \frac{563}{5947581} \end{bmatrix} \qquad (10)$$

## Zero Stability of the Method

**Definition:** (Zero-Stable) A linear multistep method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple.
The method (5) is converted into matrix form as:

$$\begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{1708219695} & -\frac{9603792}{113881313} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{7129625017}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = h$$

$$\begin{bmatrix} \frac{1298881}{341643939} & -\frac{341643939}{569406565} & \frac{72003623}{113881313} & -\frac{4260607}{3416439} \\ \frac{79696}{845265} & -\frac{41929759}{9861425} & \frac{68414023}{3944570} & -\frac{1898946}{5916855} \\ -\frac{70450}{1797393} & \frac{1295843}{1198262} & -\frac{5593225}{599131} & -\frac{676840}{105729} \\ \frac{338687}{348237} & \frac{353855969}{77076456} & -\frac{2326014617}{19269114} & \frac{49388481}{11561468} \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} + h$$

$$\begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{84571}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} \quad (11)$$

The equation above can be written in matrix form as:

$$A_0 Y_m = A_1 Y_{m-1} + h(B_0 F_{m-1} + B_1 F_m) \quad (12)$$

Where

$$
A_0 = \begin{bmatrix} 1 & -\frac{6274637}{16268759} & \frac{143998979}{170821969} & -\frac{9603792}{1138813} \\ \frac{7210474}{394457} & 1 & -\frac{21582821}{59168550} & -\frac{14016}{1972285} \\ \frac{11495780}{599131} & -\frac{6496015}{1198262} & 1 & -\frac{42690}{599131} \\ -\frac{1117145237}{19269114} & \frac{11296250717}{77076456} & -\frac{495749336}{28903671} & 1 \end{bmatrix},
$$

$$
A_1 = \begin{bmatrix} \frac{1298881}{341643939} & -\frac{341643939}{569406565} & \frac{72003623}{113881313} & -\frac{4260607}{3416439} \\ \frac{79696}{845265} & -\frac{41929759}{9861425} & \frac{68414023}{3944570} & -\frac{1898946}{591685} \\ -\frac{70450}{1797393} & \frac{1295843}{1198262} & -\frac{5593225}{599131} & -\frac{676840}{105729} \\ \frac{338687}{348237} & \frac{353855969}{77076456} & -\frac{232601461}{19269114} & \frac{74938848}{1156146} \end{bmatrix}
$$

$$
B_0 = \begin{bmatrix} 0 & \frac{9603792}{113881313} & 0 & 0 \\ 0 & 0 & \frac{19789614}{1972285} & 0 \\ 0 & 0 & 0 & \frac{84571}{46087} \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{951570371}{3211519} & 0 & 0 & 0 \end{bmatrix}
$$

$Y_{m-1}$ , $Y_m$ , $F_{m-1}$ and $F_m$ are column vectors defined by

$$
Y_m = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix} = \begin{bmatrix} y_{3m+1} \\ y_{3m+2} \\ y_{3m+3} \\ y_{3m+4} \end{bmatrix}, \; Y_{m-1} = \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} = \begin{bmatrix} y_{3(m-1)+1} \\ y_{3(m-1)+2} \\ y_{3(m-1)+3} \\ y_{3(m-1)+4} \end{bmatrix}, \; F_{m-1} = \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_{3(m-1)+1} \\ f_{3(m-1)+2} \\ f_{3(m-1)+3} \\ f_{3(m-1)+4} \end{bmatrix}
$$

$$
F_m = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} = \begin{bmatrix} f_{3m+1} \\ f_{3m+2} \\ f_{3m+3} \\ f_{3m+4} \end{bmatrix}
$$

Substituting scalar test equation $y' = \lambda y$ ($\lambda < 0$, $\lambda$ complex) into (12) and using $\lambda h = \bar{h}$ gives

$$
A_0 Y_m = A_1 Y_{m-1} + \bar{h}(B_0 Y_{m-1} + B_1 Y_m) \quad (13)
$$

The stability polynomial of (5) is obtained by evaluating

$$
det\left[(A_0 - \bar{h}B_1)t - (A_1 + \bar{h}B_0)\right] = 0 \quad (14)
$$

to get

$$R(\bar{h},t) = \frac{99609031680754755943510 33033}{13297532593636682035736546 0}h - \frac{44673770968029686867584473 1429}{10638026074909345628589236 8}t - \frac{26789683229850758208572552 49}{6648766296818341017868273}t^4h - \frac{19373318495630480438742009 6}{6648766296818341017868273}t^3h^2 - \frac{25608169462430881261642608}{6648766296818341017868273}h^2 + \frac{63817166369731096642244097 6921}{10638026074909345628589236 8}t^2 - \frac{12132642021177303935671531 27537}{1994629889045502305360481 90}th - \frac{56763008250718866056723851 59541}{39892597780910046107209638 0}t^2h - \frac{15670433886392683477783391 12886}{33243831484091705089341365}t^3h + \frac{63676135712478235030238341 44}{5114435612937185398360 21}t^2h^2 - \frac{28326438751226632796183511 36}{6648766296818341017868273}th^2 + \frac{30595712343518318988667155 247}{53190130374546728142946184 0} - \frac{92156513088949372852808209}{5114435612937185398360210}t^4 - \frac{15356062873898556875117053 83}{835009895989744554834320}t^3 \quad (15)$$

By putting $\bar{h} = 0$ in (14), we obtain the first characteristic polynomial as

$$R(0,t) = -\frac{44673770968029686867584473 1429}{10638026074909345628589236 8}t + \frac{63817166369731096642244097 6921}{10638026074909345628589236 8}t^2 - \frac{15356062873898556875117053 83}{835009895989744554834320}t^3 - \frac{92156513088949372852808209}{5114435612937185398360210}t^4 + \frac{30595712343518318988667155 247}{53190130374546728142946184 0} \quad (16)$$

Since, the roots of (16) are $t_1 = 1$ *and* $t_2, t_3, t_4 \leq 0$
Therefore, the method (5) is zero Stable.

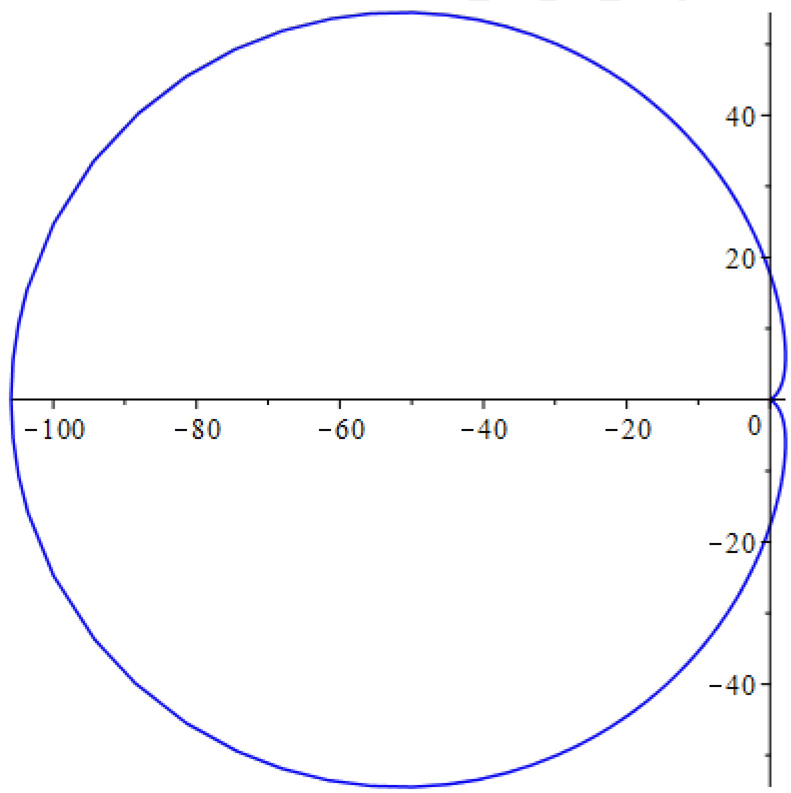


Image 1 . The A-stability region of the proposed scheme ABISBDF

## Test Problems

To validate the method developed, the following stiff IVPs are solved.

Problem 1 : $y_1' = y_2 y_1(0) = 1$
$y_2' = y_1 y_2(0) = 1 \quad 0 \leq x \leq 100$
Exact Solution
$y_1(x) = e^x$
$y_2(x) = e^x$
Source: (Bronson, 1973)

*Problem* 2: $y_1' = 198y_1 + 199y_1 y_1(0) = 1$
$0 \leq x \leq 10$
$y_2' = -398y_1 - 399y_2 \quad y_2(0) = -1$
Exact solution
$y_1(x) = e^{-x}$

$y_2(x) = -e^{-x}$
Eigen values −1 and −200
Source: (Ibrahim *et al.*, 2007);
*Problem* 3: $y_1' = y_2\ y_1(0) = 0$
$0 \leq x \leq 20$
$y_2' = -y_1\ y_2(0) = 1$
Exact solution
$y_1(x) = sinx$

$y_2(x) = cosx$
Source: (shampine*et al*, 1975)

Problem 4 : $y_1' = y_2 y_1(0) = 0$
$y_2' = -2y_2 y_2(0) = 0 \quad 0 \leq x \leq 4л$
$y_3' = y_2 + 2y_3 y_3(0) = 1$
Exact Solution
$y_1(x) = 2cosx + 6sinx - 6x - 2$
$y_2(x) = -2sinx + 6cosx - 6$
$y_3(x) = 2sinx - 2cosx + 3$
Source: (Sulaiman,1989)

## Numerical Results

The problems sampled in this research are solved using the developed scheme. The results are tabulted, compared; and the graphs highlighting the performance of these methods are plotted. The acronyms below are used in the tables.
*h* = step-size;
MAXE = Maximum Error;
T=Time in second;

3ESBBDF = 3 Point enhanced fully implicit Super Class of Block Backward Differentiation
$F_1$3ESBBDF = Family of block 3 Super class of Block Backward Differentiation
ABISBDF = A-stable block integrator scheme of Backward Differentiation Formula for solving Stiff IVPs.

Table 1: Comparison of Errors between Proposed Method and Some of the Existing Methods for Problem 1 & 2

| Numerical Result for Problem 1 | | | | Numerical Result for Problem 2 | | | |
|---|---|---|---|---|---|---|---|
| $h$ | Method | MAXE | TIME | $h$ | Method | MAXE | TIME |
| $10^{-2}$ | $F_1$3SBBDF | 3.30736e-002 | 4.23434e-1 | $10^{-2}$ | $F_1$3SBBDF | 3.23032e-002 | 3.77590e-002 |
| | 3ESBSBDF | 3.51456e-002 | 3.52416e-4 | | 3ESBSBDF | 3.98707e-002 | 2.63337e-002 |
| | ABISBDF | 5.83217e-004 | 4.23441e-5 | | ABISBDF | 5.83217e-003 | 5.68676e-002 |
| $10^{-3}$ | $F_1$3SBBDF | 5.41853e-003 | 1.81850e-3 | $10^{-3}$ | $F_1$3SBBDF | 4.76165e-003 | 5.66636e-001 |
| | 3ESBSBDF | 5.20191e-003 | 2.50367e-3 | | 3ESBSBDF | 4.40956e-003 | 2.60816e-001 |
| | ABISBDF | 6.95338e-005 | 4.65467e-4 | | ABISBDF | 6.05338e-005 | 5.64515e-001 |
| $10^{-4}$ | $F_1$3SBBDF | 5.44701e-005 | 1.71443e-2 | $10^{-4}$ | $F_1$3SBBDF | 4.66516e-004 | 5.64385e-001 |
| | 3ESBSBDF | 5.20417e-005 | 2.36918e-2 | | 3ESBSBDF | 5.08942e-005 | 2.60725e-001 |
| | ABISBDF | 6.95692e-007 | 4.48433e-3 | | 4BSBDF | 6.26692e-007 | 5.68143e+000 |
| $10^{-5}$ | $F_1$3SBBDF | 5.44971e-007 | 1.70042e-1 | $10^{-5}$ | $F_1$3SBBDF | 4.68707e005 | 5.63788e+000 |
| | 3ESBSBDF | 5.25030e-007 | 2.34808e-1 | | 3ESBSBDF | 5.21534e-007 | 2.60597e+000 |
| | ABISBDF | 6.959740e-009 | 4.58687e-2 | | ABISBDF | 6.32740e-009 | 5.59821e+001 |
| $10^{-6}$ | $F_1$3SBBDF | 5.44998e-009 | 1.70308e0 | $10^{-6}$ | $F_1$3SBBDF | 4.69123e-006 | 5.65356e+001 |
| | 3ESBSBDF | 5.25648e-009 | 2.35791e0 | | 3ESBSBDF | 5.89872e-009 | 2.60700e+001 |
| | ABISBDF | 7.186362e-011 | 4.23434e-1 | | ABISBDF | 6.33362e-011 | 5.53567e+002 |

Table 2: Comparison of Errors between Proposed Method and Some of the Existing Methods for Problem 3 & 4

| Numerical Result for Problem 3 | | | | Numerical Result for Problem 4 | | | |
|---|---|---|---|---|---|---|---|
| $h$ | Method | MAXE | TIME | $h$ | Method | MAXE | TIME |
| $10^{-2}$ | $F_1$3SBBDF | 2.07208e-002 | 1.37500e-2 | $10^{-2}$ | $F_1$3SBBDF | 2.83032e-002 | 3.67590e-002 |
| | 3ESBSBDF | 2.54347e-002 | 1.20394e-3 | | 3ESBSBDF | 2.48705e-002 | 2.63337e-002 |
| | ABISBDF | 2.83117e-004 | 7.36289e-2 | | ABISBDF | 3.83217e-003 | 5.58676e-002 |
| $10^{-3}$ | $F_1$3SBBDF | 3.20160e-004 | 2.72200e-2 | $10^{-3}$ | $F_1$3SBBDF | 3.76163e-003 | 8.56636e-002 |
| | 3ESBSBDF | 3.02893e-004 | 1.25972e-2 | | 3ESBSBDF | 3.40956e-003 | 2.60816e-001 |
| | ABISBDF | 4.05338e-006 | 5.81512e-2 | | ABISBDF | 4.05338e-005 | 5.54515e-001 |
| $10^{-4}$ | $F_1$3SBBDF | 3.20233e-006 | 2.02700e-1 | $10^{-4}$ | $F_1$3SBBDF | 3.76514e-005 | 8.54385e-001 |
| | 3ESBSBDF | 3.09895e-006 | 1.25148e-1 | | 3ESBSBDF | 3.48942e-005 | 2.60725e+000 |
| | ABISBDF | 4.26592e-008 | 5.81491e-1 | | ABISBDF | 4.26690e-007 | 5.58143e-001 |
| $10^{-5}$ | $F_1$3SBBDF | 3.20261e-008 | 1.92600e0 | $10^{-5}$ | $F_1$3SBBDF | 3.70705e005 | 8.53788e+000 |
| | 3ESBSBDF | 3.10157e-008 | 1.25471e0 | | 3ESBSBDF | 3.58532e-005 | 2.60597e+001 |
| | ABISBDF | 4.32640e-010 | 5.81122e0 | | ABISBDF | 4.32740e-009 | 5.49821e+000 |
| $10^{-6}$ | $F_1$3SBBDF | 3.20263e-010 | 1.91700e1 | $10^{-6}$ | $F_1$3SBBDF | 3.71121e-007 | 8.53356e+001 |
| | 3ESBSBDF | 3.41129e-010 | 1.24892e1 | | 3ESBSBDF | 3.69872e-007 | 2.60700e+002 |
| | ABISBDF | 4.33262e-012 | 5.79987e1 | | ABISBDF | 4.3335e-009 | 5.43567e+001 |



From the numerical problems solved in the Table 1 and Table 2 above, it has been shown that the proposed scheme, ABISBDF outperformed both the 3ESBSDF and $F_1$3SBBDF in terms of minimum error and less computational time. However, $F_1$3SBBDF has advantage over 3ESBSBDF in the entire tested problems with regard to errors. But, 3ESBSBDF competes closely with the $F_1$3SBBDF in terms of execution time, with 3ESBSBDF favored in most of the problem considered in this paper. Finally, the accuracy, computation time of the new methods seems to be better in comparison with the other two methods for all the problems solved.

Similarly, to highlight the performance of the proposed methods, ABISBDF in relation to the other methods, 3ESBSBDF and $F_1$3SBBDF. The graphs of *Log10(MAXE)* against the step size, *h* for the 4 problems are plotted accordingly as shown below:

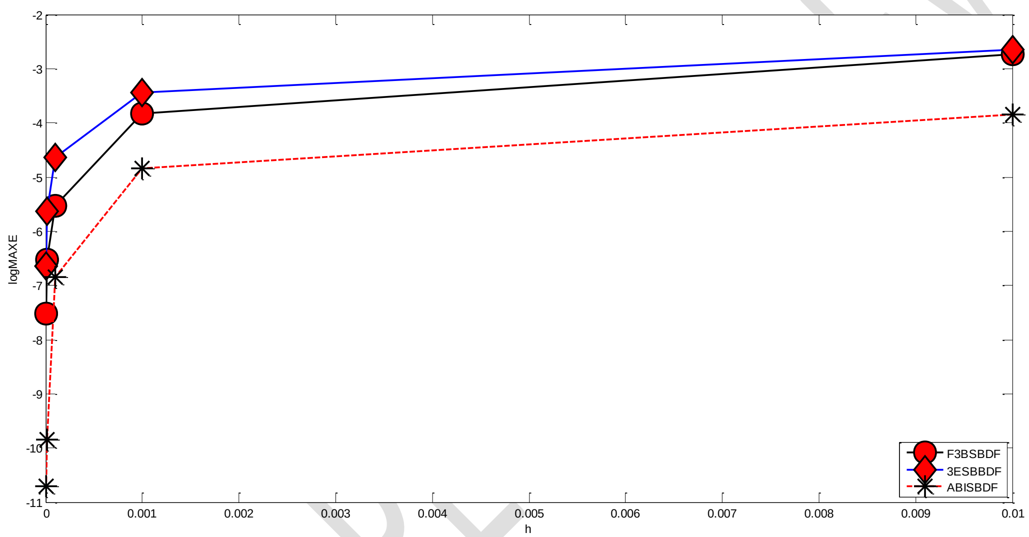


**Figure 1:** Graph of $\mathrm{Log}_{10}(MAXE)$ against h for problem 1

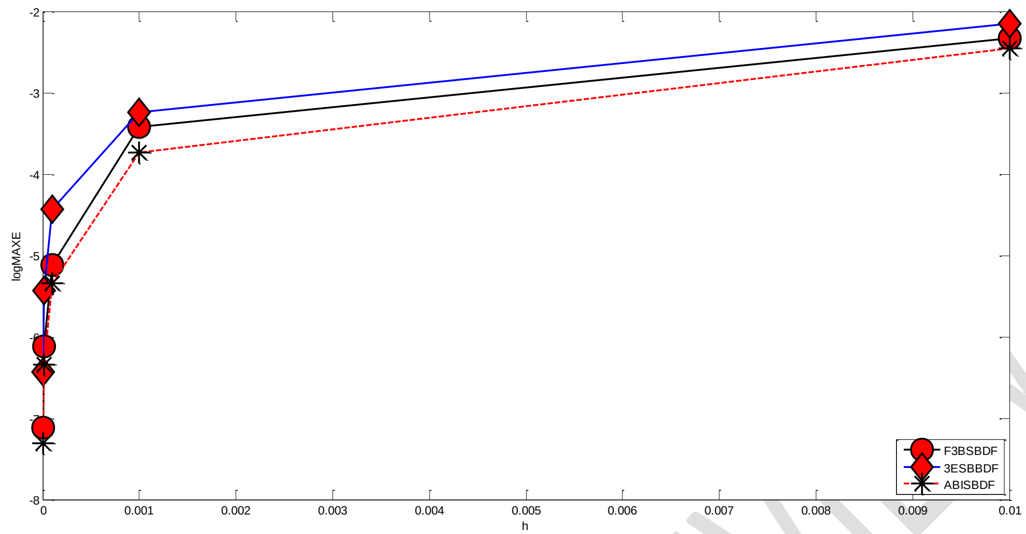


**Figure 2:** Graph of $\text{Log}_{10}(MAXE)$ against h for problem 2

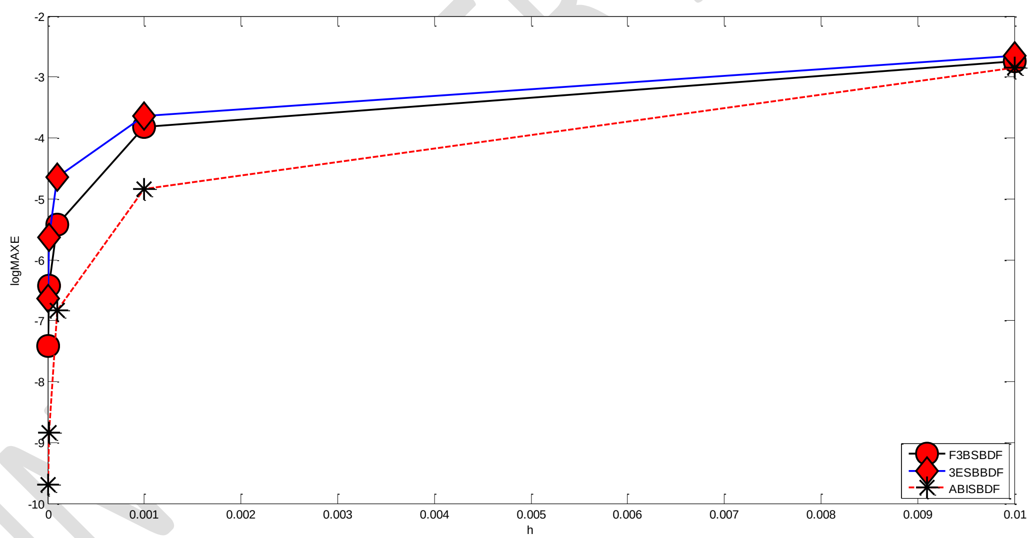


**Figure 3:** Graph of $\text{Log}_{10}(MAXE)$ against h for problem 3

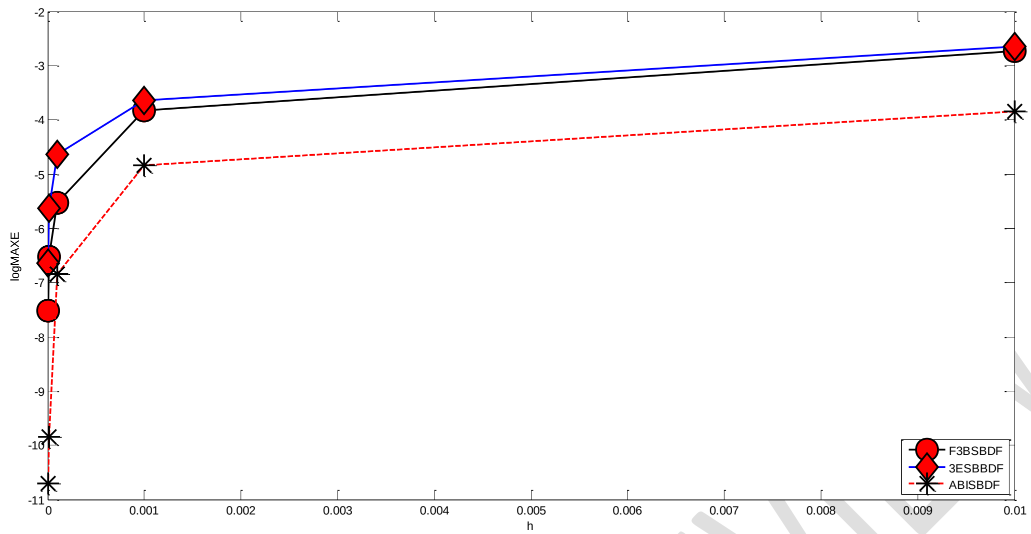


**Figure 4:** Graph of $\mathrm{Log}_{10}(MAXE)$ against h for problem 4

## Conclusion

An order sixth block numerical scheme has been proposed, the scheme have good stability properties. The developed methods are fully implicit block methods, can computes, four solution values at a time per step, concurrently. The tested problem's results shows advantage in terms of accuracy of the scaled error and computational time when compared with the 3ESBSBDF and $F_1$3SBBDF methods. The proposed scheme can be used in solving a system of first order initial value problem of ordinary differential equations.

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