UNDER PEER REVIEW

A New Generalization for Generalized Inverted Exponential Distribution with Real Data Applications

Original Research Article

Abstract

The generalization of standard and generalized distributions has become one of the concerns that the statistical theory depends on to obtain more flexible distributions. In this article, a new distribution that is considered a generalization of the generalized inverted exponential distribution called the Type II Topp Leone Generalized Inverted Exponential (TIITLGIE) distribution is introduced. Some statistical properties of this distribution are obtained. The quantile function, median, moments, moment generating function, Reliability function, hazard function, mode, harmonic mean, mean and median deviation are derived. Furthermore, important measures such Rényi entropy and the Maximum Likelihood (ML) estimation are deduced for parameters. Conduct a Monte Carlo simulation to study behavior of parameter estimates. Finally, applications on three real data sets are discussed.

Keywords: Generalized Inverted Exponential Distribution. Type II Tope – Leone. Rényi entropy. Maximum Likelihood estimation. Monte Carlo simulation.

1 Introduction

Some complex real phenomena need new lifetime distributions to modeled with. For this reason, the researchers in the area of statistics and distribution theory have been attracted to generate different lifetime models. The great importance of these models can be seen and found in many fields of study such as finance, insurance, reliability engineering and survival analysis. In the last few years, the authors have proposed new families of distribution by adding an additional parameter using generator or combining existing distributions. Some of these families are: Exponentiated-G family (7), Lomax-G family (8),the type I Topp-Leone-G family (5),Topp Leone Exponentiated-G Family (13), Type II power Topp-Leone generated family (6). Recently Elgarhy etal. (2018), introduced the type II Topp-Leone-G family (for short TIITL-G) using the half logistic distribution as a generator instead of the gamma generator in the cdf of Ristic-Balakrishnan-G (10).this family was characterized by more flexible, which made it of interest to many researchers. Many lifetime models were generated using this family. For example: type II topp-leone generalized inverse Rayleigh distribution (18) , type II topp-leone power Lomax distribution (3) and type II topp-leone inverse exponential distribution (2). The CDF of TIITL distribution is given by:

$$F(x) = 1 - [1 - G^2(x)]^\alpha \tag{1.1}$$

The corresponding PDF of (1.1) is given by :

$$f(x) = 2\alpha\ g(x)G(x)[1 - G^2(x)]^{\alpha-1} \tag{1.2}$$

where $\alpha > 0$ is a shape parameter, G(x) is a baseline CDF distribution and g(x) is the baseline PDF distribution.

A two-parameter Generalized Inverted exponential (GIE) distribution was proposed by Abouammoh and Alshingiti (1) as a generalization of the Inverted Exponential (IE) distribution which is better than the Inverted Exponential when goodness of fit was assessed using the Likelihood Ratio and Kolmogorov-Smirnov tests. Recently, there are many authors who studied the GIE distribution. For example: Dey and Nassar (9).
The probability density function (PDF) of a tow parameter GIE distribution is given by Abouammoh and Alshingiti as:

$$g(x) = (\frac{\theta\lambda}{x^2})exp(\frac{-\lambda}{x})[1 - exp(\frac{-\lambda}{x})]^{\theta-1}, x > 0, \lambda, \theta > 0, \tag{1.3}$$

and the cumulative distribution function (CDF) is given by:

$$G(x) = 1 - [1 - exp(\frac{-\lambda}{x})]^\theta, x > 0, \lambda, \theta > 0, \tag{1.4}$$

where θ is the shape parameter and λ is the scale parameter.

This article aims to combining the works of Abouammoh and Alshingiti and Elgarhy et al. in order to define and provide the basic statistical properties of our new model called Type II Topp-Leone Generalized Inverted Exponential Distribution (as short TIITLGIE). This new model show that it is more flexible in real applications using three different real data sets.

In Section 2, we introduce the TIITLGIE distribution. Statistical properties of the model are derived in Section 3. Rényi entropy derived in Section 4. In Section 5, Maximum Likelihood estimators of parameters are derived. We will provide simulation study in Section 6. Finally, three real data sets will be applied in Section 7. Various conclusions are addressed in Section 8.

2 The Type II Topp-Leone Generalized Inverted Exponential Distribution

In this section, we derived three parameter TIITLGIE Distribution. The CDF and PDF of TIITLGIE distribution with three parameters(α,λ,θ) is obtained by inserting (1.3) and (1.4) in (1.1) and (1.2):

$$F(x) = 1 - [1 - [1 - [1 - exp(\frac{-\lambda}{x})]^\theta]^2]^\alpha, x > 0, \lambda, \theta, \alpha > 0, \tag{2.1}$$

and

$$f(x) = \frac{2\alpha\theta\lambda}{x^2} exp(\frac{-\lambda}{x})[1 - exp(\frac{-\lambda}{x})]^{\theta-1}[1 - [1 - exp(\frac{-\lambda}{x})]^{\theta}] \\ \times [1 - [1 - [1 - exp(\frac{-\lambda}{x})]^{\theta}]^2]^{\alpha-1}, x > 0, \lambda, \theta, \alpha > 0, \tag{2.2}$$

where, λ is scale parameter and θ, α are shape parameters.

We can rewrite the CDF & PDF of TIITLGIE distribution using following infinite power series as follows:

$$F(x) = 1 - \sum_{s=0}^{\infty} \nu_s \; exp(\frac{-\lambda}{x} s), \tag{2.3}$$

where

$$\nu_s = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k+s}}{(\alpha+1)(2j+1)(\theta k+1)\beta(j+1, \alpha-j+1)\beta(k+1, 2j-k+1)\beta(s+1, \theta k-s+1)}. \tag{2.4}$$

and

$$f(x) = \frac{\lambda}{x^2} \sum_{s=0}^{\infty} (s+1) \; \psi_s \; exp(\frac{-\lambda}{x}(s+1)), \tag{2.5}$$

where

$$\psi_s = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{j+k+s}}{(j+1)(k+1)(s+1)\beta(j+1, \alpha-j)\beta(k+1, 2(j+1)-k)\beta(s+1, (k+1)\theta-s)}. \tag{2.6}$$

The PDF of TIITLGIED can be rewrite as

$$f(x) = \sum_{s=0}^{\infty} (s+1) \; \psi_s \; g_{\lambda(s+1)}(x), \tag{2.7}$$

where $g_{\lambda(s+1)}(x)$ is the PDF of inverted exponential distribution with scale parameter $\lambda(s+1)$.

Some Ideal Sub Models as Special Cases from Our Proposed Distribution:

- For θ = 1, the proposed distribution in (2.1) converts to Type II Topp Leone Inverse Exponential (TIITLIE) distribution.
- For θ = 1 and λ = 1, the proposed distribution reduces to Type II Topp Leone Standard Inverse Exponential (TIITLSIE) distribution.
- For λ = 1, the proposed distribution reduces to Type II Topp Leone Generalized Standard Inverse Exponential (TIITLGSIE) distribution.

Figure (1) shows that shape of the probability density function is positively skewed and unimodal for different values of the parameters.

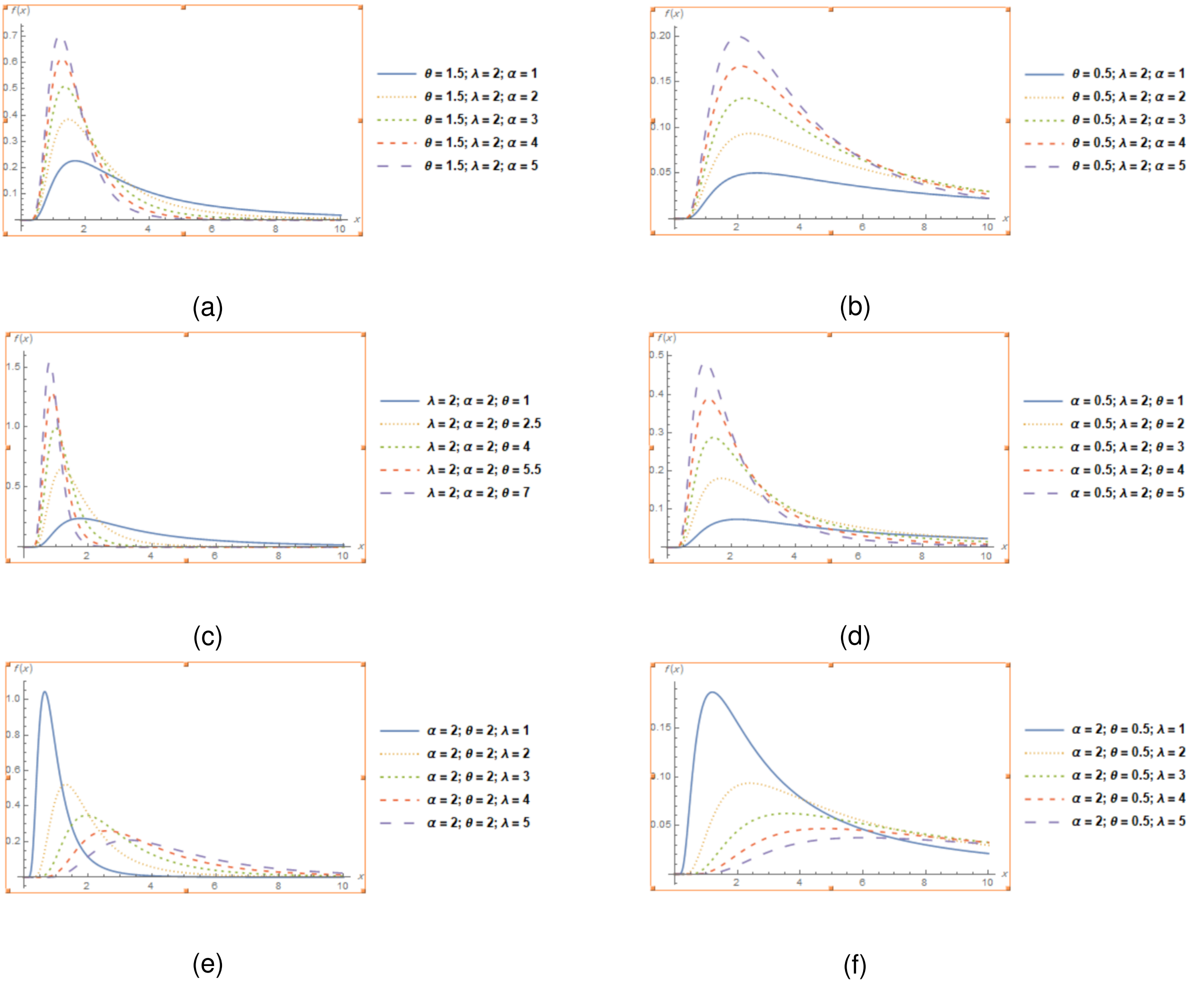


Figure 1: Plots of PDF of TIITLGIE distribution for different values of the parameters when (a,b) α increases, (c,d) θ increases, (e,f) λ increases.

3 Properties of TIITLGIE Distribution

3.1 Quantile and Median

The Quantile function of random variable X of TIITLGIE distribution is given by:

$$x = Q(u) = \frac{-\lambda}{log[1-[1-[1-[1-u]^{\frac{1}{\alpha}}]^{\frac{1}{2}}]^{\frac{1}{\theta}}]} \tag{3.1}$$

where U $\sim$ uniform (0,1), we can derive the median of TIITLGIE distribution by setting u = 0.5 in (3.1). The median (M) is given by:

$$M = \frac{-\lambda}{log[1-[1-[1-[0.5]^{\frac{1}{\alpha}}]^{\frac{1}{2}}]^{\frac{1}{\theta}}]} \tag{3.2}$$

3.2 Moments and Moment Generating Function

The r^{th} moment of TIITLGIE distribution random variable X is given by:

$$\mu_r' = \lambda \sum_{s=0}^{\infty}(s+1)\psi_s \int_0^{\infty} x^{r-2} exp(\frac{-\lambda}{x}(s+1))dx, \tag{3.3}$$

where ψ_s is defined in Equation (2.6).
By setting $u = \frac{\lambda}{x}(s+1)$
We obtain the r^{th} moment of TIITLGIE distribution:

$$\mu_r' = \lambda^r \sum_{s=0}^{\infty}(s+1)^r \psi_s [E_r(1) + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n-r+1)n!}], \tag{3.4}$$

where $E_r(1)$ is the integration exponential function, and ψ_s was known in Equation (2.6).
Substituting r = 1 in (3.4), we obtain the mean of TIITLGIE Distribution as follows :

$$\mu = \lambda \sum_{s=0}^{\infty}(s+1)\psi_s [E_1(1) + \sum_{n=0}^{\infty} \frac{(-1)^n}{nn!}]. \tag{3.5}$$

The moment generating function (MGF) of TIITLGIE distribution is given by:

$$\begin{aligned} M_x(t) &= \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{t^r}{r!} \lambda^r (s+1)^r \psi_s [E_r(1) + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n-r+1)n!}]. \end{aligned}$$

3.3 Skewness and Kurtosis

By using quantiles, the skewness and kurtosis of TIITLGIE distribution can be defined.
Bowley's skewness is based on quantiles (15) it was calculated as follows:

$$B = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}. \tag{3.6}$$

Moors' kurtosis (14) is based on octileis, and could be written as :

$$M = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}, \tag{3.7}$$

where Q(·) is the quantile function defined in Equation(3.1).

3.4 Mode

The mode of TIITLGIE distribution can be found by solving the following equation :

$$\frac{df(x)}{dx} = 0. \tag{3.8}$$

By using equation (2.2), we get :

$$f(x)[\frac{-2}{x} + \frac{\lambda}{x^2} - [\frac{\lambda}{x^2}(\theta-1)(1-exp(\frac{-\lambda}{x}))^{-1}exp(\frac{-\lambda}{x})] + \frac{\theta\lambda}{x^2}exp(\frac{-\lambda}{x})(1-exp(\frac{-\lambda}{x}))^{\theta-1}$$
$$\times [1-(1-exp(\frac{-\lambda}{x}))^{\theta}]^{-1} - [(\alpha-1)[1-[1-(1-exp(\frac{-\lambda}{x}))^{\theta}]^2]^{-1}\frac{2\theta\lambda}{x^2}exp(\frac{-\lambda}{x})(1-(1-exp(\frac{-\lambda}{x}))^{\theta})$$
$$\times (1-exp(\frac{-\lambda}{x}))^{\theta-1}] = 0. \tag{3.9}$$

Since $f(x) > 0$, the mode is the solution of the following equation :

$$\frac{-2}{x} + \frac{\lambda}{x^2} - [\frac{\lambda}{x^2}(\theta-1)(1-exp(\frac{-\lambda}{x}))^{-1}exp(\frac{-\lambda}{x})] + \frac{\theta\lambda}{x^2}exp(\frac{-\lambda}{x})(1-exp(\frac{-\lambda}{x}))^{\theta-1}$$
$$\times [1-(1-exp(\frac{-\lambda}{x}))^{\theta}]^{-1} - [(\alpha-1)[1-[1-(1-exp(\frac{-\lambda}{x}))^{\theta}]^2]^{-1}\frac{2\theta\lambda}{x^2}exp(\frac{-\lambda}{x})(1-(1-exp(\frac{-\lambda}{x}))^{\theta})$$
$$\times (1-exp(\frac{-\lambda}{x}))^{\theta-1}] = 0. \tag{3.10}$$

Equation (3.10) is a nonlinear equation and it can not be found analytically. Further, the mode of TIITLGIE distribution can be found numerically by solving (3.10) using Newton- Raphson method.

3.5 Reliability Function and Hazard function

The reliability function and hazard function are very important properties of lifetime distribution.
The reliability function is the probability of the non-failure occurring before time t. While the hazard function is the instantaneous rate of failure at a given time t. The reliability function of TIITLGIE distribution is denoted by R(x), also known as survival function obtained as follows :

$$R(x) = 1 - F(x), \tag{3.11}$$

The survival function of TIITLGIE distribution is obtained by substituting (2.1) in (3.11) to deduce :

$$R(x) = [1-[1-(1-exp(\frac{-\lambda}{x}))^{\theta}]^2]^{\alpha}. \tag{3.12}$$

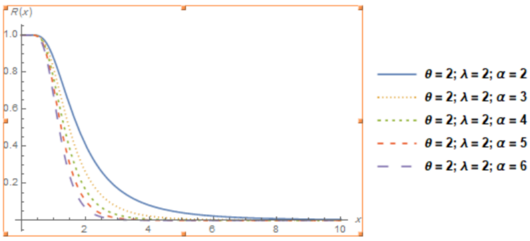
and the corresponding hazard function of TIITLGIE distribution is defined as follows:

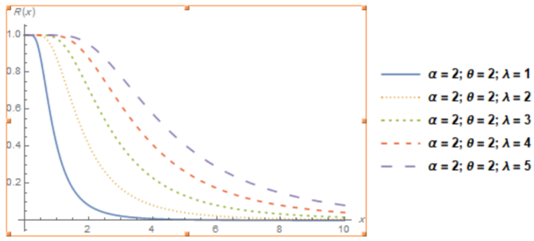
$$h(x) = \frac{f(x)}{1-F(x)}, \tag{3.13}$$

then the hazard function can be written as :

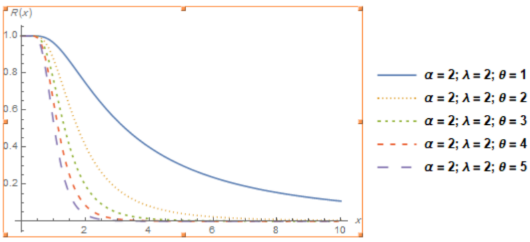
$$h(x) = \frac{2\alpha\lambda\theta}{x^2}exp(\frac{-\lambda}{x})[1-exp(\frac{-\lambda}{x})]^{\theta-1}[1-[1-exp(\frac{-\lambda}{x})]^{\theta}]$$
$$\times[1-[1-(1-exp(\frac{-\lambda}{x}))^{\theta}]^2]^{-1}. \tag{3.14}$$

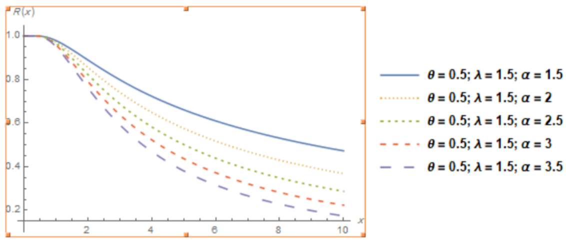
Figure (2) shows that the reliability curves are decreasing for different values of parameters for the TIITLGIE distribution, while Figure (3) shows that the hazard function of the TIITLGIE distribution is increasing at first for different values of parameters, then decreasing in shape. These kind of models are useful in survival analysis. The TIITLGIE distribution shows good statistical behavior based on these two functions.





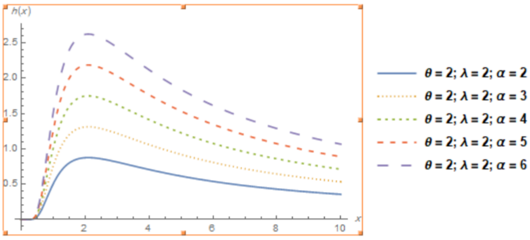
(a) (b)





(c) (d)

Figure 2: Plots of Reliability function of TIITLGIE distribution for different values of the parameters when (a,d) α increases, (b) λ increases, (c) θ increases.



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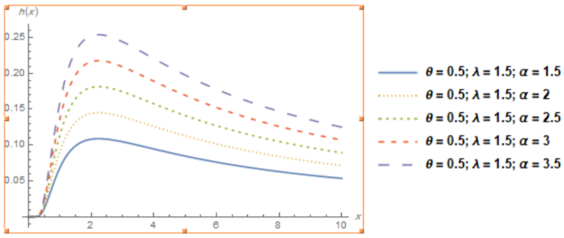

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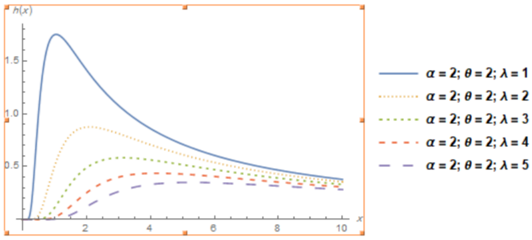
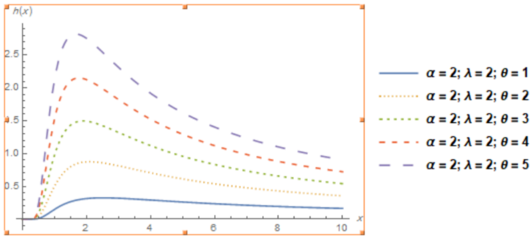


(c)



(d)

Figure 3: Plots of Hazard function of TIITLGIE distribution for different values of the parameters when(a,b) α increases, (c) λ increases, (d) θ increases.

3.6 Harmonic Mean

The Harmonic Mean defined as reciprocal of the arithmetic mean of the reciprocal of the values $x_1, x_2, ..., x_N$ and could be written as :

$$H_m(x) = \frac{1}{E(\frac{1}{x})} = [\int_0^\infty x^{-1} f(x)dx]^{-1}. \tag{3.15}$$

Using equation (2.5), the harmonic mean of TIITLGIE distribution can be derived as follows:

Let:

$$I = \int_0^\infty x^{-1} f(x)dx$$

$$= \lambda \sum_{s=0}^{\infty} (s+1)\psi_s \int_0^\infty x^{-3} exp(\frac{-\lambda}{x}(s+1))dx$$

By setting $u = \frac{\lambda}{x}(s+1)$
We obtain the harmonic mean of TIITLGIE distribution :

$$H_m(x) = [\sum_{s=0}^{\infty} \frac{1}{\lambda(s+1)} \psi_s]^{-1}. \tag{3.16}$$

Table 1: The mode, median, mean, Harmonic Mean, skewness and kurtosis for different values of the parameters.

α	θ	λ	mode	median	mean	Harmonic Mean	skewness	kurtosis
1.5	1.5	2	1.56289	2.60944	3.97169	0.875	0.302401	0.91689
2	1.5	2	1.47466	2.21353	2.94902	0.622135	0.259407	0.761012
2.5	1.5	2	1.40742	1.97581	2.46282	0.437337	0.230537	0.663612
1.5	2	2	1.38028	2.03437	2.68548	0.6875	0.256803	0.753663
2	2	2	1.30144	1.76805	2.15337	0.625595	0.220002	0.630509
2.5	2	2	1.24234	1.60362	1.87148	0.600074	0.195195	0.551819
1.5	2.5	2	1.25663	1.72018	2.11884	0.63125	0.226895	0.653764
2	2.5	2	1.18516	1.51786	1.7653	0.664596	0.19408	0.548956
2.5	2.5	2	1.1321	1.39066	1.56731	0.720075	0.171876	0.481
1	2	1.5	1.12039	1.92522	3.13949	0.611111	0.320047	0.988074
1.5	2	1.5	1.03521	1.52578	2.01411	0.916667	0.256803	0.753663
2.5	2	1.5	0.931755	1.20271	1.40361	0.800099	0.195195	0.551819
1	2	2	1.49385	2.56696	4.18599	0.458333	0.320047	0.988074
1.5	2	2	1.38028	2.03437	2.68548	0.6875	0.256803	0.753663
2.5	2	2	1.24234	1.60362	1.87148	0.600074	0.195195	0.551819
1	2	2.5	1.86732	3.2087	5.23248	0.366667	0.320047	0.988074
1.5	2	2.5	1.72535	2.54296	3.35684	0.55	0.256803	0.753663
2.5	2	2.5	1.55292	2.00452	2.33935	0.48006	0.195195	0.551819

In Table 1 the behavior of the TIITLGIE distribution can be studied, when α and θ are increasing the mode, median, mean and harmonic mean are decreasing, else the skewness and kurtosis are decreasing. By increasing the scale parameter λ the mode, median and mean are increasing but the harmonic mean is decreasing and the skewness and kurtosis remain constant.

3.7 Probability Weighted Moments

The PWMs can be calculated from the following :

$$\tau_{r,\beta} = E[x^r F(x)^\beta] = \int_{-\infty}^{\infty} x^r f(x)(F(x))^\beta dx. \tag{3.17}$$

By using (2.1) and (2.2) we get:

$$\tau_{r,\beta} = \int_{0}^{\infty} 2\alpha\lambda\theta x^{r-2} exp(\frac{-\lambda}{x})(1-exp(\frac{-\lambda}{x}))^{\theta-1}[1-(1-exp(\frac{-\lambda}{x}))^{\theta}]$$
$$\times[1-[1-(1-exp(\frac{-\lambda}{x}))^{\theta}]^2]^{\alpha-1}[1-[1-[1-(1-exp(\frac{-\lambda}{x}))^{\theta}]^2]^{\alpha}]^{\beta}dx.$$

By setting $u = \frac{\lambda}{x}$

$$\tau_{r,\beta} = 2\alpha\theta\lambda^r \int_{0}^{\infty} u^{-r} exp(-u)(1-exp(-u))^{\theta-1}[1-(1-exp(-u))^{\theta}]$$
$$\times[1-[1-(1-exp(-u))^{\theta}]^2]^{\alpha-1}[1-[1-[1-(1-exp(-u))^{\theta}]^2]^{\alpha}]^{\beta}du.$$

Now by applying the binomial expansion, we get

$$\tau_{r,\beta} = 2\alpha\theta\lambda^r \sum_{i=0}^{\infty}(-1)^i \begin{pmatrix} \beta \\ i \end{pmatrix} \int_0^{\infty} u^{-r} exp(-u)(1-exp(-u))^{\theta-1}[1-(1-exp(-u))^{\theta}]$$
$$\times[1-[1-(1-exp(-u))^{\theta}]^2]^{(\alpha-1)+\alpha i} du.$$

Using the binomial expansion again we have

$$\tau_{r,\beta} = 2\alpha\theta\lambda^r \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{s=0}^{\infty} \begin{pmatrix} \beta \\ i \end{pmatrix} \begin{pmatrix} \alpha(1+i)-1 \\ k \end{pmatrix} \begin{pmatrix} 2k+1 \\ t \end{pmatrix} \begin{pmatrix} \theta(t+1)-1 \\ s \end{pmatrix}$$
$$(-1)^{i+k+t+s} \int_0^{\infty} u^{-r} exp(-(1+s)u) du.$$

By setting $z = (1+s)u$

$$\tau_{r,\beta} = 2\alpha\theta\lambda^r \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{s=0}^{\infty} \begin{pmatrix} \beta \\ i \end{pmatrix} \begin{pmatrix} \alpha(1+i)-1 \\ k \end{pmatrix} \begin{pmatrix} 2k+1 \\ t \end{pmatrix} \begin{pmatrix} \theta(t+1)-1 \\ s \end{pmatrix}$$
$$\frac{(-1)^{i+k+t+s}}{(1+s)^{1-r}} \int_0^{\infty} z^{-r} exp(-z) dz.$$

After integrating we get

$$\tau_{r,\beta} = 2\alpha\theta\lambda^r \psi_{ikts} \frac{1}{(1+s)^{1-r}} [E_r(1) + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n-r+1)n!}]. \tag{3.18}$$

$$\psi_{ikts} = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\sum_{s=0}^{\infty} \begin{pmatrix} \beta \\ i \end{pmatrix} \begin{pmatrix} \alpha(1+i)-1 \\ k \end{pmatrix} \begin{pmatrix} 2k+1 \\ t \end{pmatrix} \begin{pmatrix} \theta(t+1)-1 \\ s \end{pmatrix} (-1)^{i+k+t+s}.$$

3.8 The Mean Deviation and the Median Deviation

The mean deviation and the median deviation are measures of dispersion derived by computing the mean of the absolute values of the differences between the observed values of a variable and the mean or the median of the variable. The mean deviation about the mean and the median are derived.

3.8.1 The mean deviation about the mean

The mean deviation about the mean can be defined as

$$\begin{aligned} D(\mu) = E|x-\mu| &= \int_0^{\infty} |x-\mu| f(x) dx \\ &= \int_0^{\mu} (\mu - x) f(x) dx + \int_{\mu}^{\infty} (x-\mu) f(x) dx \\ &= 2\mu F(\mu) - 2\int_0^{\mu} x F(x) dx \\ &= 2\int_0^{\mu} F(x) dx. \end{aligned}$$

By using the CDF from Equation (2.3), the mean deviation of TIITLGIE distribution can be derived as:

$$D(\mu) = 2\int_0^{\mu} [1 - \sum_{s=0}^{\infty} \nu_s exp(\frac{-\lambda}{x} s)] dx,$$

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where ν_s is defined in Equation (2.4).

$$= 2[\int_0^{\mu} dx - \sum_{s=0}^{\infty} \nu_s \int_0^{\mu} exp(\frac{-\lambda}{x} s)dx].$$

Then, the mean deviation about the mean is given by:

$$= 2\mu - 2\sum_{s=0}^{\infty} \nu_s \times [\mu\, exp(\frac{-\lambda}{\mu} s) - \lambda s\Gamma(0, \frac{-\lambda}{\mu} s)], \tag{3.19}$$

where $\Gamma(0, \frac{-\lambda}{\mu} s)$ is the incomplete gamma function.

3.8.2 The mean deviation about the median

The mean deviation from the median can be defined as

$$\begin{aligned} D(m) = E|x - m| &= \int_0^{\infty} |x - m| f(x)dx \\ &= 2\int_0^{m} (m - x)f(x)dx - \int_0^{m} (m - x)f(x)dx + \int_m^{\infty} (x - m)f(x)dx \\ &= 2\int_0^{m} (m - x)f(x)dx + \int_0^{\infty} (x - m)f(x)dx \\ &= \mu - 2[mF(m) - \int_0^{m} F(x)dx] \\ &= \mu - m + 2\int_0^{m} F(x)dx. \end{aligned}$$

By using the CDF from Equation (2.3),we obtain

$$D(m) = \mu - m + 2\int_0^{m} [1 - \sum_{s=0}^{\infty} \nu_s exp(\frac{-\lambda}{x} s)]dx.$$

The mean deviation about the median of TIITLGIE distribution can be obtained as the following:

$$D(m) = \mu + m - 2\sum_{s=0}^{\infty} \nu_s [m\, exp(\frac{-\lambda}{m} s) - \lambda s\Gamma(0, \frac{-\lambda}{m} s)], \tag{3.20}$$

where $\Gamma(0, \frac{-\lambda}{\mu} s)$ was known in (3.19).

3.9 Order Statistics

If $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, ..., X_n$ from the TIITLGIE distribution with CDF F(j) and PDF f(j),then the pdf of $X_{(j)}$ is given by:

$$f(x_{(j)}) = \frac{n!}{(j-1)!(n-j)!} f(x)F(x)^{j-1}[1 - F(x)]^{n-j}. \tag{3.21}$$

The PDF of the jth order statistic of TIITLGIE distribution is given by :

$$\begin{aligned} f(x_{(j)}) = \frac{n!}{(j-1)!(n-j)!} \frac{2\alpha\lambda\theta}{x_{(j)}^2} exp(\frac{-\lambda}{x_{(j)}})[1 - exp(\frac{-\lambda}{x_{(j)}})]^{\theta-1}[1 - [1 - exp(\frac{-\lambda}{x_{(j)}})]^{\theta}] \\ \times[1 - [1 - [1 - (1 - exp(\frac{-\lambda}{x_{(j)}}))^{\theta}]^2]^{\alpha}]^{j-1}[1 - [1 - (1 - exp(\frac{-\lambda}{x_{(j)}}))^{\theta}]^2]^{\alpha(1+n-j)-1}, \\ x_{(j)} > 0. \end{aligned}$$

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Therefore, the PDF of the largest order statistic $X_{(n)}$ is given by:

$$f(x_{(n)})=\frac{2n\alpha\lambda\theta}{x_{(n)}^2}exp(\frac{-\lambda}{x_{(n)}})[1-exp(\frac{-\lambda}{x_{(n)}})]^{\theta-1}[1-[1-exp(\frac{-\lambda}{x_{(n)}})]^{\theta}]$$
$$\times[1-[1-(1-exp(\frac{-\lambda}{x_{(n)}}))^{\theta}]^2]^{\alpha-1}[1-[1-[1-(1-exp(\frac{-\lambda}{x_{(n)}}))^{\theta}]^2]^{\alpha}]^{n-1},$$
$$x_{(n)}>0.$$

And the PDF of the smallest order statistic $X_{(1)}$ is:

$$f(x_{(1)})=\frac{2n\alpha\lambda\theta}{x_{(1)}^2}exp(\frac{-\lambda}{x_{(1)}})[1-exp(\frac{-\lambda}{x_{(1)}})]^{\theta-1}[1-[1-exp(\frac{-\lambda}{x_{(1)}})]^{\theta}]$$
$$\times[1-[1-(1-exp(\frac{-\lambda}{x_{(1)}}))^{\theta}]^2]^{\alpha n-1},$$
$$x_{(1)}>0.$$

4 Rényi Entropy of TIITLGIE

The Rényi entropy was introduced by (16), and is one of the several generalizations of Shannon's entropy (17), They are measures of variation of uncertainty. The theory of entropy has been successfully used in a wide diversity of applications and has also been used for characterization of numerous standard probability distributions. For the density function $f(x)$ the Rényi entropy is defined by:

$$R_{\delta}(X)=\frac{1}{1-\delta}\ (log[J(\delta)]), \tag{4.1}$$

where

$$J(\delta)=\int_0^{\infty}f^{\delta}(x)dx;\delta>0\ and\ \delta\neq 1,$$

$$J(\delta)=(\frac{2\alpha\theta\lambda}{x^2})^{\delta}\int_0^{\infty}exp(\frac{-\lambda}{x}\delta)(1-exp(\frac{-\lambda}{x}))^{(\theta-1)\delta}[1-(1-exp(\frac{-\lambda}{x}))^{\theta}]^{\delta}$$
$$\times[1-[1-(1-exp(\frac{-\lambda}{x}))^{\theta}]^2]^{(\alpha-1)\delta}dx.$$

Let $u=\frac{\lambda}{x}$

$$J(\delta)=\lambda(\frac{2\alpha\theta u^2}{\lambda})^{\delta}\int_0^{\infty}exp(-u\delta)(1-exp(-u))^{(\theta-1)\delta}[1-(1-exp(-u))^{\theta}]^{\delta}$$
$$\times[1-[1-(1-exp(-u))^{\theta}]^2]^{(\alpha-1)\delta}\frac{1}{u^2}du.$$

Now by applying the binomial expansion, we get

$$J(\delta)=\lambda(\frac{2\alpha\theta u^2}{\lambda})^{\delta}\sum_{i=0}^{\infty}\begin{pmatrix}(\alpha-1)\delta\\ i\end{pmatrix}\int_0^{\infty}exp(-u\delta)(1-exp(-u))^{(\theta-1)\delta}[1-(1-exp(-u))^{\theta}]^{\delta}$$
$$\times[1-(1-exp(-u))^{\theta}]^{2i}\frac{1}{u^2}du.$$

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Using the binomial expansion again we have

$$J(\delta)=\lambda(\frac{2\alpha\theta}{\lambda})^{\delta}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\binom{(\alpha-1)\delta}{i}\binom{2i+\delta}{k}\binom{\theta(\delta+k)-\delta}{t}(-1)^{i+k+t}\int_0^{\infty}u^{2(\delta-1)}exp(-(\delta+t)u)du.$$

By setting $y=(\delta+t)u$

$$J(\delta)=\lambda(\frac{2\alpha\theta}{\lambda})^{\delta}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\binom{(\alpha-1)\delta}{i}\binom{2i+\delta}{k}\binom{\theta(\delta+k)-\delta}{t}\frac{(-1)^{i+k+t}}{(\delta+t)^{2\delta-1}}\int_0^{\infty}y^{2(\delta-1)}exp(-y)dy.$$

After integrating we get

$$J(\delta)=\lambda(\frac{2\alpha\theta}{\lambda})^{\delta}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\binom{(\alpha-1)\delta}{i}\binom{2i+\delta}{k}\binom{\theta(\delta+k)-\delta}{t}\frac{(-1)^{i+k+t}}{(\delta+t)^{2\delta-1}}\;\Gamma(2\delta-1).$$

Then, by taking the logarithm, we have

$$log[J(\delta)]=log[\lambda(\frac{2\alpha\theta}{\lambda})^{\delta}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\binom{(\alpha-1)\delta}{i}\binom{2i+\delta}{k}\binom{\theta(\delta+k)-\delta}{t}\frac{(-1)^{i+k+t}}{(\delta+t)^{2\delta-1}}\;\Gamma(2\delta-1)]. \tag{4.2}$$

Substituting Equation (4.2) into (4.1), we get the Rényi entropy for TIITLGIE as

$$R_{\delta}(X)=\frac{1}{1-\delta}\;log[\lambda(\frac{2\alpha\theta}{\lambda})^{\delta}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{t=0}^{\infty}\binom{(\alpha-1)\delta}{i}\binom{2i+\delta}{k}\binom{\theta(\delta+k)-\delta}{t}\frac{(-1)^{i+k+t}}{(\delta+t)^{2\delta-1}}\;\Gamma(2\delta-1)]. \tag{4.3}$$

5 Maximum Likelihood Estimation Method

The maximum likelihood estimators of the unknown parameters for the TIITLGIE distribution are discussed. Let $x_1, x_2, ..., x_n$be a realization of a random sample of size n from TIITLGIE distribution then the likelihood function is written as follows:

$$L=\prod_{i=0}^{n}f(y_i)\;,$$

and the log-likelihood function is given as follows

$$\begin{aligned}\ell=log(L)=nlog(2)+nlog(\alpha)+nlog(\theta)+nlog(\lambda)-2\sum_{i=1}^{n}log(x_i)-\sum_{i=1}^{n}\frac{\lambda}{x_i}\\+(\theta-1)\sum_{i=1}^{n}log(1-exp(\frac{-\lambda}{x}))+\sum_{i=1}^{n}log(1-(1-exp(\frac{-\lambda}{x}))^{\theta})+(\alpha-1)\\\sum_{i=1}^{n}log(1-(1-(1-exp(\frac{-\lambda}{x}))^{\theta})^{2}).\end{aligned}$$

Differentiating(ℓ) with respect to each of the parameters;α, θ and λ gives:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} log(1-(1-(1-exp(\frac{-\lambda}{x}))^{\theta})^{2}) = 0, \tag{5.1}$$

$$\frac{\partial \ell}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n} log(1-exp(\frac{-\lambda}{x})) - \sum_{i=1}^{n} \frac{(1-exp(\frac{-\lambda}{x}))^{\theta} log(1-exp(\frac{-\lambda}{x}))}{1-(1-exp(\frac{-\lambda}{x}))^{\theta}} + 2(\alpha-1)\sum_{i=1}^{n} \frac{(1-(1-exp(\frac{-\lambda}{x}))^{\theta})(1-exp(\frac{-\lambda}{x}))^{\theta} log(1-exp(\frac{-\lambda}{x}))}{1-(1-(1-exp(\frac{-\lambda}{x}))^{\theta})^{2}} = 0, \tag{5.2}$$

$$\frac{\partial \ell}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} \frac{1}{x_i} + (\theta-1)\sum_{i=1}^{n} \frac{exp(\frac{-\lambda}{x})}{x_i(1-exp(\frac{-\lambda}{x}))} - \theta \sum_{i=1}^{n} \frac{exp(\frac{-\lambda}{x})(1-exp(\frac{-\lambda}{x}))^{\theta-1}}{x_i(1-(1-exp(\frac{-\lambda}{x}))^{\theta})} + 2\theta(\alpha-1)\sum_{i=1}^{n} \frac{exp(\frac{-\lambda}{x})(1-exp(\frac{-\lambda}{x}))^{\theta-1}(1-(1-exp(\frac{-\lambda}{x}))^{\theta})}{x_i(1-(1-(1-exp(\frac{-\lambda}{x}))^{\theta})^{2})} = 0, \tag{5.3}$$

The MLE of parameters $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\lambda}$ can be found numerically by equating the derivatives equations in (5.1), (5.2) and (5.3) to zero and solve them using Mathematica (V.10.2).

$$\hat{\alpha} = [-\frac{1}{n}\sum_{i=1}^{n} log(1-(1-(1-exp(\frac{-\hat{\lambda}}{x}))^{\hat{\theta}})^{2})]^{-1} \tag{5.4}$$

6 Simulation Study

In this section, we will conduct simulation to study behavior of unknown parameters (α,θ,λ) for TIITLGIE using Mathematica (V.10.2). We generate samples of size n = 10; 30; 50; 100; 200; 500 and 1000 from TIITLGIE distribution for some selected combination of parameters. This process will be repeated N = 1000 times. In each process, estimates of the parameters that will be obtained by mean estimate, mean squared error and bias. Then, the estimates of $R(x_0)$ and $h(x_0)$ from (3.12) and (3.14) at point x_0 = 0.5 were also evaluated using the estimated parameters.

We can observe from Table (2,3) that, if the sample size increases, the bias (BIAS) and mean squared error (MSE) decreases in all cases.

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Table 2: The MLE, BIAS and MSE of TIITLGIE distribution for true values (α = 1, θ = 1, λ = 1, x_0 = 0.5)

n	Parameters	MLE	BIAS	MSE
10	α	2.46842	1.46842	8.38966
	θ	2.06023	1.06023	5.9919
	λ	1.47638	0.476379	1.00841
	R(x)	0.979781	- 0.00190381	0.000773155
	h(x)	0.129346	- 0.0199127	0.0197592
30	α	2.00886	1.00886	5.1536
	θ	1.73775	0.737747	3.09241
	λ	1.15384	0.153842	0.173499
	R(x)	0.98223	0.000545434	0.000188478
	h(x)	0.13552	- 0.0137385	0.00650461
50	α	1.80555	0.80555	4.03229
	θ	1.74504	0.745041	2.84547
	λ	1.11085	0.110846	0.114978
	R(x)	0.981839	0.000154577	0.000119655
	h(x)	0.142212	- 0.00704724	0.00431343
100	α	1.74469	0.74469	3.3529
	θ	1.4406	0.440605	1.61672
	λ	1.05897	0.0589683	0.0612171
	R(x)	0.982026	0.00034169	0.0000707223
	h(x)	0.14276	- 0.0064984	0.00230038

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Table 3: The MLE, BIAS and MSE of TIITLGIE distribution for true values (α = 1, θ = 1, λ = 1, x_0 = 0.5)

n	Parameters	MLE	BIAS	MSE
200	α	1.45024	0.450238	2.04912
	θ	1.41734	0.417339	1.32852
	λ	1.04602	0.0460214	0.042614
	R(x)	0.981799	0.000114878	0.0000405748
	h(x)	0.146533	- 0.00272554	0.00120306
500	α	1.20161	0.201606	0.68862
	θ	1.21196	0.211957	0.599059
	λ	1.02016	0.0201573	0.0220294
	R(x)	0.981555	- 0.000129064	0.0000209308
	h(x)	0.14883	- 0.000429294	0.000553292
1000	α	1.10397	0.103975	0.305711
	θ	1.09735	0.0973547	0.222811
	λ	1.00855	0.0085523	0.0109286
	R(x)	0.981538	- 0.000146694	0.0000109141
	h(x)	0.14946	0.000200975	0.000289528

7 Applications

In this Section, three sets of data are presented to demonstrate the utility of using the TIITLGIE distribution. We compared the Type II Topp-Leone Generalized Inverted Exponential Distribution (TIITLGIE) with, Type II Topp Leone Standard Inverse Exponential distribution (TIITLSIE), Type II Topp Leone Generalized Standard Inverse Exponential distribution (TIITLGSIE) and Topp Leone Generalized Inverted Exponential distribution (TLGIE)(4). The parameters are estimated using maximum likelihood method, and computed using Mathematica (V.10.2).
The following statistical measures were calculated: log-likelihood(LL), Akaike information criterion(AIC), Consistent Akaike information criteria (CAIC)(12) and Hannan-Quinn information criterion (HQIC)(11).

7.1 Data set 1

The first data set, is the numbers (in million Riyals) of credit facilities provided to micro enterprises in Saudi Arabia From Q1 2018 to Q2 2021. This data is downloaded from (https://data.gov.sa/Data/en/dataset/credit-facilities-provided-to-smes).

7.2 Data set 2

The second data set represents the number of daily COVID-19 cases in Jeddah, Saudi Arabia from 2^{nd} May to 6^{th} July. These data were taken from the website of the Saudi Ministry of Health with URL: https://covid19.moh.gov.sa/.
The data is as follows :245,261,385,312,315,373,265,374,236,306,338,482,444,450,357,305, 526,311,390,403,444,474,350,327,325,360,251,247,586,293,279,418,259,459,572,351,577,

447,460,294,391,527,352,413,477, 279,300,384,421,342,388,393,214,218,243,171,121,212, 167,172,164,169,169,149,209,227.

7.3 Data set 3

The data set shows the seasonal (July 1 - June 30) rainfall in inches recorded at Los Angeles Civic Center from 1962 to 2012. These data were taken from the website of Los Angeles Almanac with URL:
http://www.laalmanac.com/weather/we13.php reported by United States National Weather Service
3.21,4.42,7.17,7.22,7.35,7.66,7.77,7.93,8.08,8.11,8.38,8.69,8.98,9.08,9.09,9.24,10.43,10.71, 11.47,11.57,12.31,12.32,12.4,12.46,12.48,12.82,13.19,13.53,13.69,14.35,14.92,16.36,16.49, 16.58,17.86,17.94,19.67,20.2,20.44,21.0,21.26,22.0,24.35,26.98,27.36,27.47,31.01,31.25,33.44, 37.25.

In Table (4-6) shows that the TIITLGIE distribution has smaller values for measure, (LL, AIC, CAIC and HQIC) compared with the values of others models for the three data sets.
Figure (4-6) shows the empirical distribution and estimated CDF of the models for three data sets.

Table 4: Statistical measures for data set 1.

Model	Parameters	LL	AIC	CAIC	HQIC
TIITLGIE	$\hat{\alpha}$ = 4.19523 $\hat{\theta}$ = 12.9575 $\hat{\lambda}$ = 4.78025	- 2.11004	10.2201	13.6486	9.46762
TIITLSIE	$\hat{\alpha}$ = 3.37693	- 10.3583	22.7165	23.161	22.4657
TIITLGSIE	$\hat{\alpha}$ = 15.6049 $\hat{\theta}$ = 0.392864	- 9.31184	22.6237	24.1237	22.122
TLGIE	$\hat{\alpha}$ = 1.42755 $\hat{\theta}$ = 0.5 $\hat{\lambda}$ = 0.992072	- 15.2647	36.5295	39.958	35.777

Table 5: Statistical measures for data set 2.

Model	Parameters	LL	AIC	CAIC	HQIC
TIITLGIE	$\hat{\alpha}$ = 15.6834 $\hat{\theta}$ = 0.962963 $\hat{\lambda}$ = 483.436	- 406.108	818.216	818.603	820.812
TIITLSIE	$\hat{\alpha}$ = 0.197486	- 553.007	1108.01	1108.08	1108.88
TIITLGSIE	$\hat{\alpha}$ = 8.5854 $\hat{\theta}$ = 0.067094	- 525.562	1055.12	1055.32	1056.86
TLGIE	$\hat{\alpha}$ = 1.39053 $\hat{\theta}$ = 0.5 $\hat{\lambda}$ = 211.596	- 450.282	906.564	906.951	909.16

Table 6: Statistical measures for data set 3.

Model	Parameters	LL	AIC	CAIC	HQIC
TIITLGIE	$\hat{\alpha}$ = 13.8009 $\hat{\theta}$ = 0.510292 $\hat{\lambda}$ = 12.4307	- 168.324	342.648	343.17	344.832
TIITLSIE	$\hat{\alpha}$ = 0.504938	-218.012	438.023	438.107	438.752
TIITLGSIE	$\hat{\alpha}$ = 6.98918 $\hat{\theta}$ = 0.164037	- 205.072	414.144	414.399	415.6
TLGIE	$\hat{\alpha}$ = 1.33702 $\hat{\theta}$ = 0.5 $\hat{\lambda}$ = 8.6618	- 186.422	378.844	379.366	381.028

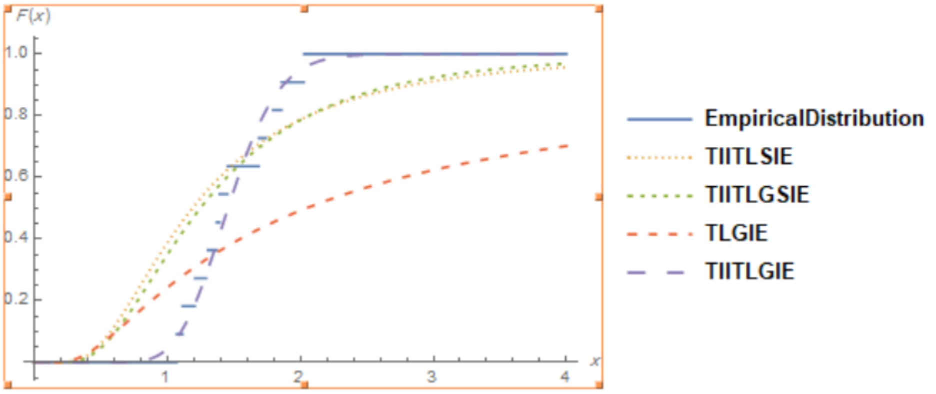

Figure 4: Plot of the Goodness of fit of TIITLGIE distribution using data set 1.

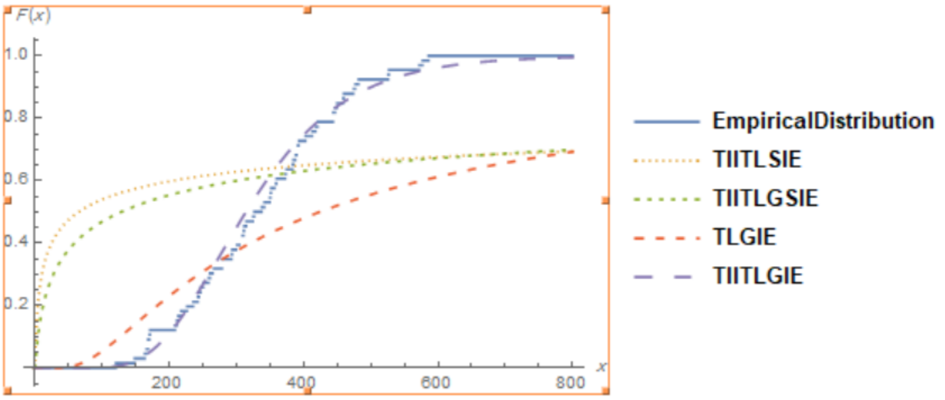

Figure 5: Plot of the Goodness of fit of TIITLGIE distribution using data set 2.

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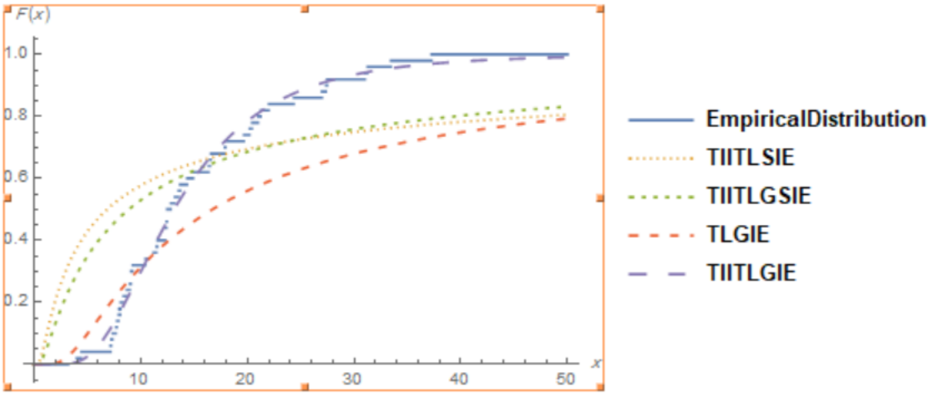

Figure 6: Plot of the Goodness of fit of TIITLGIE distribution using data set 3.

8 conclusions

In this study, we derived a three parameter Type II Topp Leone Generalized Inverted Exponential Distribution. Statistical properties of TIITLGIE distribution are computed. Maximum Likelihood estimators of TIITLGIE distribution obtained. Finally, three real data applications are analyzed, it is significantly the TIITLGIE distribution provides better result than derived models.

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