

Original Research Article

A Study on the Sum of the Squares of Generalized p-Oresme Numbers: The Sum Formula $\sum_{k=0}^n x^k W_{mk+j}^2$

Abstract. In this paper, closed forms of the sum formulas $\sum_{k=0}^n x^k W_{mk+j}^2$ for generalized Oresme numbers are presented. As special cases, we give sum formulas of Modified p-Oresme, p-Oresme-Lucas and p-Oresme numbers.

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1. Introduction

p-Oresme numbers (see, for example, [3]) are defined by the recurrence relation

$$O_{n+2} = O_{n+1} - \frac{1}{p^2} O_n, \quad O_0 = 0, O_1 = \frac{1}{p}, \quad p \neq 0.$$

The case $p = 2$, which is called the Oresme sequence, $\{O_n\}_{n \geq 0}$, was introduced by Nicole Oresme (1320–1382) in the 14-th century. Oresme obtained the sum of the rational numbers formed by the terms $0, \frac{1}{2}, \frac{2}{4}, \frac{3}{8}, \frac{4}{16}, \frac{5}{32}, \frac{6}{64}, \dots, \frac{n}{2^n}$. These numbers form a second order sequence and are defined by the recurrence relation

$$O_{n+2} = O_{n+1} - \frac{1}{4} O_n, \quad O_0 = 0, O_1 = \frac{1}{2}.$$

In [4], Horadam presented a history and obtained an abundance of properties of these numbers. Oresme numbers have many interesting properties and applications in many fields of science (see, for example, [1,2,3,4,5,6]).

For $0 \neq p \in \mathbb{R}$, a generalized p-Oresme sequence $\{W_n\}_{n \geq 0} = \{W_n(W_0, W_1)\}_{n \geq 0}$ is defined by the second-order recurrence relations

$$W_n = W_{n-1} - \frac{1}{p^2} W_{n-2} \tag{1.1}$$

with the initial values $W_0 = c_0, W_1 = c_1$ not all being zero.

The sequence $\{W_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = p^2 W_{-(n-1)} - p^2 W_{-(n-2)}$$

for $n = 1, 2, 3, \dots$. Therefore, recurrence (1.1) holds for all integer n .

Modified p-Oresme sequence $\{G_n\}_{n \geq 0}$, p-Oresme-Lucas sequence $\{H_n\}_{n \geq 0}$ and p-Oresme sequence $\{O_n\}_{n \geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = G_{n+1} - \frac{1}{p^2} G_n, \quad G_0 = 0, G_1 = 1, \quad (1.2)$$

$$H_{n+2} = H_{n+1} - \frac{1}{p^2} H_n, \quad H_0 = 2, H_1 = 1, \quad (1.3)$$

$$O_{n+2} = O_{n+1} - \frac{1}{p^2} O_n, \quad O_0 = 0, O_1 = \frac{1}{p}. \quad (1.4)$$

The sequences $\{G_n\}_{n \geq 0}$, $\{H_n\}_{n \geq 0}$ and $\{O_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = p^2 G_{-(n-1)} - p^2 G_{-(n-2)},$$

$$H_{-n} = p^2 H_{-(n-1)} - p^2 H_{-(n-2)},$$

$$O_{-n} = p^2 O_{-(n-1)} - p^2 O_{-(n-2)},$$

for $n = 1, 2, 3, \dots$ respectively. Therefore, recurrences (1.2)-(1.4) hold for all integer n . For more information on generalized Oresme numbers, see Soykan [7].

2. The Sum Formula $\sum_{k=0}^n x^k W_{mk+j}^2$

The following theorem presents sum formulas of generalized p-Oresme numbers (the case $r = 1, s = -\frac{1}{p^2}$).

THEOREM 1. *Suppose that $p^2 \neq 4$ so that $p^2 - 4 \neq 0$. Let x be a real (or complex) number. For all integers m and j , for generalized p-Oresme numbers (the case $r = 1, s = -\frac{1}{p^2}$) we have the following sum formulas:*

(a): *If $(1 + \frac{1}{p^{4m}} x^2 - x H_{2m})(\frac{1}{p^{2m}} x - 1) = (x^2 - p^{4m} H_{2m} x + p^{4m})(x - p^{2m}) p^{-6m} \neq 0$ then*

$$\sum_{k=0}^n x^k W_{mk+j}^2 = \frac{\Omega_1}{(p-2)(p+2)(x-p^{2m})(x^2-p^{4m}H_{2m}x+p^{4m})} \quad (2.1)$$

where

$$\begin{aligned} \Omega_1 = & (x - p^{4m} H_{2m})(x - p^{2m})(p-2)(p+2)x^{n+1}W_{mn+j}^2 + (x - p^{2m})(p-2)(p+2)x^{n+1}W_{mn-m+j}^2 + \\ & (x - p^{2m})(p-2)p^{4m}(p+2)W_j^2 + (p^{2m} - x)(p-2)(p+2)xW_{j-m}^2 + 2p^{-2mn+4m-2j+2}(H_{2m}p^{2m} - \\ & 2)(x^n - p^{2mn})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x \end{aligned}$$

(b): *If $(x^2 - p^{4m} H_{2m} x + p^{4m})(x - p^{2m}) = (x - a)(x - b)(x - c) = 0$ for some*

$$a = \frac{p^{4m} H_{2m} + \sqrt{p^{8m} H_{2m}^2 - 4p^{4m}}}{2}, b = \frac{p^{4m} H_{2m} - \sqrt{p^{8m} H_{2m}^2 - 4p^{4m}}}{2}, c = p^{2m} \in \mathbb{C}$$

and $a \neq b \neq c$, i.e., $x = a$ or $x = b$ or $x = c$, then,

$$\sum_{k=0}^n x^k W_{mk+j}^2 = \frac{\Omega_2}{(p-2)(p+2)(3x^2 - 2p^{2m}(p^{2m}H_{2m} + 1)x + p^{4m} + p^{6m}H_{2m})}$$

where

$$\begin{aligned} \Omega_2 = & (p^2 - 4)((x - p^{4m}H_{2m})x^{n+1} + (x - p^{2m})((n+2)x - p^{4m}(n+1)H_{2m})x^n)W_{mn+j}^2 + (p^2 - \\ & 4)((n+2)x - p^{2m}(n+1))x^n W_{mn-m+j}^2 + p^{4m}(p^2 - 4)W_j^2 - (p^2 - 4)(2x - p^{2m})W_{j-m}^2 + 2p^{-2mn+4m-2j+2}(p^{2m}H_{2m} - \\ & 2)(x^n + nx^n - p^{2mn})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0) \end{aligned}$$

(c): If $p^{8m}H_{2m}^2 - 4p^{4m} = 0$ and $(x^2 - p^{4m}xH_{2m} + p^{4m})(x - p^{2m}) = (x - a)^2(x - c) = 0$ for some $a = \frac{p^{4m}H_{2m}}{2}, c = p^{2m} \in \mathbb{C}$ and $a \neq c$ then if $x = c$ then

$$\sum_{k=0}^n x^k W_{mk+j}^2 = \frac{\Omega_3}{(p-2)(p+2)(3x^2 - 2p^{2m}(p^{2m}H_{2m} + 1)x + p^{4m} + p^{6m}H_{2m})}$$

where

$$\begin{aligned} \Omega_3 = & (p^2 - 4)((x - p^{4m}H_{2m})x^{n+1} + (x - p^{2m})((n+2)x - p^{4m}(n+1)H_{2m})x^n)W_{mn+j}^2 + (p^2 - \\ & 4)((n+2)x - p^{2m}(n+1))x^n W_{mn-m+j}^2 + p^{4m}(p^2 - 4)W_j^2 - (p^2 - 4)(2x - p^{2m})W_{j-m}^2 + 2p^{-2mn+4m-2j+2}(p^{2m}H_{2m} - \\ & 2)(x^n + nx^n - p^{2mn})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0) \end{aligned}$$

and if $x = a$ then

$$\sum_{k=0}^n x^k W_{mk+j}^2 = \frac{\Omega_4}{2(p-2)(p+2)(3x - p^{2m} - p^{4m}H_{2m})}$$

where

$$\begin{aligned} \Omega_4 = & (p^2 - 4)((n+3)(n+2)x^2 - xp^{2m}(n+2)(n+1)(p^{2m}H_{2m} + 1) + p^{6m}n(n+1)H_{2m})x^{n-1}W_{mn+j}^2 + \\ & (p^2 - 4)(n+1)((2+n)x^n - p^{2m}nx^{n-1})W_{mn-m+j}^2 - 2(p^2 - 4)W_{j-m}^2 + 2n(n+1)p^{-2mn+4m-2j+2} \\ & (p^{2m}H_{2m} - 2)(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x^{n-1} \end{aligned}$$

Proof. Take $r = 1, s = -\frac{1}{p^2}$ and $H_n = H_n$ in Soykan [??, Theorem 2.1.]. Note that $(x^2 - p^{4m}xH_{2m} + p^{4m})(x - p^{2m}) = (x - a)^3 = 0$ can not be possible for some $a = p^{2m} \in \mathbb{C}$, $m \neq 0$, i.e., $x = a$, because otherwise we must have $p^{8m}H_{2m}^2 - 4p^{4m} = 0, \frac{p^{4m}H_{2m}}{2} = p^{2m}$ and

$$\begin{aligned} \frac{p^{4m}H_{2m}}{2} - p^{2m} &= \frac{1}{2}p^{2m}(p^{2m}H_{2m} - 2) = 0 \Rightarrow p^{2m}H_{2m} - 2 = 0 \\ \Rightarrow p^{2m} \times 2 \left(\frac{1}{2}\right)^{2m} - 2 &= 0 \\ \Rightarrow \left(\frac{p}{2}\right)^{2m} - 1 &= 0 \end{aligned}$$

in which the last equality doesn't have any integer solutions m other than $m = 0$ (so $x = p^{2m} = 1$) for the case $p \neq 2$. Here we used

$$\begin{aligned} \alpha &= \frac{r}{2}, \\ H_{2m} &= 2\alpha^{2m} = 2\left(\frac{1}{2}\right)^{2m}. \end{aligned}$$

In this case (i.e. $m = 0, x = 1$), we have $\sum_{k=0}^n W_j^2 = (n+1)W_j^2$. $\square$

Note that in the last theorem, the case $p^2 \neq 4$ so that $p^2 - 4 \neq 0$ is considered. The sum formulas for the case $p^2 = 4$ so that $p^2 - 4 = 0$ is given in Soykan [8, Theorem 2.1].

Note also that (2.1) can be written in the following form

$$\sum_{k=1}^n x^k W_{mk+j}^2 = \frac{\Omega_6}{(p-2)(p+2)(x-p^{2m})(x^2-p^{4m}H_{2m}x+p^{4m})}$$

where

$$\Omega_6 = (p^2-4)(x-p^{2m})(x-p^{4m}H_{2m})x^{n+1}W_{mn+j}^2 + (p^2-4)(x-p^{2m})x^{n+1}W_{mn-m+j}^2 - (p^2-4)(x-p^{2m})(x-p^{4m}H_{2m})xW_j^2 - (p^2-4)(x-p^{2m})W_{j-m}^2x + 2p^{-2mn+4m-2j+2}(x^n-p^{2mn})(p^{2m}H_{2m}-2)(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x.$$

As special cases of m and j in the last Theorem, we obtain the following proposition.

PROPOSITION 2. *Suppose that $p^2 \neq 4$ so that $p^2 - 4 \neq 0$. For generalized p -Oresme numbers (the case $r = 1, s = -\frac{1}{p^2}$) we have the following sum formulas for $n \geq 0$:*

(a): ($m = 1, j = 0$)

If $(x-p^2)(x^2-p^2(p^2-2)x+p^4) \neq 0$, i.e.,

$$x \neq \frac{p^2(p^2-2) + \sqrt{p^6(p^2-4)}}{2}, x \neq \frac{p^2(p^2-2) - \sqrt{p^6(p^2-4)}}{2}, x \neq p^2,$$

then

$$\sum_{k=0}^n x^k W_k^2 = \frac{\Phi}{(p^2-4)(x-p^2)(x^2-p^2(p^2-2)x+p^4)}$$

where

$$\Phi = (x-p^2(p^2-2))(x-p^2)(p^2-4)x^{n+1}W_n^2 + (x-p^2)(p^2-4)x^{n+1}W_{n-1}^2 + (x-p^2)(p^2-4)p^4W_0^2 + p^4(p^2-x)(p^2-4)(W_0-W_1)^2x + 2p^{6-2n}((p^2-2)-2)(x^n-p^{2n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x,$$

and

if $(x-p^2)(x^2-p^2(p^2-2)x+p^4) = 0$, i.e.,

$$x = \frac{p^2(p^2-2) + \sqrt{p^6(p^2-4)}}{2} \text{ or } x = \frac{p^2(p^2-2) - \sqrt{p^6(p^2-4)}}{2} \text{ or } x = p^2,$$

then

$$\sum_{k=0}^n x^k W_k^2 = \frac{\Lambda}{(p^2-4)(3x^2-2p^2(p^2-1)x+p^4(p^2-1))}$$

where

$$\Lambda = (p^2-4)((x-p^2(p^2-2))x^{n+1} + (x-p^2)((n+2)x - (n+1)p^2(p^2-2))x^n)W_n^2 + (p^2-4)((n+2)x - p^2(n+1))x^nW_{n-1}^2 + p^4(p^2-4)W_0^2 - p^4(2x-p^2)(p^2-4)(W_0-W_1)^2 + 2p^{6-2n}((p^2-2)-2)(x^n+nx^n-p^{2n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0).$$

(b): ($m = 2, j = 0$)

If $(x-p^4)(x^2-p^4(p^4-4p^2+2)x+p^8) \neq 0$, i.e.,

$$x \neq \frac{p^4(p^4-4p^2+2) + \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x \neq \frac{p^4(p^4-4p^2+2) - \sqrt{p^{10}(p^2-4)(p^2-2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^n x^k W_{2k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8)}$$

where

$$\begin{aligned} \Phi &= (x - p^4(p^4 - 4p^2 + 2))(x - p^4)(p^2 - 4)x^{n+1}W_{2n}^2 + (x - p^4)(p^2 - 4)x^{n+1}W_{2n-2}^2 + (x - p^4)(p^2 - 4)p^8W_0^2 \\ &+ (p^4 - x)(p^2 - 4)p^4(p^2W_0 - W_0 - p^2W_1)^2x + 2p^{12-4n}(p^2 - 4)(x^n - p^{4n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x, \\ \text{and if } (x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) &= 0 \text{ and } p^2 \neq 2 \text{ i.e.,} \end{aligned}$$

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = p^4,$$

(and $p^2 \neq 2$) then

$$\sum_{k=0}^n x^k W_{2k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^4(p^2 - 1)(p^2 - 3)x + p^8(p^2 - 1)(p^2 - 3))}$$

where

$$\begin{aligned} \Lambda &= (p^2 - 4)((x - p^4(p^4 - 4p^2 + 2))x^{n+1} + (x - p^4)((n + 2)x - p^4(p^4 - 4p^2 + 2)(n + 1))x^n)W_{2n}^2 + \\ &(p^2 - 4)((n + 2)x - p^4(n + 1))x^nW_{2n-2}^2 + p^8(p^2 - 4)W_0^2 - (p^2 - 4)(2x - p^4)p^4(p^2W_0 - W_0 - p^2W_1)^2 + \\ &2p^{12-4n}(p^2 - 4)(x^n + nx^n - p^{4n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0), \end{aligned}$$

and if $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = (x - 4)(x + 4)^2 = 0$, $x \neq p^4 = 4$ and $p^2 = 2$ i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = -4,$$

and $x \neq p^4 = 4$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n (-4)^k W_{2k}^2 = \frac{\Theta}{32}$$

where

$$\begin{aligned} \Theta &= 64(-4)^{n-1}(n^2 - 2)W_{2n}^2 + (-4)^{n+1}(n + 1)^2W_{2n-2}^2 + 16(W_0 - 2W_1)^2 - n(n + 1)(-4)^{n-1}2^{8-2n} \\ &(W_1^2 + \frac{1}{2}W_0^2 - W_1W_0). \end{aligned}$$

(c): ($m = 2, j = 1$)

If $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) \neq 0$, i.e.,

$$x \neq \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^n x^k W_{2k+1}^2 = \frac{\Phi}{(p^2 - 4)(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8)},$$

where

$$\begin{aligned} \Phi &= (x - p^4(p^4 - 4p^2 + 2))(x - p^4)(p^2 - 4)x^{n+1}W_{2n+1}^2 + (x - p^4)(p^2 - 4)x^{n+1}W_{2n-1}^2 + (x - p^4)(p^2 - 4)p^8W_1^2 \\ &+ (p^4 - x)(p^2 - 4)p^4(W_0 - W_1)^2x + 2p^{10-4n}(p^2 - 4)(x^n - p^{4n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x, \end{aligned}$$

and if $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = 0$ and $p^2 \neq 2$ i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = p^4,$$

(and $p^2 \neq 2$) then

$$\sum_{k=0}^n x^k W_{2k+1}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^4(p^2 - 1)(p^2 - 3)x + p^8(p^2 - 1)(p^2 - 3))},$$

where

$$\begin{aligned} \Lambda = & (p^2 - 4)((x - p^4(p^4 - 4p^2 + 2))x^{n+1} + (x - p^4)((n + 2)x - p^4(p^4 - 4p^2 + 2)(n + 1))x^n)W_{2n+1}^2 + \\ & (p^2 - 4)((n + 2)x - p^4(n + 1))x^n W_{2n-1}^2 + p^8(p^2 - 4)W_1^2 - (p^2 - 4)(2x - p^4)p^4(W_0 - W_1)^2 + 2p^{10-4n} \\ & (p^2 - 4)(x^n + nx^n - p^{4n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0), \end{aligned}$$

and if $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = (x - 4)(x + 4)^2 = 0$, $x \neq p^4 = 4$ and $p^2 = 2$ i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = -4,$$

and $x \neq p^4 = 4$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n (-4)^k W_{2k+1}^2 = \frac{\Theta}{32},$$

where

$$\begin{aligned} \Theta = & 64(-4)^{n-1}(n^2 - 2)W_{2n+1}^2 + (-4)^{n+1}(n + 1)^2 W_{2n-1}^2 + 16(W_0 - W_1)^2 - n(n + 1)(-4)^{n-1}2^{7-2n} \\ & (W_1^2 + \frac{1}{2}W_0^2 - W_1W_0). \end{aligned}$$

(d): ($m = -1, j = 0$)

If $(p^2x - 1)p^{-2}(x^2 - (p^2 - 2)p^{-2}x + p^{-4}) \neq 0$, i.e.,

$$x \neq \frac{(p^2 - 2)p^{-2} + \sqrt{(p^2 - 4)p^{-2}}}{2}, x \neq \frac{(p^2 - 2)p^{-2} - \sqrt{(p^2 - 4)p^{-2}}}{2}, x \neq p^{-2},$$

then

$$\sum_{k=0}^n x^k W_{-k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-2})(x^2 - (p^2 - 2)p^{-2}x + p^{-4})},$$

where

$$\begin{aligned} \Phi = & (x - p^{2m})(p^2 - 4)x^{n+1}W_{-n+1}^2 + (x - (p^2 - 2)p^{-2})(x - p^{-2})(p^2 - 4)x^{n+1}W_{-n}^2 + (x - p^{-2})(p^2 - \\ & 4)p^{-4}W_0^2 + (p^{-2} - x)(p^2 - 4)xW_1^2 + 2p^{2n-2}(p^2 - 4)(x^n - p^{-2n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x \end{aligned}$$

and if $(p^2x - 1)p^{-2}(x^2 - (p^2 - 2)p^{-2}x + p^{-4}) = 0$, i.e.,

$$x = \frac{(p^2 - 2)p^{-2} + \sqrt{(p^2 - 4)p^{-2}}}{2} \text{ or } x = \frac{(p^2 - 2)p^{-2} - \sqrt{(p^2 - 4)p^{-2}}}{2} \text{ or } x = p^{-2},$$

then

$$\sum_{k=0}^n x^k W_{-k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)p^{-2}x + (p^2 - 1)p^{-4})},$$

where

$$\Lambda = (p^2 - 4)((n + 2)x - p^{-2}(n + 1))x^n W_{-n+1}^2 + (p^2 - 4)(p^4(n + 3)x^2 - p^2x(p^2 - 1)(n + 2) + (p^2 - 2)(n + 1))p^{-4}x^n W_{-n}^2 + p^{-4}(p^2 - 4)W_0^2 - (p^2 - 4)(2x - p^{-2})W_1^2 + 2p^{2n-2}(p^2 - 4)(x^n + nx^n - p^{-2n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0).$$

(e): ($m = -2, j = 0$)

If $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) \neq 0$, i.e.,

$$x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq p^{-4},$$

then

$$\sum_{k=0}^n x^k W_{-2k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8})},$$

where

$$\Phi = (x - p^{-4})(p^2 - 4)x^{n+1}W_{-2n+2}^2 + (x - (p^4 - 4p^2 + 2)p^{-4})(x - p^{-4})(p^2 - 4)x^{n+1}W_{-2n}^2 + (x - p^{-4})(p^2 - 4)p^{-8}W_0^2 + (p^{-4} - x)(p^2 - 4)x(-W_0 + p^2W_1)^2p^{-4} + 2p^{4n-4}(p^2 - 4)(x^n - p^{-4n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = 0$, and $p^2 \neq 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = p^{-4},$$

(and $p^2 \neq 2$, i.e., $p \neq -\sqrt{2}, p \neq \sqrt{2}$) then

$$\sum_{k=0}^n x^k W_{-2k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)(p^2 - 3)p^{-4}x + (p^2 - 1)(p^2 - 3)p^{-8})},$$

where

$$\Lambda = (p^2 - 4)((n + 2)x - p^{-4}(n + 1))x^n W_{-2n+2}^2 + (p^2 - 4)((x - (p^4 - 4p^2 + 2)p^{-4})x^{n+1} + (x - p^{-4})((n + 2)x - (p^4 - 4p^2 + 2)p^{-4}(n + 1))x^n)W_{-2n}^2 + p^{-8}(p^2 - 4)W_0^2 - (p^2 - 4)(2x - p^{-4})(-W_0 + p^2W_1)^2p^{-4} + 2p^{4n-4}(p^2 - 4)(x^n + nx^n - p^{-4n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0),$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = \frac{1}{64}(4x - 1)(4x + 1)^2 = 0, x \neq p^{-4} = \frac{1}{4}$ and $p^2 = 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = -\frac{1}{4},$$

and $x \neq p^{-4} = \frac{1}{4}$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n \left(-\frac{1}{4}\right)^k W_{-2k}^2 = \frac{\Theta}{2},$$

where

$$\Theta = -4\left(-\frac{1}{4}\right)^n (n + 1)^2 W_{-2n+2}^2 + \frac{1}{4}\left(-\frac{1}{4}\right)^{n-1} (n^2 - 2) W_{-2n}^2 + (W_0 - 2W_1)^2 - n(n + 1)2^{2n-1}\left(-\frac{1}{4}\right)^{n-1} (2W_1^2 + W_0^2 - 2W_1W_0).$$

(f): ($m = -2, j = 1$)

If $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) \neq 0$, i.e.,

$$x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq p^{-4},$$

then

$$\sum_{k=0}^n x^k W_{-2k+1}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8})},$$

where

$$\begin{aligned} \Phi = & (x - p^{-4})(p^2 - 4)x^{n+1}W_{-2n+3}^2 + (x - (p^4 - 4p^2 + 2)p^{-4})(x - p^{-4})(p^2 - 4)x^{n+1}W_{-2n+1}^2 + \\ & (x - p^{-4})(p^2 - 4)p^{-8}W_1^2 + (p^{-4} - x)(p^2 - 4)p^{-4}(-W_0 - W_1 + p^2W_1)^2x + 2p^{4n-6}(p^2 - 4)(x^n - \\ & p^{-4n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0)x, \end{aligned}$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = 0$, and $p^2 \neq 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = p^{-4},$$

(and $p^2 \neq 2$, i.e., $p \neq -\sqrt{2}, p \neq \sqrt{2}$) then

$$\sum_{k=0}^n x^k W_{-2k+1}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)(p^2 - 3)p^{-4}x + (p^2 - 1)(p^2 - 3)p^{-8})},$$

where

$$\begin{aligned} \Lambda = & (p^2 - 4)((n + 2)x - p^{-4}(n + 1))x^n W_{-2n+3}^2 + (p^2 - 4)((x - (p^4 - 4p^2 + 2)p^{-4})x^{n+1} + \\ & (x - p^{-4})((n + 2)x - (p^4 - 4p^2 + 2)p^{-4}(n + 1))x^n)W_{-2n+1}^2 + p^{-8}(p^2 - 4)W_1^2 - (p^2 - 4)(2x - p^{-4})p^{-4} \\ & (-W_0 - W_1 + p^2W_1)^2 + 2p^{4n-6}(p^2 - 4)(x^n + nx^n - p^{-4n})(W_1^2 + \frac{1}{p^2}W_0^2 - W_1W_0), \end{aligned}$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = \frac{1}{64}(4x - 1)(4x + 1)^2 = 0$, $x \neq p^{-4} = \frac{1}{4}$ and $p^2 = 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = -\frac{1}{4},$$

and $x \neq p^{-4} = \frac{1}{4}$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n \left(-\frac{1}{4}\right)^k W_{-2k+1}^2 = \frac{\Theta}{2},$$

where

$$\begin{aligned} \Theta = & -4\left(-\frac{1}{4}\right)^n (n + 1)^2 W_{-2n+3}^2 + \frac{1}{4}\left(-\frac{1}{4}\right)^{n-1} (n^2 - 2) W_{-2n+1}^2 + (W_0 - W_1)^2 - n(n + 1)2^{2n-2}(2W_1^2 + \\ & W_0^2 - 2W_1W_0)\left(-\frac{1}{4}\right)^{n-1}. \end{aligned}$$

Note that in the last proposition, the case $p^2 \neq 4$ so that $p^2 - 4 \neq 0$ is considered. The sum formulas for the case $p^2 = 4$ so that $p^2 - 4 = 0$ is given in Soykan [8, Proposition 2.1].

From the above proposition, we have the following corollary which gives sum formulas of modified p-Oresme numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1$).

COROLLARY 3. Suppose that $p^2 \neq 4$ so that $p^2 - 4 \neq 0$. For $n \geq 0$, modified p -Oresme numbers have the following properties:

(a): ($m = 1, j = 0$)

If $(x - p^2)(x^2 - p^2(p^2 - 2)x + p^4) \neq 0$, i.e.,

$$x \neq \frac{p^2(p^2 - 2) + \sqrt{p^6(p^2 - 4)}}{2}, x \neq \frac{p^2(p^2 - 2) - \sqrt{p^6(p^2 - 4)}}{2}, x \neq p^2,$$

then

$$\sum_{k=0}^n x^k G_k^2 = \frac{\Phi}{(p^2 - 4)(x - p^2)(x^2 - p^2(p^2 - 2)x + p^4)}$$

where

$$\Phi = (x - p^2(p^2 - 2))(x - p^2)(p^2 - 4)x^{n+1}G_n^2 + (x - p^2)(p^2 - 4)x^{n+1}G_{n-1}^2 - p^{-2n+4}(p^2 - 4)(p^{2n}(x + p^2) - 2p^2x^n)x,$$

and

if $(x - p^2)(x^2 - p^2(p^2 - 2)x + p^4) = 0$, i.e.,

$$x = \frac{p^2(p^2 - 2) + \sqrt{p^6(p^2 - 4)}}{2} \text{ or } x = \frac{p^2(p^2 - 2) - \sqrt{p^6(p^2 - 4)}}{2} \text{ or } x = p^2,$$

then

$$\sum_{k=0}^n x^k G_k^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^2(p^2 - 1)x + p^4(p^2 - 1))}$$

where

$$\Lambda = (p^2 - 4)((x - p^2(p^2 - 2))x^{n+1} + (x - p^2)((n + 2)x - (n + 1)p^2(p^2 - 2))x^n)G_n^2 + (p^2 - 4)((n + 2)x - p^2(n + 1))x^n G_{n-1}^2 + p^{-2n+4}(p^2 - 4)(-p^{2n}(2x + p^2) + 2p^2(n + 1)x^n).$$

(b): ($m = 2, j = 0$)

If $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) \neq 0$, i.e.,

$$x \neq \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^n x^k G_{2k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8)}$$

where

$$\Phi = (x - p^4(p^4 - 4p^2 + 2))(x - p^4)(p^2 - 4)x^{n+1}G_{2n}^2 + (x - p^4)(p^2 - 4)x^{n+1}G_{2n-2}^2 - p^{-4n+8}(p^2 - 4)(p^{4n}(x + p^4) - 2p^4x^n)x,$$

and if $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = 0$ and $p^2 \neq 2$ i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = p^4,$$

(and $p^2 \neq 2$) then

$$\sum_{k=0}^n x^k G_{2k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^4(p^2 - 1)(p^2 - 3)x + p^8(p^2 - 1)(p^2 - 3))}$$

where

$$\Lambda = (p^2 - 4)((x - p^4(p^4 - 4p^2 + 2))x^{n+1} + (x - p^4)((n + 2)x - p^4(p^4 - 4p^2 + 2)(n + 1))x^n)G_{2n}^2 + (p^2 - 4)((n + 2)x - p^4(n + 1))x^n G_{2n-2}^2 + p^{-4n+8}(p^2 - 4)(-p^{4n}(2x + p^4) + 2p^4(n + 1)x^n),$$

and if $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = (x - 4)(x + 4)^2 = 0$, $x \neq p^4 = 4$ and $p^2 = 2$ i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = -4,$$

and $x \neq p^4 = 4$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n (-4)^k G_{2k}^2 = \frac{\Theta}{32}$$

where

$$\Theta = 64(-4)^{n-1}(n^2 - 2)G_{2n}^2 + (-4)^{n+1}(n + 1)^2 G_{2n-2}^2 + 16(G_0 - 2G_1)^2 - n(n + 1)(-4)^{n-1} 2^{8-2n}$$

(c): ($m = 2, j = 1$)

If $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) \neq 0$, i.e.,

$$x \neq \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^n x^k G_{2k+1}^2 = \frac{\Phi}{(p^2 - 4)(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8)},$$

where

$$\Phi = (x - p^4(p^4 - 4p^2 + 2))(x - p^4)(p^2 - 4)x^{n+1}G_{2n+1}^2 + (x - p^4)(p^2 - 4)x^{n+1}G_{2n-1}^2 - p^{-4n+4}(p^2 - 4)(p^{4n}(x^2 - 2p^4x + 2p^6x + p^8) - 2p^6x^{n+1}),$$

and if $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = 0$ and $p^2 \neq 2$ i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = p^4,$$

(and $p^2 \neq 2$) then

$$\sum_{k=0}^n x^k G_{2k+1}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^4(p^2 - 1)(p^2 - 3)x + p^8(p^2 - 1)(p^2 - 3))},$$

where

$$\Lambda = (p^2 - 4)((x - p^4(p^4 - 4p^2 + 2))x^{n+1} + (x - p^4)((n + 2)x - p^4(p^4 - 4p^2 + 2)(n + 1))x^n)G_{2n+1}^2 + (p^2 - 4)((n + 2)x - p^4(n + 1))x^n G_{2n-1}^2 + 2p^{-4n+4}(p^2 - 4)(p^{4n}(-x - p^6 + p^4) + p^6(n + 1)x^n),$$

and if $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = (x - 4)(x + 4)^2 = 0$, $x \neq p^4 = 4$ and $p^2 = 2$ i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = -4,$$

and $x \neq p^4 = 4$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n (-4)^k G_{2k+1}^2 = \frac{\Theta}{32},$$

where

$$\Theta = 64(-4)^{n-1} (n^2 - 2) G_{2n+1}^2 + (-4)^{n+1} (n+1)^2 G_{2n-1}^2 + 16(G_0 - G_1)^2 - n(n+1) (-4)^{n-1} 2^{7-2n}$$

(d): ($m = -1, j = 0$)

If $(p^2x - 1)p^{-2}(x^2 - (p^2 - 2)p^{-2}x + p^{-4}) \neq 0$, i.e.,

$$x \neq \frac{(p^2 - 2)p^{-2} + \sqrt{(p^2 - 4)p^{-2}}}{2}, x \neq \frac{(p^2 - 2)p^{-2} - \sqrt{(p^2 - 4)p^{-2}}}{2}, x \neq p^{-2},$$

then

$$\sum_{k=0}^n x^k G_{-k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-2})(x^2 - (p^2 - 2)p^{-2}x + p^{-4})},$$

where

$$\Phi = (x - p^{-2})(p^2 - 4)x^{n+1}G_{-n+1}^2 + (x - (p^2 - 2)p^{-2})(x - p^{-2})(p^2 - 4)x^{n+1}G_{-n}^2 + (p^2 - 4)p^{-2}(2p^{2n}x^n - p^2x - 1)x$$

and if $(p^2x - 1)p^{-2}(x^2 - (p^2 - 2)p^{-2}x + p^{-4}) = 0$, i.e.,

$$x = \frac{(p^2 - 2)p^{-2} + \sqrt{(p^2 - 4)p^{-2}}}{2} \text{ or } x = \frac{(p^2 - 2)p^{-2} - \sqrt{(p^2 - 4)p^{-2}}}{2} \text{ or } x = p^{-2},$$

then

$$\sum_{k=0}^n x^k G_{-k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)p^{-2}x + (p^2 - 1)p^{-4})},$$

where

$$\Lambda = (p^2 - 4)((n+2)x - p^{-2}(n+1))x^n G_{-n+1}^2 + (p^2 - 4)(p^4(n+3)x^2 - p^2x(p^2 - 1)(n+2) + (p^2 - 2)(n+1))p^{-4}x^n G_{-n}^2 + (p^2 - 4)p^{-2}(2p^{2n}(n+1)x^n - 2p^2x - 1)$$

(e): ($m = -2, j = 0$)

If $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) \neq 0$, i.e.,

$$x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq p^{-4},$$

then

$$\sum_{k=0}^n x^k G_{-2k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8})},$$

where

$$\Phi = (x - p^{-4})(p^2 - 4)x^{n+1}G_{-2n+2}^2 + (x - (p^4 - 4p^2 + 2)p^{-4})(x - p^{-4})(p^2 - 4)x^{n+1}G_{-2n}^2 + (p^2 - 4)p^{-4}(2p^{4n}x^n - p^4x - 1)x$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = 0$, and $p^2 \neq 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = p^{-4},$$

(and $p^2 \neq 2$, i.e., $p \neq -\sqrt{2}, p \neq \sqrt{2}$) then

$$\sum_{k=0}^n x^k G_{-2k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)(p^2 - 3)p^{-4}x + (p^2 - 1)(p^2 - 3)p^{-8})},$$

where

$$\begin{aligned} \Lambda &= (p^2 - 4)((n + 2)x - p^{-4}(n + 1))x^n G_{-2n+2}^2 + (p^2 - 4)((x - (p^4 - 4p^2 + 2)p^{-4})x^{n+1} + \\ &(x - p^{-4})((n + 2)x - (p^4 - 4p^2 + 2)p^{-4}(n + 1))x^n)G_{-2n}^2 + (p^2 - 4)p^{-4}(2p^{4n}(n + 1)x^n - 2p^4x - 1) \\ &\text{and if } (x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = \frac{1}{64}(4x - 1)(4x + 1)^2 = 0, x \neq p^{-4} = \frac{1}{4} \text{ and} \\ &p^2 = 2 \text{ i.e.,} \end{aligned}$$

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = -\frac{1}{4},$$

and $x \neq p^{-4} = \frac{1}{4}$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n \left(-\frac{1}{4}\right)^k G_{-2k}^2 = \frac{\Theta}{2},$$

where

$$\begin{aligned} \Theta &= -4\left(-\frac{1}{4}\right)^n (n + 1)^2 G_{-2n+2}^2 + \frac{1}{4}\left(-\frac{1}{4}\right)^{n-1} (n^2 - 2) G_{-2n}^2 + 4(n(n + 1)2^{2n}\left(-\frac{1}{4}\right)^n + 1) \\ \text{(f): } (m &= -2, j = 1) \end{aligned}$$

If $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) \neq 0$, i.e.,

$$x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq p^{-4},$$

then

$$\sum_{k=0}^n x^k G_{-2k+1}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8})},$$

where

$$\begin{aligned} \Phi &= (x - p^{-4})(p^2 - 4)x^{n+1}G_{-2n+3}^2 + (x - (p^4 - 4p^2 + 2)p^{-4})(x - p^{-4})(p^2 - 4)x^{n+1}G_{-2n+1}^2 + \\ &(p^2 - 4)p^{-12}(-p^8(p - 1)^2(p + 1)^2x^2 + p^4(p^4 - 4p^2 + 2)x + 2p^{4n+6}x^{n+1} - 1) \\ &\text{and if } (x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = 0, \text{ and } p^2 \neq 2 \text{ i.e.,} \end{aligned}$$

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = p^{-4},$$

(and $p^2 \neq 2$, i.e., $p \neq -\sqrt{2}, p \neq \sqrt{2}$) then

$$\sum_{k=0}^n x^k G_{-2k+1}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)(p^2 - 3)p^{-4}x + (p^2 - 1)(p^2 - 3)p^{-8})},$$

where

$$\begin{aligned} \Lambda &= (p^2 - 4)((n + 2)x - p^{-4}(n + 1))x^n G_{-2n+3}^2 + (p^2 - 4)((x - (p^4 - 4p^2 + 2)p^{-4})x^{n+1} + \\ &(x - p^{-4})((n + 2)x - (p^4 - 4p^2 + 2)p^{-4}(n + 1))x^n)G_{-2n+1}^2 + (p^2 - 4)p^{-8}(-2p^4x(p - 1)^2(p + 1)^2 + \\ &2p^{4n+2}(n + 1)x^n + p^4 - 4p^2 + 2) \end{aligned}$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = \frac{1}{64}(4x - 1)(4x + 1)^2 = 0$, $x \neq p^{-4} = \frac{1}{4}$ and $p^2 = 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = -\frac{1}{4},$$

and $x \neq p^{-4} = \frac{1}{4}$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n \left(-\frac{1}{4}\right)^k G_{-2k+1}^2 = \frac{\Theta}{2},$$

where

$$\Theta = -4\left(-\frac{1}{4}\right)^n (n+1)^2 G_{-2n+3}^2 + \frac{1}{4}\left(-\frac{1}{4}\right)^{n-1} (n^2 - 2) G_{-2n+1}^2 + 2n(n+1) 2^{2n} \left(-\frac{1}{4}\right)^n + 1.$$

Note that in the last corollary, the case $p^2 \neq 4$ so that $p^2 - 4 \neq 0$ is considered. The sum formulas for the case $p^2 = 4$ so that $p^2 - 4 = 0$ is given in Soykan [8, Corollary 2.2].

Taking $W_n = H_n$ with $H_0 = 2, H_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of p-Oresme-Lucas numbers.

COROLLARY 4. *Suppose that $p^2 \neq 4$ so that $p^2 - 4 \neq 0$. For $n \geq 0$, p-Oresme-Lucas numbers have the following properties:*

(a): ($m = 1, j = 0$)

If $(x - p^2)(x^2 - p^2(p^2 - 2)x + p^4) \neq 0$, i.e.,

$$x \neq \frac{p^2(p^2 - 2) + \sqrt{p^6(p^2 - 4)}}{2}, x \neq \frac{p^2(p^2 - 2) - \sqrt{p^6(p^2 - 4)}}{2}, x \neq p^2,$$

then

$$\sum_{k=0}^n x^k H_k^2 = \frac{\Phi}{(p^2 - 4)(x - p^2)(x^2 - p^2(p^2 - 2)x + p^4)}$$

where

$$\Phi = (x - p^2(p^2 - 2))(x - p^2)(p^2 - 4)x^{n+1}H_n^2 + (x - p^2)(p^2 - 4)x^{n+1}H_{n-1}^2 + p^{-2n+4}(p^2 - 4)(p^{2n}(p^2(3x - 4) - x(x + 4)) - 2(p^2 - 4)x^{n+1}),$$

and

if $(x - p^2)(x^2 - p^2(p^2 - 2)x + p^4) = 0$, i.e.,

$$x = \frac{p^2(p^2 - 2) + \sqrt{p^6(p^2 - 4)}}{2} \text{ or } x = \frac{p^2(p^2 - 2) - \sqrt{p^6(p^2 - 4)}}{2} \text{ or } x = p^2,$$

then

$$\sum_{k=0}^n x^k H_k^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^2(p^2 - 1)x + p^4(p^2 - 1))}$$

where

$$\Lambda = (p^2 - 4)((x - p^2(p^2 - 2))x^{n+1} + (x - p^2)((n + 2)x - (n + 1)p^2(p^2 - 2))x^n)H_n^2 + (p^2 - 4)((n + 2)x - p^2(n + 1))x^n H_{n-1}^2 + p^{-2n+4}(p^2 - 4)(p^{2n}(-2x + 3p^2 - 4) - 2(n + 1)(p^2 - 4)x^n).$$

(b): ($m = 2, j = 0$)

If $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) \neq 0$, i.e.,

$$x \neq \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^n x^k H_{2k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8)}$$

where

$$\Phi = (x - p^4(p^4 - 4p^2 + 2))(x - p^4)(p^2 - 4)x^{n+1}H_{2n}^2 + (x - p^4)(p^2 - 4)x^{n+1}H_{2n-2}^2 + p^{-4n+4}(p^2 - 4)(p^{4n}(4p^2x^2 - p^4x^2 - 4x^2 + 8p^4x - 12p^6x + 3p^8x - 4p^8) - 2p^6(p^2 - 4)x^{n+1})$$

and if $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = 0$ and $p^2 \neq 2$ i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = p^4,$$

(and $p^2 \neq 2$) then

$$\sum_{k=0}^n x^k H_{2k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^4(p^2 - 1)(p^2 - 3)x + p^8(p^2 - 1)(p^2 - 3))}$$

where

$$\Lambda = (p^2 - 4)((x - p^4(p^4 - 4p^2 + 2))x^{n+1} + (x - p^4)((n + 2)x - p^4(p^4 - 4p^2 + 2)(n + 1))x^n)H_{2n}^2 + (p^2 - 4)((n + 2)x - p^4(n + 1))x^n H_{2n-2}^2 + p^{-4n+4}(p^2 - 4)(p^{4n}(-8x + 8p^2x - 2p^4x + 8p^4 - 12p^6 + 3p^8) - 2p^6(p^2 - 4)(n + 1)x^n)$$

and if $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = (x - 4)(x + 4)^2 = 0$, $x \neq p^4 = 4$ and $p^2 = 2$

i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = -4,$$

and $x \neq p^4 = 4$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n (-4)^k H_{2k}^2 = \frac{\Theta}{32}$$

where

$$\Theta = 64(-4)^{n-1}(n^2 - 2)H_{2n}^2 + (-4)^{n+1}(n + 1)^2H_{2n-2}^2 - n(n + 1)2^{-2n+8}(-4)^{n-1}$$

(c): ($m = 2, j = 1$)

If $(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) \neq 0$, i.e.,

$$x \neq \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^n x^k H_{2k+1}^2 = \frac{\Phi}{(p^2 - 4)(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8)},$$

where

$$\begin{aligned} \Phi &= (x - p^4(p^4 - 4p^2 + 2))(x - p^4)(p^2 - 4)x^{n+1}H_{2n+1}^2 + (x - p^4)(p^2 - 4)x^{n+1}H_{2n-1}^2 - p^{-4n+4}(p^2 - 4)(p^{4n}(x^2 + 6p^4x - 2p^6x + p^8) + 2p^4(p^2 - 4)x^{n+1}) \\ &\text{and if } (x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = 0 \text{ and } p^2 \neq 2 \text{ i.e.,} \\ x &= \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = p^4, \end{aligned}$$

(and $p^2 \neq 2$) then

$$\sum_{k=0}^n x^k H_{2k+1}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^4(p^2 - 1)(p^2 - 3)x + p^8(p^2 - 1)(p^2 - 3))},$$

where

$$\begin{aligned} \Lambda &= (p^2 - 4)((x - p^4(p^4 - 4p^2 + 2))x^{n+1} + (x - p^4)((n + 2)x - p^4(p^4 - 4p^2 + 2)(n + 1))x^n)H_{2n+1}^2 + \\ &(p^2 - 4)((n + 2)x - p^4(n + 1))x^n H_{2n-1}^2 - 2p^{-4n+4}(p^2 - 4)(p^{4n}(x + 3p^4 - p^6) + p^4(p^2 - 4)(n + 1)x^n) \\ &\text{and if } (x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = (x - 4)(x + 4)^2 = 0, x \neq p^4 = 4 \text{ and } p^2 = 2 \\ &\text{i.e.,} \\ x &= \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = -4, \end{aligned}$$

and $x \neq p^4 = 4$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n (-4)^k H_{2k+1}^2 = \frac{\Theta}{32},$$

where

$$\begin{aligned} \Theta &= 64(-4)^{n-1}(n^2 - 2)H_{2n+1}^2 + (-4)^{n+1}(n + 1)^2 H_{2n-1}^2 + 2^{-2n+4}(2^{2n} + 2n(n + 1)(-4)^n). \\ \text{(d): } (m &= -1, j = 0) \end{aligned}$$

If $(p^2x - 1)p^{-2}(x^2 - (p^2 - 2)p^{-2}x + p^{-4}) \neq 0$, i.e.,

$$x \neq \frac{(p^2 - 2)p^{-2} + \sqrt{(p^2 - 4)p^{-2}}}{2}, x \neq \frac{(p^2 - 2)p^{-2} - \sqrt{(p^2 - 4)p^{-2}}}{2}, x \neq p^{-2},$$

then

$$\sum_{k=0}^n x^k H_{-k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-2})(x^2 - (p^2 - 2)p^{-2}x + p^{-4})},$$

where

$$\begin{aligned} \Phi &= (x - p^{-2})(p^2 - 4)x^{n+1}H_{-n+1}^2 + (x - (p^2 - 2)p^{-2})(x - p^{-2})(p^2 - 4)x^{n+1}H_{-n}^2 - (p^2 - 4)p^{-6}(p^6x^2 + 4p^2x - 3p^4x + 2p^{2n+2}(p^2 - 4)x^{n+1} + 4) \\ &\text{and if } (p^2x - 1)p^{-2}(x^2 - (p^2 - 2)p^{-2}x + p^{-4}) = 0, \text{ i.e.,} \\ x &= \frac{(p^2 - 2)p^{-2} + \sqrt{(p^2 - 4)p^{-2}}}{2} \text{ or } x = \frac{(p^2 - 2)p^{-2} - \sqrt{(p^2 - 4)p^{-2}}}{2} \text{ or } x = p^{-2}, \end{aligned}$$

then

$$\sum_{k=0}^n x^k H_{-k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)p^{-2}x + (p^2 - 1)p^{-4})},$$

where

$$\Lambda = (p^2 - 4)((n + 2)x - p^{-2}(n + 1))x^n H_{-n+1}^2 + (p^2 - 4)(p^4(n + 3)x^2 - p^2x(p^2 - 1)(n + 2) + (p^2 - 2)(n + 1))p^{-4}x^n H_{-n}^2 + (p^2 - 4)p^{-4}(-2p^{2n}(p^2 - 4)(n + 1)x^n - 2p^4x + 3p^2 - 4)$$

(e): ($m = -2, j = 0$)

$$\text{If } (x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) \neq 0, \text{ i.e.,}$$

$$x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq p^{-4},$$

then

$$\sum_{k=0}^n x^k H_{-2k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8})},$$

where

$$\Phi = (x - p^{-4})(p^2 - 4)x^{n+1}H_{-2n+2}^2 + (x - (p^4 - 4p^2 + 2)p^{-4})(x - p^{-4})(p^2 - 4)x^{n+1}H_{-2n}^2 + (p^2 - 4)p^{-12}(-p^8(p^2 - 2)^2x^2 + p^4(3p^4 - 12p^2 + 8)x - 2p^{4n+6}(p^2 - 4)x^{n+1} - 4)$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = 0$, and $p^2 \neq 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = p^{-4},$$

(and $p^2 \neq 2$, i.e., $p \neq -\sqrt{2}, p \neq \sqrt{2}$) then

$$\sum_{k=0}^n x^k H_{-2k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)(p^2 - 3)p^{-4}x + (p^2 - 1)(p^2 - 3)p^{-8})},$$

where

$$\Lambda = (p^2 - 4)((n + 2)x - p^{-4}(n + 1))x^n H_{-2n+2}^2 + (p^2 - 4)((x - (p^4 - 4p^2 + 2)p^{-4})x^{n+1} + (x - p^{-4})((n + 2)x - (p^4 - 4p^2 + 2)p^{-4}(n + 1))x^n)H_{-2n}^2 + (p^2 - 4)p^{-8}(-2p^2x^n p^{4n}(p^2 - 4)(n + 1) - 2p^4(p^2 - 2)^2x + 3p^4 - 12p^2 + 8)$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = \frac{1}{64}(4x - 1)(4x + 1)^2 = 0, x \neq p^{-4} = \frac{1}{4}$ and $p^2 = 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = -\frac{1}{4},$$

and $x \neq p^{-4} = \frac{1}{4}$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n \left(-\frac{1}{4}\right)^k H_{-2k}^2 = \frac{\Theta}{2},$$

where

$$\Theta = -4\left(-\frac{1}{4}\right)^n (n + 1)^2 H_{-2n+2}^2 + \frac{1}{4}\left(-\frac{1}{4}\right)^{n-1} (n^2 - 2) H_{-2n}^2 - n(n + 1)2^{2n}\left(-\frac{1}{4}\right)^{n-1}$$

(f): ($m = -2, j = 1$)

$$\text{If } (x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) \neq 0, \text{ i.e.,}$$

$$x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq p^{-4},$$

then

$$\sum_{k=0}^n x^k H_{-2k+1}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8})},$$

where

$$\Phi = (x - p^{-4})(p^2 - 4)x^{n+1}H_{-2n+3}^2 + (x - (p^4 - 4p^2 + 2)p^{-4})(x - p^{-4})(p^2 - 4)x^{n+1}H_{-2n+1}^2 + (p^2 - 4)p^{-12}(-p^8(p^2 - 3)^2x^2 + p^4(p^4 - 4p^2 + 2)x - 2p^{4n+4}(p^2 - 4)x^{n+1} - 1)$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = 0$, and $p^2 \neq 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2p^{-6}}}{2} \text{ or } x = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2p^{-6}}}{2} \text{ or } x = p^{-4},$$

(and $p^2 \neq 2$, i.e., $p \neq -\sqrt{2}, p \neq \sqrt{2}$) then

$$\sum_{k=0}^n x^k H_{-2k+1}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)(p^2 - 3)p^{-4}x + (p^2 - 1)(p^2 - 3)p^{-8})},$$

where

$$\Lambda = (p^2 - 4)((n + 2)x - p^{-4}(n + 1))x^n H_{-2n+3}^2 + (p^2 - 4)((x - (p^4 - 4p^2 + 2)p^{-4})x^{n+1} + (x - p^{-4})((n + 2)x - (p^4 - 4p^2 + 2)p^{-4}(n + 1))x^n)H_{-2n+1}^2 + (p^2 - 4)p^{-8}(-2p^{4n}(p^2 - 4)(n + 1)x^n - 2p^4(p^2 - 3)^2x + p^4 - 4p^2 + 2)$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = \frac{1}{64}(4x - 1)(4x + 1)^2 = 0$, $x \neq p^{-4} = \frac{1}{4}$ and $p^2 = 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2p^{-6}}}{2} = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2p^{-6}}}{2} = -\frac{1}{4},$$

and $x \neq p^{-4} = \frac{1}{4}$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n \left(-\frac{1}{4}\right)^k H_{-2k+1}^2 = \frac{\Theta}{2},$$

where

$$\Theta = -4\left(-\frac{1}{4}\right)^n (n + 1)^2 H_{-2n+3}^2 + \frac{1}{4}\left(-\frac{1}{4}\right)^{n-1} (n^2 - 2) H_{-2n+1}^2 + n(n + 1)2^{2n+1}\left(-\frac{1}{4}\right)^n + 1.$$

Note that in the last corollary, the case $p^2 \neq 4$ so that $p^2 - 4 \neq 0$ is considered. The sum formulas for the case $p^2 = 4$ so that $p^2 - 4 = 0$ is given in Soykan [8, Corollary 2.3].

From the above proposition, we have the following corollary which gives sum formulas of p-Oresme numbers (take $W_n = O_n$ with $O_0 = 0, O_1 = \frac{1}{p}$).

COROLLARY 5. *Suppose that $p^2 \neq 4$ so that $p^2 - 4 \neq 0$. For $n \geq 0$, p-Oresme numbers have the following properties:*

(a): ($m = 1, j = 0$)

If $(x - p^2)(x^2 - p^2(p^2 - 2)x + p^4) \neq 0$, i.e.,

$$x \neq \frac{p^2(p^2 - 2) + \sqrt{p^6(p^2 - 4)}}{2}, x \neq \frac{p^2(p^2 - 2) - \sqrt{p^6(p^2 - 4)}}{2}, x \neq p^2,$$

then

$$\sum_{k=0}^n x^k O_k^2 = \frac{\Phi}{(p^2 - 4)(x - p^2)(x^2 - p^2(p^2 - 2)x + p^4)}$$

where

$$\Phi = (x - p^2(p^2 - 2))(x - p^2)(p^2 - 4)x^{n+1}O_n^2 + (x - p^2)(p^2 - 4)x^{n+1}O_{n-1}^2 - p^{-2n+2}(p^2 - 4)(p^{2n}(x + p^2) - 2p^2x^n)x$$

and

$$\text{if } (x - p^2)(x^2 - p^2(p^2 - 2)x + p^4) = 0, \text{ i.e.,}$$

$$x = \frac{p^2(p^2 - 2) + \sqrt{p^6(p^2 - 4)}}{2} \text{ or } x = \frac{p^2(p^2 - 2) - \sqrt{p^6(p^2 - 4)}}{2} \text{ or } x = p^2,$$

then

$$\sum_{k=0}^n x^k O_k^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^2(p^2 - 1)x + p^4(p^2 - 1))}$$

where

$$\Lambda = (p^2 - 4)((x - p^2(p^2 - 2))x^{n+1} + (x - p^2)((n + 2)x - (n + 1)p^2(p^2 - 2))x^n)O_n^2 + (p^2 - 4)((n + 2)x - p^2(n + 1))x^n O_{n-1}^2 + p^{-2n+2}(p^2 - 4)(-p^{2n}(2x + p^2) + 2p^2(n + 1)x^n)$$

(b): ($m = 2, j = 0$)

$$\text{If } (x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) \neq 0, \text{ i.e.,}$$

$$x \neq \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^n x^k O_{2k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8)}$$

where

$$\Phi = (x - p^4(p^4 - 4p^2 + 2))(x - p^4)(p^2 - 4)x^{n+1}O_{2n}^2 + (x - p^4)(p^2 - 4)x^{n+1}O_{2n-2}^2 - p^{-4n+6}(p^2 - 4)(p^{4n}(x + p^4) - 2p^4x^n)x$$

$$\text{and if } (x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = 0 \text{ and } p^2 \neq 2 \text{ i.e.,}$$

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = p^4,$$

(and $p^2 \neq 2$) then

$$\sum_{k=0}^n x^k O_{2k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^4(p^2 - 1)(p^2 - 3)x + p^8(p^2 - 1)(p^2 - 3))}$$

where

$$\Lambda = (p^2 - 4)((x - p^4(p^4 - 4p^2 + 2))x^{n+1} + (x - p^4)((n + 2)x - p^4(p^4 - 4p^2 + 2)(n + 1))x^n)O_{2n}^2 + (p^2 - 4)((n + 2)x - p^4(n + 1))x^n O_{2n-2}^2 + p^{-4n+6}(p^2 - 4)(-p^{4n}(2x + p^4) + 2p^4(n + 1)x^n)$$

$$\text{and if } (x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8) = (x - 4)(x + 4)^2 = 0, x \neq p^4 = 4 \text{ and } p^2 = 2$$

i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = -4,$$

and $x \neq p^4 = 4$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n (-4)^k O_{2k}^2 = \frac{\Theta}{32}$$

where

$$\Theta = 64(-4)^{n-1} (n^2 - 2) O_{2n}^2 + (-4)^{n+1} (n+1)^2 O_{2n-2}^2 + p^{-2} 2^{-2n+6} (2^{2n} + n(n+1) (-4)^n)$$

(c): ($m = 2, j = 1$)

If $(x - p^4) (x^2 - p^4 (p^4 - 4p^2 + 2) x + p^8) \neq 0$, i.e.,

$$x \neq \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2}, x \neq p^4,$$

then

$$\sum_{k=0}^n x^k O_{2k+1}^2 = \frac{\Phi}{(p^2 - 4)(x - p^4)(x^2 - p^4(p^4 - 4p^2 + 2)x + p^8)},$$

where

$$\Phi = (x - p^4(p^4 - 4p^2 + 2))(x - p^4)(p^2 - 4)x^{n+1} O_{2n+1}^2 + (x - p^4)(p^2 - 4)x^{n+1} O_{2n-1}^2 - p^{-4n+2} (p^2 - 4)(p^{4n}(p^8 + 2p^6x - 2p^4x + x^2) - 2p^6x^{n+1}),$$

and if $(x - p^4) (x^2 - p^4 (p^4 - 4p^2 + 2) x + p^8) = 0$ and $p^2 \neq 2$ i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} \text{ or } x = p^4,$$

(and $p^2 \neq 2$) then

$$\sum_{k=0}^n x^k O_{2k+1}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2p^4(p^2 - 1)(p^2 - 3)x + p^8(p^2 - 1)(p^2 - 3))},$$

where

$$\Lambda = (p^2 - 4)((x - p^4(p^4 - 4p^2 + 2))x^{n+1} + (x - p^4)((n+2)x - p^4(p^4 - 4p^2 + 2)(n+1))x^n) O_{2n+1}^2 + (p^2 - 4)((n+2)x - p^4(n+1))x^n O_{2n-1}^2 + 2p^{-4n+2} (p^2 - 4)(p^{4n}(-x - p^6 + p^4) + p^6(n+1)x^n),$$

and if $(x - p^4) (x^2 - p^4 (p^4 - 4p^2 + 2) x + p^8) = (x - 4)(x + 4)^2 = 0$, $x \neq p^4 = 4$ and $p^2 = 2$

i.e.,

$$x = \frac{p^4(p^4 - 4p^2 + 2) + \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = \frac{p^4(p^4 - 4p^2 + 2) - \sqrt{p^{10}(p^2 - 4)(p^2 - 2)^2}}{2} = -4,$$

and $x \neq p^4 = 4$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n (-4)^k O_{2k+1}^2 = \frac{\Theta}{32},$$

where

$$\Theta = 64(-4)^{n-1} (n^2 - 2) O_{2n+1}^2 + (-4)^{n+1} (n+1)^2 O_{2n-1}^2 + 2^{-2n+4} p^{-2} (2^{2n} + 2n(n+1) (-4)^n).$$

(d): ($m = -1, j = 0$)

If $(p^2x - 1)p^{-2}(x^2 - (p^2 - 2)p^{-2}x + p^{-4}) \neq 0$, i.e.,

$$x \neq \frac{(p^2 - 2)p^{-2} + \sqrt{(p^2 - 4)p^{-2}}}{2}, x \neq \frac{(p^2 - 2)p^{-2} - \sqrt{(p^2 - 4)p^{-2}}}{2}, x \neq p^{-2},$$

then

$$\sum_{k=0}^n x^k O_{-k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-2})(x^2 - (p^2 - 2)p^{-2}x + p^{-4})},$$

where

$$\Phi = (x - p^{-2})(p^2 - 4)x^{n+1}O_{-n+1}^2 + (x - (p^2 - 2)p^{-2})(x - p^{-2})(p^2 - 4)x^{n+1}O_{-n}^2 + (p^2 - 4)p^{-4}(2p^{2n}x^n - p^2x - 1)x$$

and if $(p^2x - 1)p^{-2}(x^2 - (p^2 - 2)p^{-2}x + p^{-4}) = 0$, i.e.,

$$x = \frac{(p^2 - 2)p^{-2} + \sqrt{(p^2 - 4)p^{-2}}}{2} \text{ or } x = \frac{(p^2 - 2)p^{-2} - \sqrt{(p^2 - 4)p^{-2}}}{2} \text{ or } x = p^{-2},$$

then

$$\sum_{k=0}^n x^k O_{-k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)p^{-2}x + (p^2 - 1)p^{-4})},$$

where

$$\Lambda = (p^2 - 4)((n + 2)x - p^{-2}(n + 1))x^n O_{-n+1}^2 + (p^2 - 4)(p^4(n + 3)x^2 - p^2x(p^2 - 1)(n + 2) + (p^2 - 2)(n + 1))p^{-4}x^n O_{-n}^2 + (p^2 - 4)p^{-4}(2p^{2n}(n + 1)x^n - 2p^2x - 1)$$

(e): $(m = -2, j = 0)$

If $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) \neq 0$, i.e.,

$$x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq p^{-4},$$

then

$$\sum_{k=0}^n x^k O_{-2k}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8})},$$

where

$$\Phi = (x - p^{-4})(p^2 - 4)x^{n+1}O_{-2n+2}^2 + (x - (p^4 - 4p^2 + 2)p^{-4})(x - p^{-4})(p^2 - 4)x^{n+1}O_{-2n}^2 + (p^2 - 4)p^{-6}(2p^{4n}x^n - p^4x - 1)x$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = 0$, and $p^2 \neq 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = p^{-4},$$

(and $p^2 \neq 2$, i.e., $p \neq -\sqrt{2}, p \neq \sqrt{2}$) then

$$\sum_{k=0}^n x^k O_{-2k}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)(p^2 - 3)p^{-4}x + (p^2 - 1)(p^2 - 3)p^{-8})},$$

where

$$\Lambda = (p^2 - 4)((n + 2)x - p^{-4}(n + 1))x^n O_{-2n+2}^2 + (p^2 - 4)((x - (p^4 - 4p^2 + 2)p^{-4})x^{n+1} + (x - p^{-4})((n + 2)x - (p^4 - 4p^2 + 2)p^{-4}(n + 1))x^n)O_{-2n}^2 + (p^2 - 4)p^{-6}(2p^{4n}(n + 1)x^n - 2p^4x - 1),$$

and if $(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = \frac{1}{64}(4x - 1)(4x + 1)^2 = 0, x \neq p^{-4} = \frac{1}{4}$ and

$p^2 = 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = -\frac{1}{4},$$

and $x \neq p^{-4} = \frac{1}{4}$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n \left(-\frac{1}{4}\right)^k O_{-2k}^2 = \frac{\Theta}{2},$$

where

$$\Theta = -4 \left(-\frac{1}{4}\right)^n (n+1)^2 O_{-2n+2}^2 + \frac{1}{4} \left(-\frac{1}{4}\right)^{n-1} (n^2 - 2) O_{-2n}^2 + 4p^{-2} (2^{2n} n(n+1) \left(-\frac{1}{4}\right)^n + 1).$$

(f): ($m = -2$, $j = 1$)

If $(x - p^{-4}) (x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) \neq 0$, i.e.,

$$x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2}, x \neq p^{-4},$$

then

$$\sum_{k=0}^n x^k O_{-2k+1}^2 = \frac{\Phi}{(p^2 - 4)(x - p^{-4})(x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8})},$$

where

$$\Phi = (x - p^{-4})(p^2 - 4)x^{n+1}O_{-2n+3}^2 + (x - (p^4 - 4p^2 + 2)p^{-4})(x - p^{-4})(p^2 - 4)x^{n+1}O_{-2n+1}^2 + (p^2 - 4)p^{-14}(-p^8(p-1)^2(p+1)^2x^2 + p^4(p^4 - 4p^2 + 2)x + 2p^{4n+6}x^{n+1} - 1),$$

and if $(x - p^{-4}) (x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = 0$, and $p^2 \neq 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} \text{ or } x = p^{-4},$$

(and $p^2 \neq 2$, i.e., $p \neq -\sqrt{2}, p \neq \sqrt{2}$) then

$$\sum_{k=0}^n x^k O_{-2k+1}^2 = \frac{\Lambda}{(p^2 - 4)(3x^2 - 2(p^2 - 1)(p^2 - 3)p^{-4}x + (p^2 - 1)(p^2 - 3)p^{-8})},$$

where

$$\Lambda = (p^2 - 4)((n+2)x - p^{-4}(n+1))x^n O_{-2n+3}^2 + (p^2 - 4)((x - (p^4 - 4p^2 + 2)p^{-4})x^{n+1} + (x - p^{-4})((n+2)x - (p^4 - 4p^2 + 2)p^{-4}(n+1))x^n)O_{-2n+1}^2 + (p^2 - 4)p^{-10}(2p^{4n+2}(n+1)x^n - 2p^4(p-1)^2(p+1)^2x + p^4 - 4p^2 + 2),$$

and if $(x - p^{-4}) (x^2 - (p^4 - 4p^2 + 2)p^{-4}x + p^{-8}) = \frac{1}{64}(4x-1)(4x+1)^2 = 0$, $x \neq p^{-4} = \frac{1}{4}$ and $p^2 = 2$ i.e.,

$$x = \frac{(p^4 - 4p^2 + 2)p^{-4} + \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = \frac{(p^4 - 4p^2 + 2)p^{-4} - \sqrt{(p^2 - 4)((p^2 - 2)^2 p^{-6}}}{2} = -\frac{1}{4},$$

and $x \neq p^{-4} = \frac{1}{4}$ (and $p^2 = 2$, i.e., $p = -\sqrt{2}, p = \sqrt{2}$) then

$$\sum_{k=0}^n \left(-\frac{1}{4}\right)^k O_{-2k+1}^2 = \frac{\Theta}{2},$$

where

$$\Theta = -4 \left(-\frac{1}{4}\right)^n (n+1)^2 O_{-2n+3}^2 + \frac{1}{4} \left(-\frac{1}{4}\right)^{n-1} (n^2 - 2) O_{-2n+1}^2 + p^{-2}(n(n+1)2^{2n+1} \left(-\frac{1}{4}\right)^n + 1).$$

Note that in the last corollary, the case $p^2 \neq 4$ so that $p^2 - 4 \neq 0$ is considered. The sum formulas for the case $p^2 = 4$ so that $p^2 - 4 = 0$ is given in Soykan [8, Corollary 2.4].

Taking $x = 1$ in the last three corollaries we get the following corollary.

COROLLARY 6. *Suppose that $p^2 \neq 4$ so that $p^2 - 4 \neq 0$. For $n \geq 0$, modified p -Oresme numbers and p -Oresme-Lucas numbers and p -Oresme numbers have the following properties:*

(1) *modified p -Oresme numbers:*

(a): *If $p^2 \neq 1$ then*

$$\sum_{k=0}^n G_k^2 = \frac{\Phi}{(p^2 - 4)(1 - p^2)(2p^2 + 1)}$$

where

$$\Phi = (-p^4 + 2p^2 + 1)(1 - p^2)(p^2 - 4)G_n^2 + (1 - p^2)(p^2 - 4)G_{n-1}^2 - p^{-2n+4}(p^2 - 4)(p^{2n}(1 + p^2) - 2p^2)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n G_k^2 = \frac{1}{3}(2G_n^2 + G_{n-1}^2 + 2n - 1).$$

(b): *If $p^2 \neq 1$ then*

$$\sum_{k=0}^n G_{2k}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^4)(2p^2 + 1)(2p^4 - 2p^2 + 1)}$$

where

$$\Phi = (-p^8 + 4p^6 - 2p^4 + 1)(1 - p^4)(p^2 - 4)G_{2n}^2 + (1 - p^4)(p^2 - 4)G_{2n-2}^2 - p^{-4n+8}(p^2 - 4)(p^{4n}(1 + p^4) - 2p^4)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n G_{2k}^2 = \frac{1}{3}(2G_{2n}^2 + G_{2n-2}^2 + 2n - 1).$$

(c): *If $p^2 \neq 1$ then*

$$\sum_{k=0}^n G_{2k+1}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^4)(2p^2 + 1)(2p^4 - 2p^2 + 1)}$$

where

$$\Phi = (-p^8 + 4p^6 - 2p^4 + 1)(1 - p^4)(p^2 - 4)G_{2n+1}^2 + (1 - p^4)(p^2 - 4)G_{2n-1}^2 - p^{-4n+4}(p^2 - 4)(p^{4n}(1 - 2p^4 + 2p^6 + p^8) - 2p^6)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n G_{2k+1}^2 = \frac{1}{3}(2G_{2n+1}^2 + G_{2n-1}^2 + 2n).$$

(d): *If $p^2 \neq 1$ then*

$$\sum_{k=0}^n G_{-k}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^{-2})(2p^{-2} + p^{-4})}$$

where

$$\Phi = (1 - p^{-2})(p^2 - 4)G_{-n+1}^2 + 2p^{-2}(1 - p^{-2})(p^2 - 4)G_{-n}^2 + (p^2 - 4)p^{-2}(2p^{2n} - p^2 - 1)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n G_{-k}^2 = \frac{1}{3}(G_{-n+1}^2 + 2G_{-n}^2 + 2n - 1).$$

(e): If $p^2 \neq 1$ then

$$\sum_{k=0}^n G_{-2k}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^{-4})(1 - (p^4 - 4p^2 + 2)p^{-4} + p^{-8})}$$

where

$$\Phi = (1 - p^{-4})(p^2 - 4)G_{-2n+2}^2 + (1 - (p^4 - 4p^2 + 2)p^{-4})(1 - p^{-4})(p^2 - 4)G_{-2n}^2 + (p^2 - 4)p^{-4}(2p^{4n} - p^4 - 1)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n G_{-2k}^2 = \frac{1}{3}(G_{-2n+2}^2 + 2G_{-2n}^2 + 2n - 1).$$

(f): If $p^2 \neq 1$ then

$$\sum_{k=0}^n G_{-2k+1}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^{-4})(1 - (p^4 - 4p^2 + 2)p^{-4} + p^{-8})}$$

where

$$\Phi = (1 - p^{-4})(p^2 - 4)G_{-2n+3}^2 + (1 - (p^4 - 4p^2 + 2)p^{-4})(1 - p^{-4})(p^2 - 4)G_{-2n+1}^2 + (p^2 - 4)p^{-12}(-p^8(p - 1)^2(p + 1)^2 + p^4(p^4 - 4p^2 + 2) + 2p^{4n+6} - 1)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n G_{-2k+1}^2 = \frac{1}{3}(G_{-2n+3}^2 + 2G_{-2n+1}^2 + 2n + 1).$$

(2) *p*-Oresme-Lucas numbers:

(a): If $p^2 \neq 1$ then

$$\sum_{k=0}^n H_k^2 = \frac{\Phi}{(p^2 - 4)(1 - p^2)(2p^2 + 1)}$$

where

$$\Phi = (p^2 - 1)(p^2 - 4)(p^4 - 2p^2 - 1)H_n^2 + (1 - p^2)(p^2 - 4)H_{n-1}^2 + p^{-2n+4}(p^2 - 4)(8 - 2p^2 - p^{2n}(p^2 + 5))$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n H_k^2 = \frac{1}{3}(2H_n^2 + H_{n-1}^2 + 6n + 3).$$

(b): If $p^2 \neq 1$ then

$$\sum_{k=0}^n H_{2k}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^4)(1 - p^4(p^4 - 4p^2 + 2) + p^8)}$$

where

$$\Phi = (1 - p^4(p^4 - 4p^2 + 2))(1 - p^4)(p^2 - 4)H_{2n}^2 + (1 - p^4)(p^2 - 4)H_{2n-2}^2 + p^{-4n+4}(p^2 - 4)(p^{4n}(-p^8 - 12p^6 + 7p^4 + 4p^2 - 4) - 2p^6(p^2 - 4))$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n H_{2k}^2 = \frac{1}{3}(2H_{2n}^2 + H_{2n-2}^2 + 6n + 3).$$

(c): If $p^2 \neq 1$ then

$$\sum_{k=0}^n H_{2k+1}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^4)(1 - p^4(p^4 - 4p^2 + 2) + p^8)}$$

where

$$\Phi = (1 - p^4(p^4 - 4p^2 + 2))(1 - p^4)(p^2 - 4)H_{2n+1}^2 + (1 - p^4)(p^2 - 4)H_{2n-1}^2 - p^{-4n+4}(p^2 - 4)(p^{4n}(1 + 6p^4 - 2p^6 + p^8) + 2p^4(p^2 - 4))$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n H_{2k+1}^2 = \frac{1}{3}(2H_{2n+1}^2 + H_{2n-1}^2 + 6n).$$

(d): If $p^2 \neq 1$ then

$$\sum_{k=0}^n H_{-k}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^{-2})(2p^{-2} + p^{-4})}$$

where

$$\Phi = (1 - p^{-2})(p^2 - 4)H_{-n+1}^2 + 2p^{-2}(1 - p^{-2})(p^2 - 4)H_{-n}^2 - (p^2 - 4)p^{-6}(p^6 + 4p^2 - 3p^4 + 2p^{2n+2}(p^2 - 4) + 4)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n H_{-k}^2 = \frac{1}{3}(H_{-n+1}^2 + 2H_{-n}^2 + 6n + 3).$$

(e): If $p^2 \neq 1$ then

$$\sum_{k=0}^n H_{-2k}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^{-4})(1 - (p^4 - 4p^2 + 2)p^{-4} + p^{-8})}$$

where

$$\Phi = (1 - p^{-4})(p^2 - 4)H_{-2n+2}^2 + (1 - (p^4 - 4p^2 + 2)p^{-4})(1 - p^{-4})(p^2 - 4)H_{-2n}^2 + (p^2 - 4)p^{-12}(-p^8(p^2 - 2)^2 + p^4(3p^4 - 12p^2 + 8) - 2p^{4n+6}(p^2 - 4) - 4)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n H_{-2k}^2 = \frac{1}{3}(H_{-2n+2}^2 + 2H_{-2n}^2 + 6n + 3).$$

(f): If $p^2 \neq 1$ then

$$\sum_{k=0}^n H_{-2k+1}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^{-4})(1 - (p^4 - 4p^2 + 2)p^{-4} + p^{-8})}$$

where

$$\Phi = (1 - p^{-4})(p^2 - 4)H_{-2n+3}^2 + (1 - (p^4 - 4p^2 + 2)p^{-4})(1 - p^{-4})(p^2 - 4)H_{-2n+1}^2 + (p^2 - 4)p^{-12}(-p^8(p^2 - 3)^2 + p^4(p^4 - 4p^2 + 2) - 2p^{4n+4}(p^2 - 4) - 1)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n H_{-2k+1}^2 = \frac{1}{3}(H_{-2n+3}^2 + 2H_{-2n+1}^2 + 6n - 3).$$

(3) *p-Oresme numbers:*

(a): If $p^2 \neq 1$ then

$$\sum_{k=0}^n O_k^2 = \frac{\Phi}{(p^2 - 4)(1 - p^2)(2p^2 + 1)}$$

where

$$\Phi = (-p^4 + 2p^2 + 1)(1 - p^2)(p^2 - 4)O_n^2 + (1 - p^2)(p^2 - 4)O_{n-1}^2 - p^{-2n+2}(p^2 - 4)(p^{2n}(1 + p^2) - 2p^2)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n O_k^2 = \frac{1}{3}(2O_n^2 + O_{n-1}^2 + 2n - 1).$$

(b): If $p^2 \neq 1$ then

$$\sum_{k=0}^n O_{2k}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^4)(1 - p^4(p^4 - 4p^2 + 2) + p^8)}$$

where

$$\Phi = (1 - p^4(p^4 - 4p^2 + 2))(1 - p^4)(p^2 - 4)O_{2n}^2 + (1 - p^4)(p^2 - 4)O_{2n-2}^2 - p^{-4n+6}(p^2 - 4)(p^{4n}(1 + p^4) - 2p^4)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n O_{2k}^2 = \frac{1}{3}(2O_{2n}^2 + O_{2n-2}^2 + 2n - 1).$$

(c): If $p^2 \neq 1$ then

$$\sum_{k=0}^n O_{2k+1}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^4)(1 - p^4(p^4 - 4p^2 + 2) + p^8)}$$

where

$$\Phi = (1 - p^4(p^4 - 4p^2 + 2))(1 - p^4)(p^2 - 4)O_{2n+1}^2 + (1 - p^4)(p^2 - 4)O_{2n-1}^2 + p^2(p^2 - 4)(2p^4p^{4n} - 2p^6p^{4n} - p^8p^{4n} - p^{4n} + 2p^6)p^{-4n}$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n O_{2k+1}^2 = \frac{1}{3}(2O_{2n+1}^2 + O_{2n-1}^2 + 2n).$$

(d): If $p^2 \neq 1$ then

$$\sum_{k=0}^n O_{-k}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^{-2})(2p^{-2} + p^{-4})}$$

where

$$\Phi = (1 - p^{-2})(p^2 - 4)O_{-n+1}^2 + 2p^{-2}(1 - p^{-2})(p^2 - 4)O_{-n}^2 + (p^2 - 4)p^{-4}(2p^{2n} - p^2 - 1)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n O_{-k}^2 = \frac{1}{3}(O_{-n+1}^2 + 2O_{-n}^2 + 2n - 1).$$

(e): If $p^2 \neq 1$ then

$$\sum_{k=0}^n O_{-2k}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^{-4})(1 - (p^4 - 4p^2 + 2)p^{-4} + p^{-8})}$$

where

$$\Phi = (1 - p^{-4})(p^2 - 4)O_{-2n+2}^2 + (1 - (p^4 - 4p^2 + 2)p^{-4})(1 - p^{-4})(p^2 - 4)O_{-2n}^2 + (p^2 - 4)p^{-6}(2p^{4n} - p^4 - 1)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n O_{-2k}^2 = \frac{1}{3}(O_{-2n+2}^2 + 2O_{-2n}^2 + 2n - 1).$$

(f): If $p^2 \neq 1$ then

$$\sum_{k=0}^n O_{-2k+1}^2 = \frac{\Phi}{(p^2 - 4)(1 - p^{-4})(1 - (p^4 - 4p^2 + 2)p^{-4} + p^{-8})}$$

where

$$\Phi = (1 - p^{-4})(p^2 - 4)O_{-2n+3}^2 + (1 - (p^4 - 4p^2 + 2)p^{-4})(1 - p^{-4})(p^2 - 4)O_{-2n+1}^2 + (p^2 - 4)p^{-14}(-p^8(p - 1)^2(p + 1)^2 + p^4(p^4 - 4p^2 + 2) + 2p^{4n+6} - 1)$$

and if $p^2 = 1$ then

$$\sum_{k=0}^n O_{-2k+1}^2 = \frac{1}{3}(O_{-2n+3}^2 + 2O_{-2n+1}^2 + 2n + 1).$$

Note that in the last corollary, the case $p^2 \neq 4$ so that $p^2 - 4 \neq 0$ is considered. The sum formulas for the case $p^2 = 4$ so that $p^2 - 4 = 0$ is given in Soykan [8, Corollary 2.5].

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