Fixed Point Results for Contractive Mappings in F-metric space

ABSTRACT. In this research, the existence of fixed point and it uniqueness were investigated in the setting of F-metric spaces, taking into account certain contractive conditions such as p-contraction and monotonically decreasing contraction. The findings extend and generalize a number of previously published findings.

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1. INTRODUCTION

One of the more conventional theories in functional and nonlinear analysis is fixed point theory. Numerous mathematicians have expanded the metric structure by altering Frechet's initial notion of a metric after he created metric spaces [4]. The triangle inequality of the original formulation is modified in order to make the majority of the generalizations.

In the 1960s, [5] defined 2-metric space in a series of publications, which he claimed was a generalization of ordinary metric spaces. This is one of the well-known metrics of such generalizations. This space's structure is described as follows:

Let Y be a nonempty set. A function $b : Y \times Y \to \mathbb{R}$ is said to be a 2-metric on Y if it satisfies the following properties:

1 For distinct points $i, j \in Y$, there is a point $K \in Y$ such that $H(i, j, k) \neq 0$,
2 $b(i, j, k) = 0$ if any two elements of the triplet (i, j, k) are equal,
3 (i, j, k)=(i, j, k)....(symmetry),
4 $(i, j, k) \leq (i, j, a) + (i, a, k) + (a, j, k)$ for all $i, j, k, a \in Y$ (triangle inequality).

A generalized metric space and associated fixed point theorems were defined by [7]. More recently, [8] presented another abstraction of this type, which they call F-metric.

Let $f \in F$ and $f : (0, +\infty) \to \mathbb{R}$ be such that:
$(F1)$ $0 < x < y \implies f(x) \leq f(j)$; and

$(F2)$ for $\{i_n\} \subseteq R^+, \lim_{n\to\infty} i_n = 0 \leftrightarrow \lim_{n\to\infty} f\{i_n\} = -\infty$

Definition 1 Let $d_F : M \times M \to [0, +\infty)$ be a given mapping, and let M be a nonempty set. Assume that $(f, \alpha) \in F \times [0, +\infty)$ occurs in such a way that:

(1) $H(i,j) \in M \times M, H(i,j) = 0 \implies x = y$;
(2) $H(i,j) = H(i,j)$ for all $(x,y) \in M \times M$; and
(3) for every $(x,y) \in M \times M, N \in \mathbb{N}, N \geq 2$, and $(i_i)_i^n = 1 \subset M$ with $(i_1, i_N) = (x,y)$, we get

$$H(i,j) > 0 \; implies \; f(H(i,j)) \leq f(\sum_{i=1}^{N-1} H(i_i, y_{i+1})) + \alpha$$

The pair (M,H) is then referred to as a F-metric space, and H is a F-metric on M.

The approach of successive approximations used to prove the existence of solutions of differential equations, which was independently introduced by [9] and [10], is where fixed point theory got its start. However, it was formally introduced as a significant component of analysis at the turn of the 20th century. The famous Polish mathematician's groundbreaking work is the abstraction of this classical theory.

[11] The concept of sequential F-metric spaces was introduced as an extension of normal metric spaces, b-metric spaces, JS-metric spaces, and mainly F-metric spaces. We looked at some of these spaces' topological properties. Considering this notion, they proved fixed-point theorems for some classes of contractive mappings over such spaces. Their fixed-point theorems are supported by examples, which also verify that the underlying space is valid. Furthermore, their fixed-point theorem is used to solve a system of linear algebraic equations.

[12], presented a new class of Picard operators for such mappings in the framework of F -metric space and examined a fixed point problem associated with specific contraction mappings, producing some interesting and original findings. They also showed that the integral equation and fixed point issue are well-posed, and they examined the Hyers-Ulam stability of an integral equation, a Cauchy functional equation, and a fixed point problem as applications of their findings (see. [1], [6], [2], [3]).

1. PRELIMINARY

This section will examine definitions, examples, lemmas, propositions, and properties that are crucial in producing our main findings.

[8] coined the concept of F-metric spaces by utilizing a particular class of auxiliary functions, which we start with.

Let $f \in F$ and $f : (0, +\infty) \to \mathbb{R}$ be such that:

$(F1)$ $0 < i < j \implies f(i) \leq f(j)$; and

$(F2)$ for $\{i_n\} \subseteq R^+$, $\lim\limits_{n\to\infty} i_n = 0 \leftrightarrow \lim\limits_{n\to\infty} f\{i_n\} = -\infty$

Example 2.1 [8] Examples of the type of auxiliary functions that were previously addressed include the following:

- i $-\frac{1}{t}$ where $t \in (0, \infty)$;
- ii $-\exp^{\frac{1}{t}}$ for all $t \in (0, \infty)$.

The authors developed the idea of F -metric spaces by using such functions to broaden the idea of conventional metric spaces. [8]

Definition 2.1 Let $d_F : M \times M \to [0, +\infty)$ be a given mapping, and let M be a nonempty set. Assume that $(f, \alpha) \in F \times [0, +\infty)$ exists in such a way that:

- H1 $H(i, j) \in M \times M, H(i, j) = 0 \implies x = y$;
- H2 $H(i, j) = H(i, j)$ for all $H(i, j) \in M \times M$; and
- H3 for every $(i, j) \in M \times M, N \in \mathbb{N}, N \geq 2$, and $(i_i)_i^n = 1 \subset M$ with $(i_1, i_N) = (i, j)$, we get

$$H(i, j) > 0 \textit{ implies } f(H(i, j)) \leq f(\sum_{i=1}^{N-1} H(i_n, y_{n+1})) + \alpha$$

Then H is an F-metric on M, and the pair (M, H) is said to be an F-metric space.

It is observed that any metric on Y is an F -metric, but the converse is not true.

Proposition 2.1.[8]

Let (j, H) be a space with F metrics. Let $i \in Y$ and $\{i_n\}$ be a sequence in Y. The statements that follow are interchangeable:

- i $\{i_n\}$ is F-convergent to i,
- ii $\lim\limits_{i_n\to\infty} H(i_n, i) = 0$

The next result shows that the limit of an F-convergent sequence is unique.

Proposition 2.2 [8]

Let (j, H) be an F-metric space. Let $\{i_n\}$ be a sequence in Y.

Then

$$(i,j) \in Y \times Y, \lim_{i_n \to \infty} H(i_n, i) = \lim_{i_n \to \infty} H(i_n, j) = 0 \Rightarrow i = j$$

Definition 2.2 [8]
Let (j, H) be an F-metric space. Let $\{i_n\}$ be a sequence in Y.

i We say that $\{i_n\}$ is F-convergent, if

$$\lim_{i_n \to \infty} H(i_n, x) = 0$$

ii We say that i_n is F-Cauchy, if

$$\lim_{i_n \to \infty} H(i_n, i_m) = 0$$

iii We say that (j, H) is F-complete, if every F-Cauchy sequence in Y is F- convergent to a certain element in Y.

Proposition 2.3 [8]
Let (j, H) be an F-metric space. If $\{i_n\} \subset Y$ is F-convergent, then it is F-Cauchy.

2. MAIN RESULT

We state our main results as follows.

Theorem 3.1
Let (j, H) be an F-metric space and $g : Y \to Y$ be a continuous p-contraction mapping. Then g has a unique fixed point in Y

$$H(gi, gj) \leq k[H(i,j) + |H(i, gi) - H(j, gj)|] \tag{1}$$

Proof
Let $i_0 \in Y$ be an arbitrary point and define a sequence $\{i_n\}$ by $i_{n+1} = gi_n$ for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ such that $i_{n_0} = i_{n_0+1}$ is a fixed point of g. Suppose $i_n \neq i_{n+1}$ for all $n \in \mathbb{N}$ hence $H(i_n, i_{n+1}) > 0$
Now we have.

$$\begin{aligned} H(i_{n+1}, i_{n+2}) &= H(gi_n, gi_{n+1}) && (2)\\ &\leq k[H(i_n, i_{n+1}) + |H(i_n, gi_n) - H(i_{n+1}, gi_{n+1})|] && (3)\\ &= k[H(i_n, i_{n+1}) + |H(i_n, i_{n+1}) - H(i_{n+1}, i_{n+2})|] && (4) \end{aligned}$$

for all $n \in \mathbb{N}$ if $H(i_n, i_{n+1}) \geq H(i_{n+1}, i_{n+2})$ for some n then from (1)

$$\begin{aligned} H(i_{n+1}, i_{n+2}) &\leq k[H(i_n, i_{n+1}) + |H(i_n, i_{n+1}) - H(i_{n+1}, i_{n+2})|] && (5)\\ &\leq k[H(i_n, i_{n+1}) + H(i_n, i_{n+1}) - H(i_{n+1}, i_{n+2})] && (6)\\ &= 2kH(i_n, i_{n+1}) - kH(i_{n+1}, i_{n+2}) && (7) \end{aligned}$$

$$H(i_{n+1}, i_{n+2}) + kH(i_{n+1}, i_{n+2}) \leq 2kH(i_n, i_{n+1}) \tag{8}$$

$$(1+k)H(i_{n+1}, i_{n+2}) \leq 2kH(i_n, i_{n+1}) \tag{9}$$

$$H(i_{n+1}, i_{n+2}) \leq \frac{2k}{1+k} H(i_n, i_{n+1}) \tag{10}$$

for all $n \in \mathbb{N}$
Let $\frac{2k}{1+k} = \lambda$, then $0 < \lambda < 1$ and so we have

$$\begin{aligned} H(i_{n+1}, i_{n+2}) &\leq \lambda H(i_n, i_{n+1}) && (11)\\ &\leq \lambda^2 H(i_{n-1}, i_n) && (12)\\ &\leq \lambda^3 H(i_{n-3}, i_{n-1}) && (13)\\ &. && (14)\\ &. && (15)\\ &. && (16)\\ &\leq \lambda^n H(i_1, i_0) && (17) \end{aligned}$$

for all value of $n \in \mathbb{N}$
We have for $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned} \sum_{i=n}^{m-1} H(i_n, i_{n+1}) &\leq \sum_{i=n}^{m-1} \lambda^n H(i_0, i_1) && (18)\\ &\leq \frac{\lambda^n}{1-\lambda} H(i_0, i_1) && (19) \end{aligned}$$

since $0 < \lambda < 1$ for all $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n > no_0$

$$0 < \frac{\lambda^n}{1-\lambda} H(i_0, i_1) < \delta \tag{20}$$

Now, let $(f, \alpha) \in F \times [0, \infty)$ be such that (H3) is satisfied.
let $\epsilon > 0$ be fixed, then by (f2) there exists $\eta > 0$ such that

$$0 < t < \eta \Rightarrow f(t) < f(\epsilon) - \alpha \tag{21}$$

considering δ as η we get

$$f\left(\frac{\lambda^n}{1-\lambda} H(i_0, i_1)\right) < f(\epsilon) - \alpha \tag{22}$$

By (f1) we have

$$f\left(\sum_{i=1}^{N-1} H(i_i, i_{i+1})\right) \leq f\left(\frac{\lambda^n}{1-\lambda} H(i_0, i_1)\right) < f(\epsilon) - \alpha \tag{23}$$

for all $m, n \in \mathbb{N}$ with $m > n \geq n_0$.
Using (H3) and equation (23) we get for $m > n \geq n_0$

$$H(i_n, i_m) > 0 \Rightarrow f\left(H(i_n, i_m)\right) \leq f\left(\sum_{i=1}^{m-1} H(i_i, i_{i+1})\right) + \alpha < f(\epsilon) \tag{24}$$

which implies by (f1) that $H(i_n, i_m) < \epsilon$ for $m > n \geq n_0$. Therefore the sequence $\{i_n\}$ is F-Cauchy.
Since $H(i, j)$ is F-complete there exists $k \in Y$ such that $\{i_n\}$ is F-convergent to z, that is $\lim_{n\to\infty} H(i_{n+1}, z) = 0$
Since g is continuous, and the $\lim_{n\to\infty}(gi_n, gk) = \lim_{n\to\infty} H(i_{n+1}, gk) = 0$
The uniqueness of the limit we have $z = gk$.
Now suppose w is another fixed point of g, then $gw = w$ and $H(k, w) > 0$.
Hence

$$\begin{align} H(k, w) &= H(gk, gw) \tag{25} \\ &\leq k[H(k, w) + |H(k, gk) - H(w, gw)|] \tag{26} \\ &= kH(k, w) \tag{27} \end{align}$$

which is a contradiction, hence the fixed point of g is unique.

Theorem 3.2
Let g be a self mapping on an F-metric space $H(j, H)$. Suppose there exists $i_0 \in Y$ such that

$$\phi(H(gi, gj)) < \frac{1}{2}\{\phi(H(i, gi)) + \phi(H(j, gj))\} \tag{28}$$

holds for every $i, j \in Y$ with $i \neq j$, then g has a unique fixed point i^* in g.
Proof
Let the sequence $\{i_n\}$ be defined as $i_{n+1} = gi_n$ for $n \in \mathbb{N}$. If $i_n = i_{n+1}$ for some n, then g has a fixed point. So, let $i_n \neq i_{n+1}$ for every $n \in \mathbb{N}$.
Let $\alpha_n = \phi(H(i_n, i_{n+1}))$ for all $n \in \mathbb{N}$.

Therefore, it follows that

$$\begin{aligned} \alpha_{n+1} &= \phi(H(i_{n+1}, i_{n+2})) && (29) \\ &= \phi(H(gi_n, gi_{n+1})) && (30) \\ &\leq \frac{1}{2}\{\phi(H(i_n, gi_n)) + \phi(H(i_{n+1}, gi_{n+1}))\} && (31) \\ &= \frac{1}{2}\{\phi(H(i_n, i_{n+1})) + \phi(H(i_{n+1}, i_{n+2}))\} && (32) \\ &= \frac{1}{2}\{\alpha_n + \alpha_{n+1}\} && (33) \\ \alpha_{n+1} &< \frac{1}{2}\alpha_n + \frac{1}{2}\alpha_{n+1} && (34) \\ \alpha_{n+1} &< \alpha_n && (35) \end{aligned}$$

Hence,$\{\alpha_n\}$ is a strictly decreasing sequence of positive reals and hence converges to some nonnegative real number a.
Now, we claim that $a = 0$. If $a \neq 0$, then we have

$$o < a = \lim_{n\to\infty} \phi(H(i_n, i_{n+1})) = \phi(H(i^*, gi^*)) \tag{36}$$

which is a contradiction, hence $a = 0$.
Thus, the sequence$\{\alpha_n\}$ converges to zero.
Let there exists $(f, \alpha) \in F \times [0, \infty)$ satisfying the conditions $(H1-H3)$ of definition 2.1. Then, by $(f2)$, for a given $\epsilon > 0$, there exists a $\delta > 0$ such that.

$$0 < t < \delta \Rightarrow f(t) < f(\phi(\epsilon)) - \alpha \tag{37}$$

Now,

$$\begin{aligned} \phi(H(i_n, i_{n+1})) &< \frac{1}{2}\{\phi(H(i_{n-1}, gi_{n-1})) + \phi(H(i_n, gi_n))\} && (38) \\ &= \frac{1}{2}\{\phi(H(i_{n-1}, i_n)) + \phi(H(i_n, i_{n+1}))\} && (39) \\ &= \frac{1}{2}\{\phi(H(g^{n-1}i_0, g^n i_0)) + \phi(H(g^n i_0, g^{n+1}i_0))\} && (40) \end{aligned}$$

Similarly, we obtain,

$$\sum_{i=n}^{m-1} \phi(H(i_i, i_{i+1})) < \sum_{i=n}^{m-1} \frac{1}{2}\{\phi(H(g^{i-1}i_0, g^i i_0)) + \phi(H(g^i i_0, g^{i+1}i_0))\} \tag{41}$$

Since

$$\lim_{n\to\infty}\{\phi(H(g^{n-1}i_0, g^n i_0)) + \phi(H(g^n i_0, g^{n+1}i_0))\} = 0 \tag{42}$$

There exists some $N \in \mathbb{N}$ such that

$$0 < \sum_{i=n}^{m-1} \phi(H(i_i, i_{i+1})) < \delta$$

holds for all $n \geq N$. Hence by (37) and $(f1)$ we have

$$f\left(\sum_{i=n}^{m-1} \phi(H(i_i, i_{i+1}))\right) < f(\phi(\epsilon)) - \alpha \tag{43}$$

Now, we show that

$$H(i_n, i_m) < \epsilon \tag{44}$$

for all $m > n \geq \mathbb{N}$. Let $m, n \in \mathbb{N}$ be fixed but arbitrary such that $m > n \geq \mathbb{N}$. If$H(i_n, i_m) = 0$, then clearly $H(i_n, i_m) < \epsilon$ and if $H(i_n, i_m) > 0$, then using $(H3)$ and (43), we have

$$H(i_n, i_m) > 0 \tag{45}$$

$$\Rightarrow f\left(\phi(H(i_n, i_m))\right) \leq f\left(\sum_{i=n}^{m-1} \phi(H(i_i, i_{i+1}))\right) + \alpha < f(\phi(\epsilon)) \tag{46}$$

which gives by $(f1)$ that

$$(\phi(H(i_n, i_m))) < \phi(\epsilon) \tag{47}$$

$$\Rightarrow H(i_n, i_m) < \epsilon \tag{48}$$

This proves that $\{i_n\}$ is F-Cauchy. Since $\{i_n\}$ converges to i_0, then the limit of $\{i_n\}$ will be i^*. This implies.

$$\lim_{n\to\infty} H(i_n, i^*) = 0. \tag{49}$$

Since $gi_n = i_{n+1}$, we have, by the uniqueness of limit of sequence $i^* = gi^*$. Hence i^* is a fixed point of g. For uniqueness, let j^* be another fixed point of g. Then

$$\phi(H(j, j^*)) = \phi(H(gj, gj^*)) \tag{50}$$

$$\leq \frac{1}{2}\{\phi(H(j, gj)) + \phi(H(j^*, gj^*))\} \tag{51}$$

$$< 0 \tag{52}$$

a contradiction. Therefore i^* is the unique fixed point of g.

Theorem 3.3

Let (j, H) be an F-metric space and let $g : Y \to Y$ be a given mapping. Suppose that the following conditions are satisfied

i (j, H) is F- complete,

ii there exists monotonically decreasing functions a, b, c from $(0, \infty)$ into$[0, 1)$ satisfying $a(H(i, j))+b(H(i, j))+c(H(i, j)) < 1$ such that, for each $x, y \in X,\ x \neq y$

$$H(gi, gj) \leq a(H(i, j))H(i, gi) + b(H(i, j))H(j, gj) + c(H(i, j))H(i, j) \tag{53}$$

for all $(x, y) \in X \times Y$

then g has a unique fixed point $i^* \in Y$. Moreover for any $i_0 \in Y$, the $\{i_n\} \in Y$ defined by $i_{n+1} = gi_n\ n \in \mathbb{N}$ is F-convergent to i^*.

Proof

By observation g has at most one fixed point. Indeed, if $(u, v) \in X \times Y$ are two fixed point of g with $u \neq v$ i.e
$H(u, v) > 0\ gu = u\ and\ gv = v$
Then from (ii) we have

$$H(u, v) = H(gu, gv) \leq a(H(u, u))H(u, g) + b(H(u, v))H(v, gv) + c(H(u, v))H(u, v) \tag{54}$$

$$\leq c(H(u, v))H(u, v) \tag{55}$$

which is a contradiction.
Hence, $u = v$
Next, let $(f, \alpha) \in F \times [o, \infty)$ be such that (H3) is satisfied, let $\epsilon > 0$ be fixed.
$0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha$
Let $i_0 \in Y$ be an arbitrary element, let $\{i_n\} \in Y$ be sequence defined by $i_{n+1} = gi_n$ for $n \in \mathbb{N}$

$$H(gi_n, gi_{n+1}) \leq a(H(i, j))H(i_n, gi_n) + b(H(i, j))H(i_{n+1}, gi_{n+1}) \tag{56}$$

$$+ c(H(i, j))H(i_n, i_{n+1})H(i_{n+1}, i_{n+2}) \leq a(H(i, j))H(i_n, i_{n+1}) \tag{57}$$

$$+ b(H(i, j))H(i_{n+1}, i_{n+2}) + c(H(i, j))H(i_n, i_{n+1}) \tag{58}$$

$$H(i_{n+1}, i_{n+2}) - b(H(i, j))H(i_{n+1}, i_{n+2}) \leq [a(H(i, j)) + c(H(i, j))]H(i_n, i_{n+1}) \tag{59}$$

$$[1 - b(H(i, j))]H(i_{n+1}, i_{n+2}) \leq [a(H(i, j)) + c(H(i, j))]H(i_n, i_{n+1}) \tag{60}$$

$$H(i_{n+1}, i_{n+2}) \leq \frac{[a(H(i, j)) + c(H(i, j))]}{[1 - b(H(i, j))]}H(i_n, i_{n+1}) \tag{61}$$

$$let\ q = \frac{[a(H(i, j)) + c(H(i, j))]}{[1 - b(H(i, j))]} \tag{62}$$

then, $0 < q < 1$, since $a(H(i,j)) + b(H(i,j)) + c(H(i,j)) < 1$
Hence,

$$H(i_{n+1}, i_{n+2}) \leq qH(i_n, i_{n+1}) \tag{63}$$

Continue the above argument iteratively we will have

$$H(i_{n+1}, i_{n+2}) \leq \frac{q^n}{1-q} H(i_0, i_1) \tag{64}$$

which yields

$$\sum_{i=n}^{m-1} H(i_i, i_{i+1}) \leq \frac{q^n}{1-q} H(i_0, i_1) \ m > n \tag{65}$$

since $\lim\limits_{n\to\infty} \frac{q^n}{1-q} H(i_0, i_1) = 0$
there exists some $N \in \mathbb{N}$ such that

$$0 < \frac{q^n}{1-q} H(i_0, i_1) < \delta, \ n \geq \mathbb{N} \tag{66}$$

Hence, by $0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha$ and $(f1)$, we have

$$f\left(\sum_{i=n}^{m-1} H(i_i, i_{i+1})\right) \leq f\left(\frac{q^n}{1-q}(H(i_0, i_1))\right) < f(\epsilon) - \alpha \tag{67}$$

using (H3) and (67), we obtain

$$H(i_n, i_m) > 0, \ m > n \geq N \Rightarrow f(H(i_n, i_m)) < f\left(\sum_{i=n}^{m-1} H(i_i, i_{i+1})\right) + \alpha < f(\epsilon) \tag{68}$$

which implies $H(i_n, i_m) < \epsilon, \ m, n > N$
Therefore the $\{i_n\}$ if F-Cauchy, since (j, H) is F-complete there exists $i^* \in Y$ such that $\{i_n\}$ is F-convergent to i^* i.e

$$\lim_{n\to\infty} H(i_n, i^*) = 0 \tag{69}$$

we shall show that i^* is a fixed point of g. We argue by contradiction by supposing that $H(gi^*, i^*) > 0$. By (H3), we obtain

$$f(H(gi^*, i^*)) \leq f(H(gi^*, gi_n)) + f(H(gi_n, i^*)) + \alpha \tag{70}$$

using (ii) and $(f1)$, we obtain

$$f(H(gi^*, i^*)) \leq f(a(H(i,j))[H(i^*, i_n) + H(i_n, gi^*)] + b(H(i,j))[H(i^*, i_n) \tag{71}$$

$$+ H(i_n, gi^*)] + c(H(i,j))[H(i^*, i_n) + H(i_n, i^*)]) + \alpha \tag{72}$$

on the other hand, using $(f2)$ and $\lim\limits_{n\to\infty} H(i_n, i^*) = 0$ we have

$$\lim_{n\to\infty} f(a(H(i,j))[H(i^*, i_n) + H(i_n, gi^*)] + b(H(i,j))[H(i^*, i_n) + H(i_n, gi^*)] \tag{73}$$

$$+ c(H(i,j))[H(i^*, i_n) + H(i_n, i^*)]) + \alpha = -\infty \tag{74}$$

which is a contradiction, therefore, we have
$H(gi^*, i^*) = 0$ i.e $gi^* = i^*$. As a consequence, $i^* \in Y$ is the unique fixed point of g.

Theorem 3.4
Let (j, H) be an F-metric space and let $g : Y \to Y$ be a given mapping. Suppose that the following conditions are satisfied.

i (j, H) is F-complete,

ii for each $i, j \in Y$ $i \neq j$ such that

$$H(gi, gj) < h \max(H(i, gi), H(j, gj), H(i, j)) \tag{75}$$

Then g has a unique fixed point $i^* \in Y$.

Proof
Let $i_0 \in Y$ be arbitrary but fixed and let $\{i_n\}$ $n \geq 0$ be the Picard sequence of g based on i_0, that is

$$i_{n+1} = gi_n \; for \; all \; n \geq 0 \tag{76}$$

If $i_n = i_{n+1}$ for some n, then it is easily noticeable that i_n is a fixed point of g
Let $i_n \neq i_{n+1}$ for all $n \in \mathbb{N}$.
Putting $x = i_n$, $y = i_{n+1}$ in (75) and define sequence of a real number as $s_n = H(i_n, i_{n+1})$

$$H(gi_n, gi_{n+1}) < h \max(H(i_n, gi_n), H(i_{n+1}, gi_{n+1}) H(i_n, i_{n+1}), H(i_{n+1}, i_{n+2}),) \tag{77}$$

$$H(i_{n+1}, i_{n+2}) < h \max((i_n, i_{n+1}), H(i_{n+1}, i_{n+2}), H(i_n, i_{n+1})) \tag{78}$$

$$s_{n+1} < h \max(s_n, s_{n+1}, s_n) \tag{79}$$

$$s_{n+1} < h \max(s_n) \tag{80}$$

$$s_{n+1} < h(s_n) \tag{81}$$

$$\forall \; 0 < h < 1 \tag{82}$$

Therefore, $\{s_n\}$ is a monotone decreasing sequence of nonnegative real number.
Observe that g has at most one fixed point if $(u, v) \in X \times Y$ are two fixed points of g with $u \neq v$ i.e

$(u,v) > 0,\ gu = u\ and gv = v$
then from equation(75) we have

$$H(u,v) < h \max(H(u,gu), H(v,gv), H(u,v)) \tag{83}$$
$$H(u,v) < h \max(H(u,v)) \tag{84}$$
$$H(u,v) < h(H(u,v)) \tag{85}$$

which is a contradiction hence, $u = v$.
Next, let $(f,\alpha) \in F \times [0,\infty)$ be such that (H3) is satisfied let $\epsilon > 0$ be fixed. By $(F2)$, there exists $\delta > 0$ such that

$$0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha \tag{86}$$

let $i_0 \in Y$ be an arbitrary element, let $\{i_n\} \in Y$ be sequence defined by $i_{n+1} = gi_n\ n \in \mathbb{N}$

$$H(gi_n, gi_{n+1}) < h \max(H(i_n, gi_n), H(i_{n+1}, gi_{n+1})H(i_n, i_{n+1}), H(i_{n+1}, i_{n+2}),) \tag{87}$$
$$H(i_{n+1}, i_{n+2}) < h \max((i_n, i_{n+1}), H(i_{n+1}, i_{n+2}), H(i_n, i_{n+1})) \tag{88}$$

By (81) we have
$H(i_{n+1}, i_{n+2}) < H(i_n, i_{n+1})$
therefore

$$H(i_{n+1}, i_{n+2}) < hH(i_n, i_{n+1}) \tag{89}$$
$$= hH(gi_{n-1}, gi_n) \tag{90}$$
$$\leq h^2 H(i_{n-1}, i_n) \tag{91}$$
$$= h^2 H(gi_{n-2}, gi_{n-}) \tag{92}$$
$$\leq h^3 H(i_{n-2}, i_{n-1}) \tag{93}$$
$$\tag{94}$$

consequently, by induction for all $n \in \mathbb{N}$, we have
$H(i_{n+1}, i_{n+2}) < h^n H(i_0, i_1)$ which yield

$$\sum_{i=n}^{m-1} H(i_i, i_{i+1}) \leq \frac{h^n}{1-h} H(i_0, i_1)\ m > n \tag{95}$$

since $\lim\limits_{n\to\infty} \frac{h^n}{1-h} H(i_0, i_1) = 0$
there exists some $N \in \mathbb{N}$ such that

$$0 < \frac{h^n}{1-h} H(i_0, i_1) < \delta,\ n \geq \mathbb{N} \tag{96}$$

Hence, by $0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha$ and $(f1)$, we have

$$f\left(\sum_{i=n}^{m-1} H(i_i, i_{i+1})\right) \leq f\left(\frac{q^n}{1-q}(H(i_0, i_1))\right) < f(\epsilon) - \alpha \tag{97}$$

using (H3) and (97), we obtain

$$H(i_n, i_m) > 0,\ m > n \geq N \Rightarrow f(H(i_n, i_m)) < f\left(\sum_{i=n}^{m-1} H(i_i, i_{i+1})\right) + \alpha < f(\epsilon) \tag{98}$$

which implies $H(i_n, i_m) < \epsilon,\ m, n > N$
Therefore the $\{i_n\}$ if F-Cauchy, since (j, H) is F-complete there exists $i^* \in Y$ such that $\{i_n\}$ is F-convergent to i^* i.e

$$\lim_{n\to\infty} H(i_n, i^*) = 0 \tag{99}$$

which shows that $\{i_n\}$ is F-Cauchy.
Hence, F-completeness of (j, H) implies that there exists $i^* \in Y$ such that $i_n \to i^*$ as we shall proof that i^* is a fixed point of g. We argue by contradiction by supposing that $H(gi^*, i^*) > 0$. By (H3) we have

$$f(H(gi^*, i^*)) \leq f(H(gi^*, gi_n)) + f(H(gi_n, i^*)) + \alpha \tag{100}$$

using(75) and $(f1)$, we have.

$$f(H(gi^*, i^*)) \leq f(h \max[H(i^*, i_n) + H(i_n, gi^*)], [H(i^*, i_{n+1}) + H(i^*, i_{n+1})], \tag{101}$$

$$[H(i^*, i_{n+1}) + H(i_n, i^*)]]) + \alpha \tag{102}$$

on the other hand using $(f2)$ and (99), we have

$$\lim_{n\to\infty} f(H(gi^*, i^*)) \leq f(h \max[H(i^*, i_n) + H(i_n, gi^*)], [H(i^*, i_{n+1}) + H(i^*, i_{n+1})], \tag{103}$$

$$[H(i^*, i_{n+1}) + H(i_n, i^*)]]) + \alpha = -\infty \tag{104}$$

which is a contradiction, therefore, we have
$H(gi^*, i^*) = 0$ i.e $gi^* = i^*$. As a consequence, $i^* \in Y$ is the unique fixed point of g.

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