

## New Fixed point results in $F$ -metric space

ABSTRACT. In this research, existence and uniqueness of a fixed point were studied, considering some contractive conditions in the context of  $F$ -metric spaces. The obtained results generalize and extend some known results in the literature.

**Keywords and phrases:**  $F$ -metric space, contraction, Existence and uniqueness,  $F$ -Cauchy.

2020 Mathematical Subject Classification: 32H50

### 1. Introduction

Fixed point theory is one of the traditional theory in functional and nonlinear analysis. After the invention of metric spaces by [1], many mathematicians have generalized the metric structure by making some changes in the original definition of a metric given by Frechet. Most of the generalizations are made by making some changes in the triangle inequality of the original definition. Some well-known metrics of such generalizations are 2-metric space defined by [2] in the sixties, in a series of papers which he claimed to be a generalization of ordinary metric spaces. The structure of this space is defined as follows:

Let  $X$  be a nonempty set. A function  $d : X \times X \rightarrow \mathbb{R}$  is said to be a 2-metric on  $X$  if it satisfies the following properties:

- 1 For distinct points  $x, y \in X$ , there is a point  $z \in X$  such that  $d(x, y, z) \neq 0$ ,
- 2  $d(x, y, z) = 0$  if any two elements of the triplet  $(x, y, z)$  are equal,
- 3  $(x, y, z) = (x, z, y) \dots$  (symmetry),
- 4  $(x, y, z) \leq (x, y, a) + (x, a, z) + (a, y, z)$  for all  $x, y, z, a \in X$  (triangle inequality).

[3] defined a generalized metric space and related fixed point theorems, recently [4] introduced another such abstraction, which they denominate as  $F$ -metric. They defined this metric structure by means of a certain class  $F$ , which contains the set of functions.  $F$ -metric space as follows:

Let  $f \in F$  and  $f : (0, +\infty) \rightarrow \mathbb{R}$  be such that:

(F1)  $0 < x < y \implies f(x) \leq f(y)$ ; and

(F2) for  $\{x_n\} \subseteq \mathbb{R}^+$ ,  $\lim_{n \rightarrow \infty} x_n = 0 \leftrightarrow \lim_{n \rightarrow \infty} f\{x_n\} = -\infty$

**Definition 1** Let  $M$  be a nonempty set, and let  $d_F : M \times M \rightarrow [0, +\infty)$  be a given mapping. Suppose that there exists  $(f, \alpha) \in F \times [0, +\infty)$  such that:

1.  $(x, y) \in M \times M, D(x, y) = 0 \iff x = y$ ;
2.  $D(x, y) = D(x, y)$  for all  $(x, y) \in M \times M$ ; and

3. for every  $(x, y) \in M \times M, N \in \mathbb{N}, N \geq 2$ , and  $(x_i)_i^n = 1 \subset M$  with  $(x_1, x_N) = (x, y)$ , we get

$$D(x, y) > 0 \text{ implies } f(D(x, y)) \leq f\left(\sum_{i=1}^{N-1} D(x_i, y_{i+1})\right) + \alpha$$

Then  $D$  is an  $F$ -metric on  $M$ , and the pair  $(M, D)$  is said to be an  $F$ -metric space.

The origin of fixed point theory lies in the method of successive approximations used for proving existence of solutions of differential equations introduced independently by [5] and [6]. But formally it was started in the beginning of twentieth century as an important part of analysis. The abstraction of this classical theory is the pioneering work of the great Polish mathematician.

[7] As a generalization of normal metric spaces, b-metric spaces, JS-metric spaces, and primarily  $F$ -metric spaces, the idea of sequential  $F$ -metric spaces was presented. A few topological characteristics of these spaces were examined. They demonstrated fixed-point theorems for certain classes of contractive mappings over such spaces by taking this idea into account. Examples have been provided to support their fixed-point theorems and to check the correctness of the underlying space. Additionally, a system of linear algebraic equations is solved by applying their fixed-point theorem.

[8], presented a new class of Picard operators for such mappings in the framework of  $F$ -metric space and examined a fixed point problem associated with specific contraction mappings, producing some intriguing and original findings. They examined the Hyers-Ulam stability of an integral equation, a Cauchy functional equation, and a fixed point problem as applications of their findings. They also demonstrated that the integral equation and fixed point issue are well-posed. Their novel conclusions were also supported by a few illustrated situations for more work on  $F$ -metric space (see. [9], [10], [11], [12]). This paper, considered the existence and uniqueness of a fixed point using some contractive conditions in  $F$ -metric space.

## 2. Preliminary

In this section, we will look at definitions, examples, lemmas, propositions and properties that are important in generating our main results.

[4] made use of a certain class of auxiliary functions to coin the idea of  $F$ -metric spaces. We begin with the collection of such functions.

Let  $f \in F$  and  $f : (0, +\infty) \rightarrow \mathbb{R}$  be such that:

(F1)  $0 < x < y \implies f(x) \leq f(y)$ ; and

(F2) for  $\{x_n\} \subseteq R^+, \lim_{n \rightarrow \infty} x_n = 0 \leftrightarrow \lim_{n \rightarrow \infty} f\{x_n\} = -\infty$

**Example 2.1** [4] The following are some examples of the previously discussed kind of auxiliary functions:

- i  $-\frac{1}{t}$  where  $t \in (0, \infty)$ ;
- ii  $-\exp^{\frac{1}{t}}$  for all  $t \in (0, \infty)$ .

Utilizing such functions, the authors generalized the concept of usual metric spaces and originated the notion of  $F$ -metric spaces as follows:

**Definition 2.1** [4] Let  $M$  be a nonempty set, and let  $d_F : M \times M \rightarrow [0, +\infty)$  be a given mapping. Suppose that there exists  $(f, \alpha) \in F \times [0, +\infty)$  such that:

- D1  $(x, y) \in M \times M, D(x, y) = 0 \Leftrightarrow x = y$ ;
- D2  $D(x, y) = D(y, x)$  for all  $(x, y) \in M \times M$ ; and
- D3 for every  $(x, y) \in M \times M, N \in \mathbb{N}, N \geq 2$ , and  $(x_i)_i^n = 1 \subset M$  with  $(x_1, x_N) = (x, y)$ , we get

$$D(x, y) > 0 \text{ implies } f(D(x, y)) \leq f\left(\sum_{i=1}^{N-1} D(x_i, y_{i+1})\right) + \alpha$$

Then  $D$  is an  $F$ -metric on  $M$ , and the pair  $(M, D)$  is said to be an  $F$ -metric space.

It is observed that any metric on  $X$  is an  $F$ -metric, but the converse is not true.

**Proposition 2.1**. [4]

Let  $(X, D)$  be an  $F$ -metric space. Let  $\{x_n\}$  be a sequence in  $X$ , and  $x \in X$ . The following statements are equivalent:

- i  $\{x_n\}$  is  $F$ -convergent to  $x$ ,
- ii  $\lim_{x_n \rightarrow \infty} D(x_n, x) = 0$

The next result shows that the limit of an  $F$ -convergent sequence is unique.

**Proposition 2.2** [4]

Let  $(X, D)$  be an  $F$ -metric space. Let  $\{x_n\}$  be a sequence in  $X$ .

Then

$$(x, y) \in X \times X, \lim_{x_n \rightarrow \infty} D(x_n, x) = \lim_{x_n \rightarrow \infty} D(x_n, y) = 0 \Rightarrow x = y$$

**Definition 2.2** [4]

Let  $(X, D)$  be an  $F$ -metric space. Let  $\{x_n\}$  be a sequence in  $X$ .

- i We say that  $\{x_n\}$  is  $F$ -convergent, if

$$\lim_{x_n \rightarrow \infty} D(x_n, x) = 0$$

- ii We say that  $x_n$  is  $F$ -Cauchy, if

$$\lim_{x_n \rightarrow \infty} D(x_n, x_m) = 0$$

- iii We say that  $(X, D)$  is  $F$ -complete, if every  $F$ -Cauchy sequence in  $X$  is  $F$ -convergent to a certain element in  $X$ .

**Proposition 2.3** [4]

Let  $(X, D)$  be an  $F$ -metric space. If  $\{x_n\} \subset X$  is  $F$ -convergent, then it is  $F$ -Cauchy.

### 3. Main Result

#### Theorem 3.1

Let  $(X, D)$  be an  $F$ -metric space and  $g : X \rightarrow X$  be a continuous  $p$ -contraction mapping. Then  $g$  has a unique fixed point in  $X$

$$D(gx, gy) \leq k[D(x, y) + |D(x, gx) - D(y, gy)|] \quad (1)$$

#### Proof

Let  $x_0 \in X$  be an arbitrary point and define a sequence  $\{x_n\}$  by  $x_{n+1} = gx_n$  for all  $n \in \mathbb{N}$ . If there exists  $n_0 \in \mathbb{N}$  such that  $x_{n_0} = x_{n_0+1}$  is a fixed point of  $g$ . Suppose  $x_n \neq x_{n+1}$  for all  $n \in \mathbb{N}$  hence  $D(x_n, x_{n+1}) > 0$

Now we have.

$$D(x_{n+1}, x_{n+2}) = D(gx_n, gx_{n+1}) \quad (2)$$

$$\leq k[D(x_n, x_{n+1}) + |D(x_n, gx_n) - D(x_{n+1}, gx_{n+1})|] \quad (3)$$

$$= k[D(x_n, x_{n+1}) + |D(x_n, x_{n+1}) - D(x_{n+1}, x_{n+2})|] \quad (4)$$

for all  $n \in \mathbb{N}$  if  $D(x_n, x_{n+1}) \geq D(x_{n+1}, x_{n+2})$  for some  $n$  then from (1)

$$D(x_{n+1}, x_{n+2}) \leq k[D(x_n, x_{n+1}) + |D(x_n, x_{n+1}) - D(x_{n+1}, x_{n+2})|] \quad (5)$$

$$\leq k[D(x_n, x_{n+1}) + D(x_n, x_{n+1}) - D(x_{n+1}, x_{n+2})] \quad (6)$$

$$= 2kD(x_n, x_{n+1}) - kD(x_{n+1}, x_{n+2}) \quad (7)$$

$$D(x_{n+1}, x_{n+2}) + kD(x_{n+1}, x_{n+2}) \leq 2kD(x_n, x_{n+1}) \quad (8)$$

$$(1 + k)D(x_{n+1}, x_{n+2}) \leq 2kD(x_n, x_{n+1}) \quad (9)$$

$$D(x_{n+1}, x_{n+2}) \leq \frac{2k}{1+k}D(x_n, x_{n+1}) \quad (10)$$

for all  $n \in \mathbb{N}$

Let  $\frac{2k}{1+k} = \lambda$ , then  $0 < \lambda < 1$  and so we have

$$D(x_{n+1}, x_{n+2}) \leq \lambda D(x_n, x_{n+1}) \quad (11)$$

$$\leq \lambda^2 D(x_{n-1}, x_n) \quad (12)$$

$$\leq \lambda^3 D(x_{n-3}, x_{n-1}) \quad (13)$$

$$\cdot \quad (14)$$

$$\cdot \quad (15)$$

$$\cdot \quad (16)$$

$$\leq \lambda^n D(x_1, x_0) \quad (17)$$

for all value of  $n \in \mathbb{N}$

We have for  $m, n \in \mathbb{N}$  with  $m > n$ ,

$$\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \lambda^i D(x_0, x_1) \quad (18)$$

$$\leq \frac{\lambda^n}{1-\lambda} D(x_0, x_1) \quad (19)$$

since  $0 < \lambda < 1$  for all  $\delta > 0$ , there exists  $n_0 \in \mathbb{N}$  such that for all  $n > n_0$

$$0 < \frac{\lambda^n}{1-\lambda} D(x_0, x_1) < \delta \quad (20)$$

Now, let  $(f, \alpha) \in F \times [0, \infty)$  be such that (D3) is satisfied. let  $\epsilon > 0$  be fixed, then by (f2) there exists  $\eta > 0$  such that

$$0 < t < \eta \Rightarrow f(t) < f(\epsilon) - \alpha \quad (21)$$

considering  $\delta$  as  $\eta$  we get

$$f\left(\frac{\lambda^n}{1-\lambda} D(x_0, x_1)\right) < f(\epsilon) - \alpha \quad (22)$$

By (f1) we have

$$f\left(\sum_{i=1}^{N-1} D(x_i, x_{i+1})\right) \leq f\left(\frac{\lambda^n}{1-\lambda} D(x_0, x_1)\right) < f(\epsilon) - \alpha \quad (23)$$

for all  $m, n \in \mathbb{N}$  with  $m > n \geq n_0$ .

Using (D3) and equation (23) we get for  $m > n \geq n_0$

$$D(x_n, x_m) > 0 \Rightarrow f(D(x_n, x_m)) \leq f\left(\sum_{i=1}^{m-1} D(x_i, x_{i+1})\right) + \alpha < f(\epsilon) \quad (24)$$

which implies by (f1) that  $D(x_n, x_m) < \epsilon$  for  $m > n \geq n_0$ . Therefore the sequence  $\{x_n\}$  is  $F$ -Cauchy.

Since  $D(x, y)$  is  $F$ -complete there exists  $z \in X$  such that  $\{x_n\}$  is  $F$ -convergent to  $z$ , that is  $\lim_{n \rightarrow \infty} D(x_{n+1}, z) = 0$

Since  $g$  is continuous, and the  $\lim_{n \rightarrow \infty} (gx_n, gz) = \lim_{n \rightarrow \infty} D(x_{n+1}, gz) = 0$

The uniqueness of the limit we have  $z = gz$ .

Now suppose  $w$  is another fixed point of  $g$ , then  $gw = w$  and  $D(z, w) > 0$ .

Hence

$$D(z, w) = D(gz, gw) \quad (25)$$

$$\leq k[D(z, w) + |D(z, gz) - D(w, gw)|] \quad (26)$$

$$= kD(z, w) \quad (27)$$

which is a contradiction, hence the fixed point of  $g$  is unique.

### Theorem 3.2

Let  $g$  be a self mapping on an  $F$ -metric space  $D(X, D)$ . Suppose there exists  $x_0 \in X$  such that

$$\phi(D(gx, gy)) < \frac{1}{2}\{\phi(D(x, gx)) + \phi(D(y, gy))\} \quad (28)$$

holds for every  $x, y \in X$  with  $x \neq y$ , then  $g$  has a unique fixed point  $x^*$  in  $g$ .

#### Proof

Let the sequence  $\{x_n\}$  be defined as  $x_{n+1} = gx_n$  for  $n \in \mathbb{N}$ . If  $x_n = x_{n+1}$  for

some  $n$ , then  $g$  has a fixed point. So, let  $x_n \neq x_{n+1}$  for every  $n \in \mathbb{N}$ .  
 Let  $\alpha_n = \phi(D(x_n, x_{n+1}))$  for all  $n \in \mathbb{N}$ .

Therefore, it follows that

$$\alpha_{n+1} = \phi(D(x_{n+1}, x_{n+2})) \quad (29)$$

$$= \phi(D(gx_n, gx_{n+1})) \quad (30)$$

$$\leq \frac{1}{2} \{ \phi(D(x_n, gx_n)) + \phi(D(x_{n+1}, gx_{n+1})) \} \quad (31)$$

$$= \frac{1}{2} \{ \phi(D(x_n, x_{n+1})) + \phi(D(x_{n+1}, x_{n+2})) \} \quad (32)$$

$$= \frac{1}{2} \{ \alpha_n + \alpha_{n+1} \} \quad (33)$$

$$\alpha_{n+1} < \frac{1}{2} \alpha_n + \frac{1}{2} \alpha_{n+1} \quad (34)$$

$$\alpha_{n+1} < \alpha_n \quad (35)$$

Hence,  $\{\alpha_n\}$  is a strictly decreasing sequence of positive reals and hence converges to some nonnegative real number  $a$ .

Now, we claim that  $a = 0$ . If  $a \neq 0$ , then we have

$$0 < a = \lim_{n \rightarrow \infty} \phi(D(x_n, x_{n+1})) = \phi(D(x^*, gx^*)) \quad (36)$$

which is a contradiction, hence  $a = 0$ .

Thus, the sequence  $\{\alpha_n\}$  converges to zero.

Let there exists  $(f, \alpha) \in F \times [0, \infty)$  satisfying the conditions (D1 – D3) of definition 2.1. Then, by (f2), for a given  $\epsilon > 0$ , there exists a  $\delta > 0$  such that.

$$0 < t < \delta \Rightarrow f(t) < f(\phi(\epsilon)) - \alpha \quad (37)$$

Now,

$$\phi(D(x_n, x_{n+1})) < \frac{1}{2} \{ \phi(D(x_{n-1}, gx_{n-1})) + \phi(D(x_n, gx_n)) \} \quad (38)$$

$$= \frac{1}{2} \{ \phi(D(x_{n-1}, x_n)) + \phi(D(x_n, x_{n+1})) \} \quad (39)$$

$$= \frac{1}{2} \{ \phi(D(g^{n-1}x_0, g^n x_0)) + \phi(D(g^n x_0, g^{n+1} x_0)) \} \quad (40)$$

Similarly, we obtain,

$$\sum_{i=n}^{m-1} \phi(D(x_i, x_{i+1})) < \sum_{i=n}^{m-1} \frac{1}{2} \{ \phi(D(g^{i-1}x_0, g^i x_0)) + \phi(D(g^i x_0, g^{i+1} x_0)) \} \quad (41)$$

Since

$$\lim_{n \rightarrow \infty} \{ \phi(D(g^{n-1}x_0, g^n x_0)) + \phi(D(g^n x_0, g^{n+1} x_0)) \} = 0 \quad (42)$$

There exists some  $N \in \mathbb{N}$  such that

$$0 < \sum_{i=n}^{m-1} \phi(D(x_i, x_{i+1})) < \delta$$

holds for all  $n \geq N$ . Hence by (37) and (f1) we have

$$f \left( \sum_{i=n}^{m-1} \phi(D(x_i, x_{i+1})) \right) < f(\phi(\epsilon)) - \alpha \quad (43)$$

Now, we show that

$$D(x_n, x_m) < \epsilon \quad (44)$$

for all  $m > n \geq \mathbb{N}$ . Let  $m, n \in \mathbb{N}$  be fixed but arbitrary such that  $m > n \geq \mathbb{N}$ . If  $D(x_n, x_m) = 0$ , then clearly  $D(x_n, x_m) < \epsilon$  and if  $D(x_n, x_m) > 0$ , then using (D3) and (43), we have

$$D(x_n, x_m) > 0 \quad (45)$$

$$\Rightarrow f(\phi(D(x_n, x_m))) \leq f \left( \sum_{i=n}^{m-1} \phi(D(x_i, x_{i+1})) \right) + \alpha < f(\phi(\epsilon)) \quad (46)$$

which gives by (f1) that

$$(\phi(D(x_n, x_m))) < \phi(\epsilon) \quad (47)$$

$$\Rightarrow D(x_n, x_m) < \epsilon \quad (48)$$

This proves that  $\{x_n\}$  is  $F$ -Cauchy. Since  $\{x_n\}$  converges to  $x_0$ , then the limit of  $\{x_n\}$  will be  $x^*$ . This implies.

$$\lim_{n \rightarrow \infty} D(x_n, x^*) = 0. \quad (49)$$

Since  $gx_n = x_{n+1}$ , we have, by the uniqueness of limit of sequence  $x^* = gx^*$ . Hence  $x^*$  is a fixed point of  $g$ . For uniqueness, let  $y^*$  be another fixed point of  $g$ . Then

$$\phi(D(y, Y^*)) = \phi(D(gy, gy^*)) \quad (50)$$

$$\leq \frac{1}{2} \{ \phi(D(y, gy)) + \phi(D(y^*, gy^*)) \} \quad (51)$$

$$< 0 \quad (52)$$

a contradiction. Therefore  $x^*$  is the unique fixed point of  $g$ .

**Theorem 3.3**

Let  $(X, D)$  be an  $F$ -metric space and let  $g : X \rightarrow X$  be a given mapping. Suppose that the following conditions are satisfied

- i  $(X, D)$  is  $F$ - complete,
- ii there exists monotonically decreasing functions  $a, b, c$  from  $(0, \infty)$  into  $[0, 1)$  satisfying  $a(D(x, y)) + b(D(x, y)) + c(D(x, y)) < 1$  such that, for each  $x, y \in X, x \neq y$

$$D(gx, gy) \leq a(D(x, y))D(x, gx) + b(D(x, y))D(y, gy) + c(D(x, y))D(x, y) \quad (53)$$

for all  $(x, y) \in X \times X$

then  $g$  has a unique fixed point  $x^* \in X$ . Moreover for any  $x_0 \in X$ , the  $\{x_n\} \in X$  defined by  $x_{n+1} = gx_n$   $n \in \mathbb{N}$  is  $F$ -convergent to  $x^*$ .

**Proof**

By observation  $g$  has at most one fixed point. Indeed, if  $(u, v) \in X \times X$  are two fixed point of  $g$  with  $u \neq v$  i.e

$$D(u, v) > 0 \quad gu = u \quad \text{and} \quad gv = v$$

Then from (ii) we have

$$D(u, v) = D(gu, gv) \leq a(D(u, u))D(u, g) + b(D(u, v))D(v, gv) + c(D(u, v))D(u, v) \tag{54}$$

$$\leq c(D(u, v))D(u, v) \tag{55}$$

which is a contradiction.

Hence,  $u = v$

Next, let  $(f, \alpha) \in F \times [0, \infty)$  be such that (D3) is satisfied, let  $\epsilon > 0$  be fixed.

$$0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha$$

Let  $x_0 \in X$  be an arbitrary element, let  $\{x_n\} \in X$  be sequence defined by  $x_{n+1} = gx_n$  for  $n \in \mathbb{N}$

$$D(gx_n, gx_{n+1}) \leq a(D(x, y))D(x_n, gx_n) + b(D(x, y))D(x_{n+1}, gx_{n+1}) \tag{56}$$

$$+ c(D(x, y))D(x_n, x_{n+1})D(x_{n+1}, x_{n+2}) \leq a(D(x, y))D(x_n, x_{n+1}) \tag{57}$$

$$+ b(D(x, y))D(x_{n+1}, x_{n+2}) + c(D(x, y))D(x_n, x_{n+1}) \tag{58}$$

$$D(x_{n+1}, x_{n+2}) - b(D(x, y))D(x_{n+1}, x_{n+2}) \leq [a(D(x, y)) + c(D(x, y))]D(x_n, x_{n+1}) \tag{59}$$

$$[1 - b(D(x, y))]D(x_{n+1}, x_{n+2}) \leq [a(D(x, y)) + c(D(x, y))]D(x_n, x_{n+1}) \tag{60}$$

$$D(x_{n+1}, x_{n+2}) \leq \frac{[a(D(x, y)) + c(D(x, y))]}{[1 - b(D(x, y))]}D(x_n, x_{n+1}) \tag{61}$$

$$\text{let } q = \frac{[a(D(x, y)) + c(D(x, y))]}{[1 - b(D(x, y))]} \tag{62}$$

then,  $0 < q < 1$ , since  $a(D(x, y)) + b(D(x, y)) + c(D(x, y)) < 1$

Hence,

$$D(x_{n+1}, x_{n+2}) \leq qD(x_n, x_{n+1}) \tag{63}$$

Continue the above argument iteratively we will have

$$D(x_{n+1}, x_{n+2}) \leq \frac{q^n}{1 - q}D(x_0, x_1) \tag{64}$$

which yields

$$\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \leq \frac{q^n}{1 - q}D(x_0, x_1) \quad m > n \tag{65}$$

since  $\lim_{n \rightarrow \infty} \frac{q^n}{1 - q}D(x_0, x_1) = 0$

there exists some  $N \in \mathbb{N}$  such that

$$0 < \frac{q^n}{1 - q}D(x_0, x_1) < \delta, \quad n \geq N \tag{66}$$

Hence, by  $0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha$  and (f1), we have

$$f\left(\sum_{i=n}^{m-1} D(x_i, x_{i+1})\right) \leq f\left(\frac{q^n}{1-q}(D(x_0, x_1))\right) < f(\epsilon) - \alpha \quad (67)$$

using (D3) and (67), we obtain

$$D(x_n, x_m) > 0, \quad m > n \geq N \Rightarrow f(D(x_n, x_m)) < f\left(\sum_{i=n}^{m-1} D(x_i, x_{i+1})\right) + \alpha < f(\epsilon) \quad (68)$$

which implies  $D(x_n, x_m) < \epsilon$ ,  $m, n > N$

Therefore the  $\{x_n\}$  is  $F$ -Cauchy, since  $(X, D)$  is  $F$ -complete there exists  $x^* \in X$  such that  $\{x_n\}$  is  $F$ -convergent to  $x^*$  i.e

$$\lim_{n \rightarrow \infty} D(x_n, x^*) = 0 \quad (69)$$

we shall show that  $x^*$  is a fixed point of  $g$ . We argue by contradiction by supposing that  $D(gx^*, x^*) > 0$ . By (D3), we obtain

$$f(D(gx^*, x^*)) \leq f(D(gx^*, gx_n)) + f(D(gx_n, x^*)) + \alpha \quad (70)$$

using (ii) and (f1), we obtain

$$f(D(gx^*, x^*)) \leq f(a(D(x, y))[D(x^*, x_n) + D(x_n, gx^*)] + b(D(x, y))[D(x^*, x_n) \quad (71)$$

$$+ D(x_n, gx^*)] + c(d(x, y))[D(x^*, x_n) + D(x_n, x^*)]) + \alpha \quad (72)$$

on the other hand, using (f2) and  $\lim_{n \rightarrow \infty} D(x_n, x^*) = 0$  we have

$$\lim_{n \rightarrow \infty} f(a(D(x, y))[D(x^*, x_n) + D(x_n, gx^*)] + b(D(x, y))[D(x^*, x_n) + D(x_n, gx^*)] \quad (73)$$

$$+ c(d(x, y))[D(x^*, x_n) + D(x_n, x^*)]) + \alpha = -\infty \quad (74)$$

which is a contradiction, therefore, we have

$D(gx^*, x^*) = 0$  i.e  $gx^* = x^*$ . As a consequence,  $x^* \in X$  is the unique fixed point of  $g$ .

### Theorem 3.4

Let  $(X, D)$  be an  $F$ -metric space and let  $g : X \rightarrow X$  be a given mapping. Suppose that the following conditions are satisfied.

i  $(X, D)$  is  $F$ -complete,

ii for each  $x, y \in X$   $x \neq y$  such that

$$D(gx, gy) < h \max(D(x, gx), D(y, gy), D(x, y)) \quad (75)$$

Then  $g$  has a unique fixed point  $x^* \in X$ .

**Proof**

Let  $x_0 \in X$  be arbitrary but fixed and let  $\{x_n\} \ n \geq 0$  be the Picard sequence of  $g$  based on  $x_0$ , that is

$$x_{n+1} = gx_n \text{ for all } n \geq 0 \quad (76)$$

If  $x_n = x_{n+1}$  for some  $n$ , then it is easily noticeable that  $x_n$  is a fixed point of  $g$

Let  $x_n \neq x_{n+1}$  for all  $n \in \mathbb{N}$ .

Putting  $x = x_n, y = x_{n+1}$  in (75) and define sequence of a real number as  $s_n = D(x_n, x_{n+1})$

$$D(gx_n, gx_{n+1}) < h \max(D(x_n, gx_n), D(x_{n+1}, gx_{n+1}), D(x_n, x_{n+1}), D(x_{n+1}, x_{n+2}), ) \quad (77)$$

$$D(x_{n+1}, x_{n+2}) < h \max(D(x_n, x_{n+1}), D(x_{n+1}, x_{n+2}), D(x_n, x_{n+1})) \quad (78)$$

$$s_{n+1} < h \max(s_n, s_{n+1}, s_n) \quad (79)$$

$$s_{n+1} < h \max(s_n) \quad (80)$$

$$s_{n+1} < h(s_n) \quad (81)$$

$$\forall \ 0 < h < 1 \quad (82)$$

Therefore,  $\{s_n\}$  is a monotone decreasing sequence of nonnegative real number. Observe that  $g$  has at most one fixed point if  $(u, v) \in X \times X$  are two fixed points of  $g$  with  $u \neq v$  i.e

$$(u, v) > 0, \quad gu = u \text{ and } gv = v$$

then from equation(75) we have

$$D(u, v) < h \max(D(u, gu), D(v, gv), D(u, v)) \quad (83)$$

$$D(u, v) < h \max(D(u, v)) \quad (84)$$

$$D(u, v) < h(D(u, v)) \quad (85)$$

which is a contradiction hence,  $u = v$ .

Next, let  $(f, \alpha) \in F \times [0, \infty)$  be such that (D3) is satisfied let  $\epsilon > 0$  be fixed. By (F2), there exists  $\delta > 0$  such that

$$0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha \quad (86)$$

let  $x_0 \in X$  be an arbitrary element, let  $\{x_n\} \in X$  be sequence defined by  $x_{n+1} = gx_n \ n \in \mathbb{N}$

$$D(gx_n, gx_{n+1}) < h \max(D(x_n, gx_n), D(x_{n+1}, gx_{n+1}), D(x_n, x_{n+1}), D(x_{n+1}, x_{n+2}), ) \quad (87)$$

$$D(x_{n+1}, x_{n+2}) < h \max(D(x_n, x_{n+1}), D(x_{n+1}, x_{n+2}), D(x_n, x_{n+1})) \quad (88)$$

By (81) we have

$$D(x_{n+1}, x_{n+2}) < D(x_n, x_{n+1})$$

therefore

$$D(x_{n+1}, x_{n+2}) < hD(x_n, x_{n+1}) \quad (89)$$

$$= hD(gx_{n-1}, gx_n) \quad (90)$$

$$\leq h^2 D(x_{n-1}, x_n) \quad (91)$$

$$= h^2 D(gx_{n-2}, gx_{n-1}) \quad (92)$$

$$\leq h^3 D(x_{n-2}, x_{n-1}) \quad (93)$$

$$(94)$$

consequently, by induction for all  $n \in \mathbb{N}$ , we have  $D(x_{n+1}, x_{n+2}) < h^n D(x_0, x_1)$  which yield

$$\sum_{i=n}^{m-1} D(x_i, x_{i+1}) \leq \frac{h^n}{1-h} D(x_0, x_1) \quad m > n \quad (95)$$

since  $\lim_{n \rightarrow \infty} \frac{h^n}{1-h} D(x_0, x_1) = 0$

there exists some  $N \in \mathbb{N}$  such that

$$0 < \frac{h^n}{1-h} D(x_0, x_1) < \delta, \quad n \geq N \quad (96)$$

Hence, by  $0 < t < \delta \Rightarrow f(t) < f(\epsilon) - \alpha$  and (f1), we have

$$f\left(\sum_{i=n}^{m-1} D(x_i, x_{i+1})\right) \leq f\left(\frac{q^n}{1-q}(D(x_0, x_1))\right) < f(\epsilon) - \alpha \quad (97)$$

using (D3) and (97), we obtain

$$D(x_n, x_m) > 0, \quad m > n \geq N \Rightarrow f(D(x_n, x_m)) < f\left(\sum_{i=n}^{m-1} D(x_i, x_{i+1})\right) + \alpha < f(\epsilon) \quad (98)$$

which implies  $D(x_n, x_m) < \epsilon$ ,  $m, n > N$

Therefore the  $\{x_n\}$  is  $F$ -Cauchy, since  $(X, D)$  is  $F$ -complete there exists  $x^* \in X$  such that  $\{x_n\}$  is  $F$ -convergent to  $x^*$  i.e

$$\lim_{n \rightarrow \infty} D(x_n, x^*) = 0 \quad (99)$$

which shows that  $\{x_n\}$  is  $F$ -Cauchy.

Hence,  $F$ -completeness of  $(X, D)$  implies that there exists  $x^* \in X$  such that  $x_n \rightarrow x^*$  as we shall proof that  $x^*$  is a fixed point of  $g$ . We argue by contradiction by supposing that  $D(gx^*, x^*) > 0$ . By (D3) we have

$$f(D(gx^*, x^*)) \leq f(D(gx^*, gx_n)) + f(D(gx_n, x^*)) + \alpha \quad (100)$$

using(75) and (f1), we have.

$$f(D(gx^*, x^*)) \leq f(h \max[D(x^*, x_n) + D(x_n, gx^*), [D(x^*, x_{n+1}) + D(x^*, x_{n+1})], \quad (101)$$

$$[D(x^*, x_{n+1}) + D(x_n, x^*)]) + \alpha \quad (102)$$

on the other hand using (f2) and (99), we have

$$\lim_{n \rightarrow \infty} f(D(gx^*, x^*)) \leq f(h \max[D(x^*, x_n) + D(x_n, gx^*)], [D(x^*, x_{n+1}) + D(x^*, x_{n+1})]), \quad (103)$$

$$[D(x^*, x_{n+1}) + D(x_n, x^*)] + \alpha = -\infty \quad (104)$$

which is a contradiction, therefore, we have

$D(gx^*, x^*) = 0$  i.e  $gx^* = x^*$ . As a consequence,  $x^* \in X$  is the unique fixed point of  $g$ .

## References

- [1] M. M. Fréchet, Sur quelques points du calcul fonctionnel, Rendiconti del Circolo Matematico di Palermo (1884-1940) 22 (1) (1906) 1–72.
- [2] S. Ghaher, 2-metrische raume und ihre topologische strukture, Math. Nachr 26 (1963) 115–148.
- [3] M. Jleli, B. Samet, A generalized metric space and related fixed point theorems, Fixed point theory and Applications 2015 (1) (2015) 1–14.
- [4] M. Jleli, B. Samet, On a new generalization of metric spaces, Journal of Fixed Point Theory and Applications 20 (3) (2018) 128.
- [5] J. Liouville, Second mémoire sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équation différentielle du second ordre, contenant un paramètre variable, Journal de mathématiques pures et appliquées 2 (1837) 16–35.
- [6] É. Picard, Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives, Journal de Mathématiques pures et appliquées 6 (1890) 145–210.
- [7] K. Roy, H. Alaeidizaji, M. Saha, B. Mohammadi, V. Parvaneh, Some fixed-point theorems over a generalized f-metric space, Advances in Mathematical Physics 2021 (2021) 1–7.
- [8] R. K. Sharma, S. Chandok, Well-posedness and ulam’s stability of functional equations in f-metric space with an application, Filomat 36 (16) (2022) 5573–5589.
- [9] M. Alansari, S. S. Mohammed, A. Azam, Fuzzy fixed point results in f-metric spaces with applications, Journal of Function Spaces 2020 (2020) 1–11.
- [10] H. Işık, N. Hussain, A. R. Khan, Endpoint results for weakly contractive mappings in  $\mathcal{F}$ -metric spaces with application, International Journal of Nonlinear Analysis and Applications 11 (2) (2020) 351–361.
- [11] L. A. Alnaser, D. Lateef, H. A. Fouad, J. Ahmad, Relation theoretic contraction results in f-metric spaces, J. Nonlinear Sci. Appl 12 (5) (2019) 337–344.

- [12] B. D. . A. C. Ashis. B., Hiranmoy. G., Fuzzy fixed point results in f-metric spaces with applications, Published by Faculty of Sciences and Mathematics, University of Niš, Serbia 2019 (2019) 54–63.