FIXED POINT RESULTS IN MODULAR S-METRIC SPACE

ABSTRACT. In this work, some definitions, properties and propositions in modular S-metric space were given, existence, uniqueness and coincidence point of a fixed point results in the space were also provided using some general contractive conditions defined in modular S-metric space settings. Our results shows that existence and uniqueness of $[L, \phi]$ contraction is possible and also the coincidence of a fixed point for a specific rational contraction can be determined which improves on some results in the literature.

Keywords and phrases: Modular S-metric space, coincidence point, Modular convergent, Modular Cauchy.
2020 Mathematical Subject Classification: 32H50

1. INTRODUCTION

The concept of distance space was first studied by Frechet [6], where he defined the gap between two points as a metric d on a non-empty set X satisfying some certain conditions. As a result of his work, many researchers have come up with several generalization of metric space, among them is Gahler [7] who introduced the notion of 2-metric space which is a generalization of ordinary metric space.
In [5] Dhage introduced a new structure of a generalized metric space called D-metric space; also in 2003, Mustafa and Sim [10] defined another generalization of metric space, and claimed that properties of D-metric space were wrong and as a result, they gave a new properties of a distance function called G-metric space. In 2012, Sedghi *et at*[14] introduced S-metric spaces and gave some properties. They proved the fixed point theorem for a self-mapping on a complete S-metric space.
[17] proved common fixed point results for monotone asymptotic pointwise ρ- nonexpansive semigroups in modular function spaces. Thus, utilizing the monotonicity of semigroups, there findings generalized and extend some prominent recent results of the literature in the setting of modular function spaces.
The idea of modular space was conceived by Nakano in [11] which conform with the theory of order spaces.
Musielak and Orlic [9] build on this idea of modular space by proving some fixed point theorems. Chistyakov in [3] introduced the notion of modular metric on an ordinary set and the corresponding

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modular space which is more general than a metric space.
In line with the concept of Chistyakov, many researchers have come up with several generalization of modular metric spaces, such as Azadifer *et al* in [1] who gave a generalization of modular metric space to develop a space called modular G-metric space. Several results of fixed point were gotten using contractive mappings defined on modular G-metric spaces.

[16] obtained common fixed point theorems for three mappings of contractive type in the setting of generalized modular metric spaces. There results generalized many results available in the literature including common fixed point theorems.
In the present paper, we introduce the concept of modular S-metric space, define modular S-metric space, prove some properties of modular S-metric space, existence and uniqueness as well as coincidence point of a modular S-metric spaces were determined.

2. PRELIMINARY

Here, we shall define the Modular S-metric space based on the concept of modular metric space defined in [3] and that of modular G-metric space define in [1].
Definition 2.1: Let X be a nonempty set, and $W^s_\lambda : (0,\infty) \times X \times X \times X \to [0,\infty]$ be a function satisfying:

(1) $W^s_\lambda(x,y,z) \geq 0$ for all $x,y,z \in X$ and $\lambda > 0$;
(2) $W^s_\lambda(x,y,z) = 0$ for all $x,y,z \in X$ and $\lambda > 0$ if and if $x = y = z$; and
(3) $W^s_{\lambda+\mu+\gamma}(x,y,z) \leq W^s_\lambda(x,x,a) + W^s_\mu(y,y,a) + W^s_\gamma(z,z,a)$ for all $x,y,z,a \in X$ and $\lambda,\mu,\gamma > 0$.
Then the function $w^s_\lambda(.,.)$ is called a modular S-metric space on X and the pair (X, W^s_λ) is called a modular S- metric space and $W^s_\lambda, W^s_\mu, W^s_\gamma$ are velocities with respect to time $\lambda, \mu \; and \; \gamma$ respectively.

Example 2.1: Let $X = [0,\infty)$ be the interval of nonnegative real numbers and let W^s_λ be defined by

$$W^s_\lambda(x,y,z) = \{^{0,\ if\ x=y=z.}_{max(x,y,z),\ otherwise}$$

Then W^s_λ is a complete W^s_λ-metric on X.

In this section, we introduce some definitions and notion of W^s_λ-convergent sequence and W^s_λ- Cauchy sequence.
Definition 2.2: A sequence$\{x_n\}$ in X is said to be modular S convergent to $x \in X$ if $W^s_\lambda(x_n,x_n,x) \to 0$ as $n \to \infty$, that is

for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $W^s_\lambda(x_n, x_n, x) < \epsilon$.. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x \; as \; n \to \infty$

Definition 2.3: A sequence$\{x_n\}$ in X is said to be modular S Cauchy sequence if $W^s_\lambda(x_n, x_n, x_m) \to 0$ as $n, m \to \infty$, that is for every $\epsilon > 0$, there exist $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$ we have $W^s_\lambda(x_n, x_n, x_m) < \epsilon$.

Definition 2.4: The modular S-metric space (X, W^s_λ) is called complete if every modular S-Cauchy sequence in it is a modular S-convergent sequence.

Definition 2.5: Let (X, W^s_λ) be a modular S-metric space,and $\{x_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ if $\lim_{n,m\to\infty}(x, x_n, x_n)$ and one say the sequence $\{x_n\}$ is a modular S-convergent to x. Thus, that if $x_n \to 0$ in a modular S-metric space (X, W^s_λ), then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $W^s_\lambda(x, x_n, x_m) < \epsilon$, for all $n, m, \in N$.

3. THE HEART OF THE MATTER

Definition 3.1 There exists a real number $L \geq 0$ and a strict comparison function $\phi : \mathbb{R} \to \mathbb{R}$ for all $x, y, z \in X$ such that

$$W^s_\lambda(Tx, Ty, Tz) \leq LW^s_\lambda(x, Tx, Tx) + \phi W^s_\lambda(x, y, z) \qquad (1)$$

where $\lambda > 0$ and $0 \leq L + \phi < 1$.

Definition 3.2 There exists a real number a, b, c, where $0 \leq a + b + c < 1$ such that for each $x, y, z \in X; \lambda > 0$

$$W^s_\lambda(Tx, Ty, Tz) \leq aW^s_\lambda(x, Tx, Tx) + W^s_\lambda(y, Ty, Ty) + cW^s_\lambda(x, y, z) \qquad (2)$$

Definition 3.3 There exists a real number α, β where $0 \leq \alpha + \beta < 1$ such that for each $x, y, z \in X; \lambda > 0$ we have

$$W^s_\lambda(Tx, Ty, Tz) \leq \alpha \frac{W^s_\lambda(x, Tx, Tx)W^s_\lambda(y, Ty, Ty)}{W^s_\lambda(x, y, z)} + \beta W^s_\lambda(x, y, z) \qquad (3)$$

Proposition 3.1: Let (X, W^s_λ) be a modular S-metric space then the following are equivalent:

(1) $\{x_n\}$ is w^s_λ-convergent to x for a function $F : X \to X$;
(2) $W^s_\lambda(x_n, x_n, x) = 0$; as $n \to \infty$;
(3) $W^s_\lambda(x_n, x, x) = 0$; as $n \to \infty$; and
(4) $W^s_\lambda(x_n, x_m, x) = 0$; as $n, m, \to \infty$.

Lemma 3.1: Let (X, W^s_λ) be a modular S-metric space. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ then $W^s_\lambda(x_n, x_n, y_n) \to W^s_\lambda(x, x, y)$ as $n \to \infty$

Lemma 3.2: Let T be a self mapping on a modular S-metric space (X, W^s_λ). Then T is said to be continuous at $x \in X$ if for any sequence $\{x_n\}$ in X, $x_n \to x$ implies $Tx_n \to Tx$ as $n \to \infty$.
Lemma 3.3: In a modular S-metric space, we have $W^s_\lambda(x, x, y) = W^s_\lambda(y, y, x)$ for all $x, y \in X$ $\lambda > 0$.

Proof. By the third condition of modular S-metric space, we have $W^s_\lambda(x, y, y) \leq W^s_\lambda(x, x, a) + W^s_\lambda(y, y, a) + W^s_\lambda(z, z, a)$

$$\begin{aligned} W^s_\lambda(x, x, y) &\leq W^s_\lambda(x, x, x) + W^s_\lambda(x, x, x) + W^s_\lambda(y, y, x) \\ &\leq W^s_\lambda(y, y, x) \\ W^s_\lambda(x, x, y) &\leq W^s_\lambda(y, y, x) \end{aligned}$$

Similarly

$$\begin{aligned} W^s_\lambda(y, y, x) &\leq W^s_\lambda(y, y, y) + W^s_\lambda(y, y, y) + W^s_\lambda(x, x, y) \\ &\leq W^s_\lambda(x, x, y) \\ W^s_\lambda(y, y, x) &\leq W^s_\lambda(x, x, y) \end{aligned}$$

Therefore, $W^s_\lambda(y, y, x) = W^s_\lambda(x, x, y)$ □

Proposition 3.2: The limit of a W^s_λ-convergent sequence in a W^s_λ-metric space is unique.

Proof. Let (X, W^s_λ) be a W^s_λ-metric space and let $\{x_n\} \subseteq X$ be a sequence that converges at the same time to $x \in X$ and to $y \in X$. We claim that $W^s_\lambda(x, y, y) < \epsilon$ for all $\epsilon > 0$. Let $\epsilon > 0$ be arbitrary, there exist natural numbers $n_1, n_2 \in \mathbb{N}$ such that

$$\begin{aligned} W^s_\lambda(x_n, x_m, x) &\leq \frac{\epsilon}{3} \ for\ all\ n, m \geq n_1 \\ W^s_\lambda(x_n, x_m, y) &\leq \frac{\epsilon}{3} \ for\ all\ n, m \geq n_2 \end{aligned}$$

Let $n_0 = max(n_1, n_2)$. Then by the third axioms of W^s_λ-metric space we have

$$\begin{aligned} W^s_\lambda(x, x, y) &\leq W^s_\lambda(x, x, x_n) + W^s_\lambda(x, x, x_n) + W^s_\lambda(y, y, x_n) \\ &\leq 2W^s_\lambda(x, x, x_n) + W^s_\lambda(y, y, x_n) \\ &= 2\frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Consequently, we have that $W^s_\lambda(x, x, y) = 0$ and by axiom two of W^s_λ-metric space, we conclude that $x = y$. □

Proposition 3.3: Every convergent sequence in a W^s_λ S-metric space is a Cauchy sequence.

Proof. Let (X, W^s_λ) be a W^s_λ S-metric space and let $\{x_n\} \subseteq X$ be a sequence that converges to $x \in X$. Let $\epsilon > 0$ be arbitrary. By definition, there exist $n_0 \in \mathbb{N}$ such that
$W^s_\lambda(x_n, x_n, x) \leq \frac{\epsilon}{3}$ for all $n, m \geq n_0$ By the third axiom of modular S-metric space we have that for all $n, m, k \geq n_0$

$$W^s_\lambda(x_n, x_m, x_k) \leq W^s_\lambda(x_n, x_n, x) + W^s_\lambda(x_m, x_m, x) + W^s_\lambda(x_k, x_k, x) \tag{4}$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \tag{5}$$

$$= \epsilon \tag{6}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in (X, W^s_λ) □

We state our main results as follows.

Theorem 3.1

Let W^s_λ be a complete S-metric space. Let $T : X \to X$ be a continuous mapping of modular S-contraction where there exists a real number $L \geq 0$ and a strict comparison function $\phi : \mathbb{R} \to \mathbb{R}$ for all $x, y, z \in X$ such that

$$W^s_\lambda(Tx, Ty, Tz) \leq LW^s_\lambda(x, Tx, Tx) + \phi W^s_\lambda(x, y, z) \tag{7}$$

where $\lambda > 0$ and $0 \leq L + \phi < 1$. Then T has a unique fixed point p in X.

Proof. If $x = x_n;\ y = z = y_n$
Let $x_n \neq x_{n+1}$ and $x_{n+1} = y_n$
then

$$W^s_\lambda(x_n, x_{n+1}, x_{n+1}) = W^s_\lambda(Tx_{n-1}, Tx_n, Tx_n) \tag{8}$$

Let $x_0 \in X$ be an arbitrary point in X and $x_{n+1} = Tx_n$ be a Picard iteration. Then for $n = 0,\ x_1 = Tx_0$.

$$W^s_\lambda(x_n, x_{n+1}, x_{n+1}) = W^s_\lambda(Tx_{n-1}, Tx_n, Tx_n) \tag{9}$$

if n=1

$$W^s_\lambda(x_n, x_{n+1}, x_{n+1}) \leq LW^s_\lambda(x_0, Tx_0, Tx_0) + \phi W^s_\lambda(x_0, Tx_0, Tx_0) \tag{10}$$

$$= LW^s_\lambda(x_0, x_1, x_1) + \phi W^s_\lambda(x_0, x_1, x_1) \tag{11}$$

$$\leq [L + \phi]W^s_\lambda(x_0, x_1, x_1) \tag{12}$$

for n=2

$$W_\lambda^s(x_2,x_3,x_3) \leq LW_\lambda^s(x_1,Tx_0,Tx_0) + \phi W_\lambda^s(x_0,Tx_0,Tx_0) \quad (13)$$
$$\leq [L+\phi]W_\lambda^s(x_1,x_2,x_2) \quad (14)$$
$$\leq [L+\phi]^2 W_\lambda^s(x_0,x_1,x_1) \quad (15)$$

for n=3

$$W_\lambda^s(x_3,x_4,x_4) \leq LW_\lambda^s(x_2,x_3,x_3) + \phi W_\lambda^s(x_2,x_3,x_3) \quad (16)$$
$$\leq [L+\phi]W_\lambda^s(x_2,x_3,x_3) \quad (17)$$
$$\leq [L+\phi]^3 W_\lambda^s(x_0,x_1,x_1) \quad (18)$$

iteratively to n we have;

$$W_\lambda^s(x_n,x_{n+1},x_{n+1}) \leq [L+\phi]^n W_\lambda^s(x_0,x_1,x_1) \quad (19)$$

Let $\beta = [L+\phi] < 1$
Hence;

$$W_\lambda^s(x_n,x_{n+1},x_{n+1}) \leq [L+\phi]^n W_\lambda^s(x_0,x_1,x_1) \quad (20)$$
$$= \beta^n W_\lambda^s(x_0,x_1,x_1) \quad (21)$$

which prove the convergence of the sequence $\{x_n\}$.
Next we show the sequence $\{x_n\}$ is a Cauchy sequence for any positive integer $m,n \in N$ and $N \in \mathbb{N}$ such that $m > n$.
Then,

$$W_\lambda^s(x_n,x_m,x_m) \leq W_\lambda^s(x_n,x_{n+1},x_{n+1}) + W_\lambda^s(x_{n+1},x_{n+2},x_{n+2}) + W_\lambda^s(x_{n+2},x_{n+3},x_{n+3}) \quad (22)$$
$$+ ... + W_\lambda^s(x_{m-1},x_m,x_m) \quad (23)$$
$$\leq \beta^n W_\lambda^s(x_0,x_1,x_1) + \beta^{n+1} W_\lambda^s(x_0,x_1,x_1) + \beta^{n+2} W_\lambda^s(x_0,x_1,x_1) + ... \quad (24)$$
$$\leq (\beta^n + \beta^{n+1} + \beta^{n+2} + ...)W_\lambda^s(x_0,x_1,x_1) \quad (25)$$
$$W_\lambda^s(x_n,x_m,x_m) \leq (\frac{\beta^n}{1-\beta})W_\lambda^s(x_0,x_1,x_1) \to 0 \ as \ n \to \infty \quad (26)$$

Therefore, the sequence $\{x_n\}$ is a W_λ^s Cauchy sequence. Then the sequence is a W_λ^s-complete.
If the limit of the sequence $\{x_n\}$ is defined as p then the contraction becomes. $W_\lambda^s(x_{n+1},Tp,Tp) \leq LW_\lambda^s(x_n,Tp,Tp) + \phi W_\lambda^s(x_n,Tp,Tp)$.
Taking the limit of the sequence as $n \to \infty$, and with the fact that

W_λ^s continuous by Lemma2.2. Then,

$$W_\lambda^s(x_{n+1}, Tp, Tp) \leq LW_\lambda^s(p, Tp, Tp) + \phi W_\lambda^s(p, Tp, Tp) \quad (27)$$
$$\leq \beta W_\lambda^s(p, Tp, Tp) \quad (28)$$

Since $0 \leq \beta < 1$ hence, $p = Tp$.
Suppose q is also a fixed point of T and $p \neq q$. Then the contraction becomes

$$W_\lambda^s(p, q, q) \leq LW_\lambda^s(p, q, q) + \phi W_\lambda^s(p, q, q) \quad (29)$$
$$W_\lambda^s(p, q, q) \leq \beta W_\lambda^s(p, q, q) < W_\lambda^s(p, q, q) \quad (30)$$

which is only possible if $p = q$ therefore, T has a unique fixed point p in X. □

Theorem 3.2
Let (X, W_λ^s) be a complete modular S-metric space and let $T : X \to X$ be a continuous modular S-contraction such that, there exists a real number a, b, c $0 \leq a + b + c < 1$ such that for each $x, y, z \in X;$ $\lambda > 0$

$$W_\lambda^s(Tx, Ty, Tz) \leq aW_\lambda^s(x, Tx, Tx) + bW_\lambda^s(y, Ty, Ty) + cW_\lambda^s(x, y, z) \quad (31)$$

Then T has a unique fixed point in$X$.

Proof. If $x = x_n;$ $y = z = y_n$ Let $x_n \neq x_{n+1}$ and $x_{n+1} = y_n$
Then,

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) = W_\lambda^s(Tx_{n-1}, Tx_n, Tx_n) \quad (32)$$
$$\leq aW_\lambda^s(x_{n-1}, x_n, x_n) + bW_\lambda^s(x_n, x_{n+1}, x_{n+1}) + cW_\lambda^s(x_n, x_{n+1}, x_{n+1}) \quad (33)$$
$$= aW_\lambda^s(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) + bW_\lambda^s(Tx_{n-1}, Tx_n, Tx_n) \quad (34)$$
$$+ cW_\lambda^s(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) \quad (35)$$
$$\leq a^2 W_\lambda^s(x_{n-2}, x_{n-1}, x_{n-1}) + b^2 W_\lambda^s(x_{n-1}, x_n, x_n) \quad (36)$$
$$+ c^2 W_\lambda^s(x_{n-2}, x_{n-1}, x_{n-1}) \quad (37)$$

Iteratively to n we have

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq a^n W_\lambda^s(x_0, x_1, x_1) + b^n W_\lambda^s(x_1, x_2, x_2) \quad (38)$$
$$+ c^n W_\lambda^s(x_0, x_1, x_1) \quad (39)$$
$$\leq [a^n + b^{n+1} + c^n] W_\lambda^s(x_0, x_1, x_1) \quad (40)$$

let $\beta = [a^n + b^{n+1} + c^n] < 1$

$$W^s_\lambda(x_n, x_{n+1}, x_{n+1}) \leq \beta^n W^s_\lambda(x_0, x_1, x_1) \quad (41)$$

This proves that the sequence $\{x_n\}$ is W^s_λ-converges.
For every $m > n \in N \; and \; N \in \mathbb{N}$

$$W^s_\lambda(x_n, x_m, x_m) \leq W^s_\lambda(x_n, x_{n+1}, x_{n+1}) + W^s_\lambda(x_{n+1}, x_{n+2}, x_{n+2}) + W^s_\lambda(x_{n+2}, x_{n+3}, x_{n+3}) \quad (42)$$

$$+ ... + W^s_\lambda(x_{m-1}, x_m, x_m) \quad (43)$$

$$\leq \beta^n W^s_\lambda(x_0, x_1, x_1) + \beta^{n+1} W^s_\lambda(x_0, x_1, x_1) + \beta^{n+2} W^s_\lambda(x_0, x_1, x_1) + ... \quad (44)$$

$$\leq (\beta^n + \beta^{n+1} + \beta^{n+2} + ...) W^s_\lambda(x_0, x_1, x_1) \quad (45)$$

$$W^s_\lambda(x_n, x_m, x_m) \leq (\frac{\beta^n}{1-\beta}) W^s_\lambda(x_0, x_1, x_1) \to 0 \; as \; n \to \infty \quad (46)$$

This proves the sequence $\{x_n\}$ is W^s_λ-Cauchy. Hence, $\{x_n\}$ is W^s_λ-complete.
If the limit of the sequence $\{x_n\}$ be defined as p then the contraction becomes. $W^s_\lambda(x_{n+1}, Tp, Tp) \leq aW^s_\lambda(x_{n-1}, Tp, Tp) + bW^s_\lambda(x_n, Tp, Tp) + aW^s_\lambda(x_{n-1}, Tp, Tp)$ Taking the limit of the sequence at $n \to \infty$, and with the fact that W^s_λ continuous. Then,

$$W^s_\lambda(p, Tp, Tp) \leq aW^s_\lambda(p, Tp, Tp) + bW^s_\lambda(p, Tp, Tp) + aW^s_\lambda(p, Tp, Tp) \quad (47)$$

$$W^s_\lambda(p, Tp, Tp) \leq \beta W^s_\lambda(p, Tp, Tp) \quad (48)$$

since $0 \leq \beta < 1$ Hence,

$$W^s_\lambda(p, Tp, Tp) \leq \beta W^s_\lambda(p, Tp, Tp) = W^s_\lambda(p, Tp, Tp) \quad (49)$$

$$Tp = p. \quad (50)$$

Since $0 \leq \beta < 1$ Hence, $p = Tp$.
Suppose q is also a fixed point of T and $p \neq q$. Then the contraction becomes

$$W^s_\lambda(p, q, q) \leq aW^s_\lambda(p, q, q) + bW^s_\lambda(p, q, q) + cW^s_\lambda(p, q, q) \quad (51)$$

$$W^s_\lambda(p, q, q) \leq \beta W^s_\lambda(p, q, q) < W^s_\lambda(p, q, q) \quad (52)$$

Since $0 \leq \beta < 1$ implies $p = q$.
therefore, T has a unique fixed point p in X. □

Here, we consider the coincidence point of a fixed point of modular S-metric space.

Theorem 3.3 Let (X, W^s_λ) be a modular S-metric space and $T, g : X \to X$ be two self-mapping of W^s_λ-contraction were there exists a real number α, β where $0 \leq \alpha + \beta < 1$ such that for each $x, y, z \in X; \lambda > 0$ we have

$$W^s_\lambda(Tx, Ty, Tz) \leq \alpha \frac{W^s_\lambda(x, Tx, Tx) W^s_\lambda(y, Ty, Ty)}{W^s_\lambda(x, y, z)} + \beta W^s_\lambda(x, y, z) \tag{53}$$

Assume that the following conditions are fulfilled:

(1) (X, W^s_λ) is complete;
(2) $T(X) \subseteq g(x)$; and
(3) g is W^s_λ is continuous and commutes with T.

Then T and g have a unique coincidence point. That is a unique point $p \in X$ such that $gp = Tp = p$ exists.

Proof. Let x_0 be an arbitrary element of X constructing a sequence $\{x_n\}$ defined as follows
$gx_{n+1} = Tx_n$ for all $n \in \mathbb{N}$.
Here, we wish show that the sequence $\{x_n\}$ is W^s_λ-convergent.

$$W^s_\lambda(gx_n, gx_{n+1}, gx_{n+1}) = (Tgx_{n-1}, Tgx_n, Tgx_n) \tag{54}$$

$$W^s_\lambda(gx_n, gx_{n+1}, gx_{n+1}) \leq \alpha \frac{W^s_\lambda(gx_{n-1}, gx_n, gx_n) W^s_\lambda(gx_n, gx_{n+1}, gx_{n+1})}{W^s_\lambda(gx_{n-1}, gx_n, gx_n)} \tag{55}$$

$$+ \beta W^s_\lambda(gx_{n-1}, gx_n, gx_n) \tag{56}$$

$$\leq \alpha W^s_\lambda(gx_n, gx_{n+1}, gx_{n+1}) + \beta W^s_\lambda(gx_{n-1}, gx_n, gx_n) \tag{57}$$

$$W^s_\lambda(gx_n, gx_{n+1}, gx_{n+1}) - \alpha W^s_\lambda(gx_n, gx_{n+1}, gx_{n+1}) \tag{58}$$

$$\leq \beta W^s_\lambda(gx_{n-1}, gx_n, gx_n) \tag{59}$$

$$W^s_\lambda(gx_n, gx_{n+1}, gx_{n+1}) \leq (\frac{\beta}{1-\beta}) W^s_\lambda(gx_{n-1}, gx_n, gx_n) \tag{60}$$

$$= (\frac{\beta}{1-\alpha}) W^s_\lambda(Tgx_{n-2}, Tgx_{n-1}, Tgx_{n-1}) \tag{61}$$

$$\leq (\frac{\beta}{1-\alpha})^2 W^s_\lambda(gx_{n-1}, gx_n, gx_n) \tag{62}$$

iteratively, we have

$$W^s_\lambda(gx_n, gx_{n+1}, gx_{n+1}) \leq (\frac{\beta}{1-\alpha})^n W^s_\lambda(gx_0, gx_1, gx_1) \tag{63}$$

let

$$\frac{\beta}{1-\alpha} = k < 1$$

$$W^s_\lambda(gx_n, gx_{n+1}, gx_{n+1}) \leq k^n W^s_\lambda(gx_0, gx_1, gx_1). \tag{64}$$

we conclude that, $W^s_\lambda(gx_n, gx_{n+1}, gx_{n+1})$ converges to $W^s_\lambda(gx_0, gx_1, gx_1)$ as $n \to \infty$. Then, the sequence $\{x_n\}$ is W^s_λ-convergent.
Now, we shall show that the sequence $\{x_n\}$ is W^s_λ-Cauchy.

$$W^s_\lambda(x_n, x_m, x_m) \leq W^s_\lambda(x_n, x_{n+1}, x_{n+1}) + W^s_\lambda(x_{n+1}, x_{n+2}, x_{n+2}) + W^s_\lambda(x_{n+2}, x_{n+3}, x_{n+3}) \tag{65}$$

$$+ ... + W^s_\lambda(x_{m-1}, x_m, x_m) \tag{66}$$

$$\leq k^n W^s_\lambda(x_0, x_1, x_1) + k^{n+1} W^s_\lambda(x_0, x_1, x_1) + k^{n+2} W^s_\lambda(x_0, x_1, x_1) + ... \tag{67}$$

$$\leq (k^n + k^{n+1} + k^{n+2} + ...) W^s_\lambda(x_0, x_1, x_1) \tag{68}$$

$$W^s_\lambda(x_n, x_m, x_m) \leq (\frac{k^n}{1-k}) W^s_\lambda(x_0, x_1, x_1) \to 0 \ as \ n \to \infty \tag{69}$$

This, confirm the sequence $\{x_n\}$ is W^s_λ-Cauchy.
Here, we show that p is a coincidence point of T and g.

$$W^s_\lambda(gx_{n+1}, gp, gp) = (Tgx_n, Tgp, Tgp) \tag{70}$$

$$\leq \alpha \frac{W^s_\lambda(gx_n, gp, gp) W^s_\lambda(gp, gp, gp)}{W^s_\lambda(gx_n, gp, gp)} + \beta W^s_\lambda(gx_{n+1}, gp, gp) \tag{71}$$

Hence,

$$W^s_\lambda(gx_{n+1}, gp, gp) \leq \beta W^s_\lambda(gx_{n+1}, gp, gp) \tag{72}$$

Since g is W^s_λ-continuous, $\{gx_n\} \to gp$ also g and T commute, we have $gx_{n+1} = Tgx_n = gTx_n$ for all $n \geq 0$.
letting $n \to \infty$ and the fact that T is continuous we have,
$W^s_\lambda(gp, Tp, Tp) \leq \beta W^s_\lambda(gp, gp, gp) = 0$
Hence, $gp = Tp$
We now show the uniqueness of the coincidence point of T and g.
Suppose, there exists another coincidence point $q \in X$ with $q \neq p$.
from our contraction we have;

$$W^s_\lambda(p, q, q) \leq \beta W^s_\lambda(Tp, Tq, Tq) = 0 \tag{73}$$

which is a contradiction since $\beta < 1$.
Hence, the coincidence point of T and g is unique. □

Conclusion

Results in modular S-metric have been established, using some generalized contractions defined in modular space, the existence and uniqueness where there exists a real number $L \geq 0$ and a strict comparison function $\phi : \mathbb{R} \to \mathbb{R}$ for all $x, y, y \in X$ were established, and coincidence point of a fixed point results for Rational-Type contraction were also established.

NOMENCLATURE

W_λ^s Modular S-metric
λ Time taking to cover a distance between two points

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