

# FIXED POINT RESULTS IN MODULAR S-METRIC SPACE

ABSTRACT. In this work, some definitions, properties and propositions in modular S-metric space were given, existence, uniqueness and coincidence point of a fixed point results in the space were also provided using some general contractive conditions defined in modular S-metric space settings. Our results improve on some results in the literature.

**Keywords and phrases:** Modular S-metric space, coincidence point, Modular convergent, Modular Cauchy.  
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## 1. INTRODUCTION

The concept of distance space was first studied by Frechet [?], where he defined the gap between two points as a metric  $d$  on a non-empty set  $X$  satisfying some certain conditions. As a result of his work, many researchers have come up with several generalization of metric space, among them is Gähler [?] who introduced the notion of 2-metric space which is a generalization of ordinary metric space. In [?] Dhage introduced a new structure of a generalized metric space called D-metric space; also in 2003, Mustafa and Sim [?] defined another generalization of metric space, and claimed that properties of D-metric space were wrong and as a result, they gave a new properties of a distance function called G-metric space. In 2012, Sedghi *et al*[?] introduced S-metric spaces and gave some properties. They proved the fixed point theorem for a self-mapping on a complete S-metric space.

The idea of modular space was conceived by Nakano in [?] which conform with the theory of order spaces.

Musielak and Orlic [?] build on this idea of modular space by proving some fixed point theorems. Chistyakov in [?] introduced the notion of modular metric on an ordinary set and the corresponding modular space which is more general than a metric space.

In line with the concept of Chistyakov, many researchers have come up with several generalization of modular metric spaces, such as Azadifer *et al* in [?] who gave a generalization of modular metric space to develop a space called modular G-metric space. Several results of fixed point were gotten using contractive mappings defined on modular G-metric spaces.

In the present paper, we introduce the concept of modular S-metric space, define modular S-metric space, prove some properties of modular S-metric space, existence and uniqueness as well as coincidence point of a modular S-metric spaces were determined.

## 2. PRELIMINARY

Here, we shall define the Modular  $S$ -metric space based on the concept of modular metric space defined in [?] and that of modular  $G$ -metric space define in [?].

**Definition 2.1:** Let  $X$  be a nonempty set, and  $W_\lambda^s : (0, \infty) \times X \times X \times X \rightarrow [0, \infty]$  be a function satisfying:

- (1)  $W_\lambda^s(x, y, z) \geq 0$  for all  $x, y, z \in X$  and  $\lambda > 0$ ;
- (2)  $W_\lambda^s(x, y, z) = 0$  for all  $x, y, z \in X$  and  $\lambda > 0$  if and if  $x = y = z$ ; and
- (3)  $W_{\lambda+\mu+\gamma}^s(x, y, z) \leq W_\lambda^s(x, x, a) + W_\mu^s(y, y, a) + W_\gamma^s(z, z, a)$  for all  $x, y, z, a \in X$  and  $\lambda, \mu, \gamma > 0$ .

Then the function  $w_\lambda^s(\cdot, \cdot)$  is called a modular  $S$ -metric space on  $X$  and the pair  $(X, W_\lambda^s)$  is called a modular  $S$ -metric space and  $W_\lambda^s, W_\mu^s, W_\gamma^s$  are velocities with respect to time  $\lambda, \mu$  and  $\gamma$  respectively.

**Example 2.1:** Let  $X = [0, \infty)$  be the interval of nonnegative real numbers and let  $W_\lambda^s$  be defined by

$$W_\lambda^s(x, y, z) = \begin{cases} 0, & \text{if } x=y=z. \\ \max(x, y, z), & \text{otherwise} \end{cases}$$

Then  $W_\lambda^s$  is a complete  $W_\lambda^s$ -metric on  $X$ .

In this section, we introduce some definitions and notion of  $W_\lambda^s$ -convergent sequence and  $W_\lambda^s$ -Cauchy sequence.

**Definition 2.2:** A sequence  $\{x_n\}$  in  $X$  is said to be modular  $S$  convergent to  $x \in X$  if  $W_\lambda^s(x_n, x_n, x) \rightarrow 0$  as  $n \rightarrow \infty$ , that is for every  $\epsilon > 0$ , there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$  we have  $W_\lambda^s(x_n, x_n, x) < \epsilon$ . We denote this by  $\lim_{n \rightarrow \infty} x_n = x$  or  $x_n \rightarrow x$  as  $n \rightarrow \infty$

**Definition 2.3:** A sequence  $\{x_n\}$  in  $X$  is said to be modular  $S$  Cauchy sequence if  $W_\lambda^s(x_n, x_n, x_m) \rightarrow 0$  as  $n, m \rightarrow \infty$ , that is for every  $\epsilon > 0$ , there exist  $n_0 \in \mathbb{N}$  such that for all  $n, m \geq n_0$  we have  $W_\lambda^s(x_n, x_n, x_m) < \epsilon$ .

**Definition 2.4:** The modular  $S$ -metric space  $(X, W_\lambda^s)$  is called complete if every modular  $S$ -Cauchy sequence in it is a modular  $S$ -convergent sequence.

**Definition 2.5:** Let  $(X, W_\lambda^s)$  be a modular  $S$ -metric space, and  $\{x_n\}$  be a sequence of points of  $X$ . A point  $x \in X$  is said to be the limit

of the sequence  $\{x_n\}$  if  $\lim_{n,m \rightarrow \infty} (x, x_n, x_n)$  and one say the sequence  $\{x_n\}$  is a modular S-convergent to  $x$ . Thus, that if  $x_n \rightarrow 0$  in a modular S-metric space  $(X, W_\lambda^s)$ , then for any  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that  $W_\lambda^s(x, x_n, x_m) < \epsilon$ , for all  $n, m, \in N$ .

### 3. THE HEART OF THE MATTER

**Definition 3.1** There exists a real number  $L \geq 0$  and a strict comparison function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  for all  $x, y, z \in X$  such that

$$W_\lambda^s(Tx, Ty, Tz) \leq LW_\lambda^s(x, Tx, Tx) + \phi W_\lambda^s(x, y, z) \quad (1)$$

where  $\lambda > 0$  and  $0 \leq L + \phi < 1$ .

**Definition 3.2** There exists a real number  $a, b, c$ , where  $0 \leq a + b + c < 1$  such that for each  $x, y, z \in X; \lambda > 0$

$$W_\lambda^s(Tx, Ty, Tz) \leq aW_\lambda^s(x, Tx, Tx) + W_\lambda^s(y, Ty, Ty) + cW_\lambda^s(x, y, z) \quad (2)$$

**Definition 3.3** There exists a real number  $\alpha, \beta$  where  $0 \leq \alpha + \beta < 1$  such that for each  $x, y, z \in X; \lambda > 0$  we have

$$W_\lambda^s(Tx, Ty, Tz) \leq \alpha \frac{W_\lambda^s(x, Tx, Tx)W_\lambda^s(y, Ty, Ty)}{W_\lambda^s(x, y, z)} + \beta W_\lambda^s(x, y, z) \quad (3)$$

**Proposition 3.1:** Let  $(X, W_\lambda^s)$  be a modular S-metric space then the following are equivalent:

- (1)  $\{x_n\}$  is  $w_\lambda^s$ -convergent to  $x$  for a function  $F : X \rightarrow X$ ;
- (2)  $W_\lambda^s(x_n, x_n, x) = 0$ ; as  $n \rightarrow \infty$ ;
- (3)  $W_\lambda^s(x_n, x, x) = 0$ ; as  $n \rightarrow \infty$ ; and
- (4)  $W_\lambda^s(x_n, x_m, x) = 0$ ; as  $n, m, \rightarrow \infty$ .

**Lemma 3.1:** Let  $(X, W_\lambda^s)$  be a modular S-metric space. If  $x_n \rightarrow x$  and  $y_n \rightarrow y$  as  $n \rightarrow \infty$  then  $W_\lambda^s(x_n, x_n, y_n) \rightarrow W_\lambda^s(x, x, y)$  as  $n \rightarrow \infty$

**Lemma 3.2:** Let  $T$  be a self mapping on a modular S-metric space  $(X, W_\lambda^s)$ . Then  $T$  is said to be continuous at  $x \in X$  if for any sequence  $\{x_n\}$  in  $X$ ,  $x_n \rightarrow x$  implies  $Tx_n \rightarrow Tx$  as  $n \rightarrow \infty$ .

**Lemma 3.3:** In a modular S-metric space, we have  $W_\lambda^s(x, x, y) = W_\lambda^s(y, y, x)$  for all  $x, y \in X \lambda > 0$ .

*Proof.* By the third condition of modular S-metric space, we have  $W_\lambda^s(x, y, y) \leq W_\lambda^s(x, x, a) + W_\lambda^s(y, y, a) + W_\lambda^s(z, z, a)$

$$\begin{aligned} W_\lambda^s(x, x, y) &\leq W_\lambda^s(x, x, x) + W_\lambda^s(x, x, x) + W_\lambda^s(y, y, x) \\ &\leq W_\lambda^s(y, y, x) \\ W_\lambda^s(x, x, y) &\leq W_\lambda^s(y, y, x) \end{aligned}$$

Similarly

$$\begin{aligned} W_\lambda^s(y, y, x) &\leq W_\lambda^s(y, y, y) + W_\lambda^s(y, y, y) + W_\lambda^s(x, x, y) \\ &\leq W_\lambda^s(x, x, y) \\ W_\lambda^s(y, y, x) &\leq W_\lambda^s(x, x, y) \end{aligned}$$

Therefore,  $W_\lambda^s(y, y, x) = W_\lambda^s(x, x, y)$   $\square$

**Proposition 3.2:** The limit of a  $W_\lambda^s$ -convergent sequence in a  $W_\lambda^s$ -metric space is unique.

*Proof.* Let  $(X, W_\lambda^s)$  be a  $W_\lambda^s$ -metric space and let  $\{x_n\} \subseteq X$  be a sequence that converges at the same time to  $x \in X$  and to  $y \in X$ . We claim that  $W_\lambda^s(x, y, y) < \epsilon$  for all  $\epsilon > 0$ . Let  $\epsilon > 0$  be arbitrary, there exist natural numbers  $n_1, n_2 \in \mathbb{N}$  such that

$$\begin{aligned} W_\lambda^s(x_n, x_m, x) &\leq \frac{\epsilon}{3} \text{ for all } n, m \geq n_1 \\ W_\lambda^s(x_n, x_m, y) &\leq \frac{\epsilon}{3} \text{ for all } n, m \geq n_2 \end{aligned}$$

Let  $n_0 = \max(n_1, n_2)$ . Then by the third axioms of  $W_\lambda^s$ -metric space we have

$$\begin{aligned} W_\lambda^s(x, x, y) &\leq W_\lambda^s(x, x, x_n) + W_\lambda^s(x, x, x_n) + W_\lambda^s(y, y, x_n) \\ &\leq 2W_\lambda^s(x, x, x_n) + W_\lambda^s(y, y, x_n) \\ &= 2\frac{\epsilon}{3} + \frac{\epsilon}{3} \\ &= \epsilon \end{aligned}$$

Consequently, we have that  $W_\lambda^s(x, x, y) = 0$  and by axiom two of  $W_\lambda^s$ -metric space, we conclude that  $x = y$ .  $\square$

**Proposition 3.3:** Every convergent sequence in a  $W_\lambda^s$  S-metric space is a Cauchy sequence.

*Proof.* Let  $(X, W_\lambda^s)$  be a  $W_\lambda^s$  S-metric space and let  $\{x_n\} \subseteq X$  be a sequence that converges to  $x \in X$ . Let  $\epsilon > 0$  be arbitrary. By definition, there exist  $n_0 \in \mathbb{N}$  such that

$W_\lambda^s(x_n, x_n, x) \leq \frac{\epsilon}{3}$  for all  $n, m \geq n_0$  By the third axiom of modular S-metric space we have that for all  $n, m, k \geq n_0$

$$W_\lambda^s(x_n, x_m, x_k) \leq W_\lambda^s(x_n, x_n, x) + W_\lambda^s(x_m, x_m, x) + W_\lambda^s(x_k, x_k, x) \quad (4)$$

$$\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad (5)$$

$$= \epsilon \quad (6)$$

Therefore,  $\{x_n\}$  is a Cauchy sequence in  $(X, W_\lambda^s)$   $\square$

We state our main results as follows.

**Theorem 1**

Let  $W_\lambda^s$  be a complete S-metric space. Let  $T : X \rightarrow X$  be a continuous mapping of modular S-contraction where there exists a real number  $L \geq 0$  and a strict comparison function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  for all  $x, y, z \in X$  such that

$$W_\lambda^s(Tx, Ty, Tz) \leq LW_\lambda^s(x, Tx, Tx) + \phi W_\lambda^s(x, y, z) \quad (7)$$

where  $\lambda > 0$  and  $0 \leq L + \phi < 1$ . Then  $T$  has a unique fixed point  $p$  in  $X$ .

*Proof.* If  $x = x_n$ ;  $y = z = y_n$

Let  $x_n \neq x_{n+1}$  and  $x_{n+1} = y_n$

then

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) = W_\lambda^s(Tx_{n-1}, Tx_n, Tx_n) \quad (8)$$

Let  $x_0 \in X$  be an arbitrary point in  $X$  and  $x_{n+1} = Tx_n$  be a Picard iteration. Then for  $n = 0$ ,  $x_1 = Tx_0$ .

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) = W_\lambda^s(Tx_{n-1}, Tx_n, Tx_n) \quad (9)$$

if  $n=1$

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq LW_\lambda^s(x_0, Tx_0, Tx_0) + \phi W_\lambda^s(x_0, Tx_0, Tx_0) \quad (10)$$

$$= LW_\lambda^s(x_0, x_1, x_1) + \phi W_\lambda^s(x_0, x_1, x_1) \quad (11)$$

$$\leq [L + \phi]W_\lambda^s(x_0, x_1, x_1) \quad (12)$$

for  $n=2$

$$W_\lambda^s(x_2, x_3, x_3) \leq LW_\lambda^s(x_1, Tx_0, Tx_0) + \phi W_\lambda^s(x_0, Tx_0, Tx_0) \quad (13)$$

$$\leq [L + \phi]W_\lambda^s(x_1, x_2, x_2) \quad (14)$$

$$\leq [L + \phi]^2 W_\lambda^s(x_0, x_1, x_1) \quad (15)$$

for  $n=3$

$$W_\lambda^s(x_3, x_4, x_4) \leq LW_\lambda^s(x_2, x_3, x_3) + \phi W_\lambda^s(x_2, x_3, x_3) \quad (16)$$

$$\leq [L + \phi]W_\lambda^s(x_2, x_3, x_3) \quad (17)$$

$$\leq [L + \phi]^3 W_\lambda^s(x_0, x_1, x_1) \quad (18)$$

iteratively to  $n$  we have;

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq [L + \phi]^n W_\lambda^s(x_0, x_1, x_1) \quad (19)$$

Let  $\beta = [L + \phi] < 1$   
Hence;

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq [L + \phi]^n W_\lambda^s(x_0, x_1, x_1) \quad (20)$$

$$= \beta^n W_\lambda^s(x_0, x_1, x_1) \quad (21)$$

which prove the convergence of the sequence  $\{x_n\}$ .

Next we show the sequence  $\{x_n\}$  is a Cauchy sequence for any positive integer  $m, n \in N$  and  $N \in \mathbb{N}$  such that  $m > n$ .

Then,

$$W_\lambda^s(x_n, x_m, x_m) \leq W_\lambda^s(x_n, x_{n+1}, x_{n+1}) + W_\lambda^s(x_{n+1}, x_{n+2}, x_{n+2}) + W_\lambda^s(x_{n+2}, x_{n+3}, x_{n+3}) \quad (22)$$

$$+ \dots + W_\lambda^s(x_{m-1}, x_m, x_m) \quad (23)$$

$$\leq \beta^n W_\lambda^s(x_0, x_1, x_1) + \beta^{n+1} W_\lambda^s(x_0, x_1, x_1) + \beta^{n+2} W_\lambda^s(x_0, x_1, x_1) + \dots \quad (24)$$

$$\leq (\beta^n + \beta^{n+1} + \beta^{n+2} + \dots) W_\lambda^s(x_0, x_1, x_1) \quad (25)$$

$$W_\lambda^s(x_n, x_m, x_m) \leq \left(\frac{\beta^n}{1-\beta}\right) W_\lambda^s(x_0, x_1, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty \quad (26)$$

Therefore, the sequence  $\{x_n\}$  is a  $W_\lambda^s$  Cauchy sequence. Then the sequence is a  $W_\lambda^s$ -complete.

If the limit of the sequence  $\{x_n\}$  is defined as  $p$  then the contraction becomes.  $W_\lambda^s(x_{n+1}, Tp, Tp) \leq LW_\lambda^s(x_n, Tp, Tp) + \phi W_\lambda^s(x_n, Tp, Tp)$ . Taking the limit of the sequence as  $n \rightarrow \infty$ , and with the fact that  $W_\lambda^s$  continuous by Lemma2.2. Then,

$$W_\lambda^s(x_{n+1}, Tp, Tp) \leq LW_\lambda^s(p, Tp, Tp) + \phi W_\lambda^s(p, Tp, Tp) \quad (27)$$

$$\leq \beta W_\lambda^s(p, Tp, Tp) \quad (28)$$

Since  $0 \leq \beta < 1$  hence,  $p = Tp$ .

Suppose  $q$  is also a fixed point of  $T$  and  $p \neq q$ . Then the contraction becomes

$$W_\lambda^s(p, q, q) \leq LW_\lambda^s(p, q, q) + \phi W_\lambda^s(p, q, q) \quad (29)$$

$$W_\lambda^s(p, q, q) \leq \beta W_\lambda^s(p, q, q) < W_\lambda^s(p, q, q) \quad (30)$$

which is only possible if  $p = q$  therefore,  $T$  has a unique fixed point  $p$  in  $X$ .  $\square$

### Theorem 2

Let  $(X, W_\lambda^s)$  be a complete modular S-metric space and let  $T : X \rightarrow X$  be a continuous modular S-contraction such that, there

exists a real number  $a, b, c$   $0 \leq a + b + c < 1$  such that for each  $x, y, z \in X$ ;  $\lambda > 0$

$$W_\lambda^s(Tx, Ty, Tz) \leq aW_\lambda^s(x, Tx, Tx) + W_\lambda^s(y, Ty, Ty) + cW_\lambda^s(x, y, z) \quad (31)$$

Then  $T$  has a unique fixed point in  $X$ .

*Proof.* If  $x = x_n$ ;  $y = z = y_n$  Let  $x_n \neq x_{n+1}$  and  $x_{n+1} = y_n$   
Then,

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) = W_\lambda^s(Tx_{n-1}, Tx_n, Tx_n) \quad (32)$$

$$\leq aW_\lambda^s(x_{n-1}, x_n, x_n) + bW_\lambda^s(x_n, x_{n+1}, x_{n+1}) + cW_\lambda^s(x_n, x_{n+1}, x_{n+1}) \quad (33)$$

$$= aW_\lambda^s(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) + bW_\lambda^s(Tx_{n-1}, Tx_n, Tx_n) \quad (34)$$

$$+ cW_\lambda^s(Tx_{n-2}, Tx_{n-1}, Tx_{n-1}) \quad (35)$$

$$\leq a^2W_\lambda^s(x_{n-2}, x_{n-1}, x_{n-1}) + b^2W_\lambda^s(x_{n-1}, x_n, x_n) \quad (36)$$

$$+ c^2W_\lambda^s(x_{n-2}, x_{n-1}, x_{n-1}) \quad (37)$$

Iteratively to  $n$  we have

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq a^nW_\lambda^s(x_0, x_1, x_1) + b^nW_\lambda^s(x_1, x_2, x_2) \quad (38)$$

$$+ c^nW_\lambda^s(x_0, x_1, x_1) \quad (39)$$

$$\leq [a^n + b^{n+1} + c^n]W_\lambda^s(x_0, x_1, x_1) \quad (40)$$

let  $\beta = [a^n + b^{n+1} + c^n] < 1$

$$W_\lambda^s(x_n, x_{n+1}, x_{n+1}) \leq \beta^nW_\lambda^s(x_0, x_1, x_1) \quad (41)$$

This proves that the sequence  $\{x_n\}$  is  $W_\lambda^s$ -converges.

For every  $m > n \in \mathbb{N}$  and  $N \in \mathbb{N}$

$$W_\lambda^s(x_n, x_m, x_m) \leq W_\lambda^s(x_n, x_{n+1}, x_{n+1}) + W_\lambda^s(x_{n+1}, x_{n+2}, x_{n+2}) + W_\lambda^s(x_{n+2}, x_{n+3}, x_{n+3}) \quad (42)$$

$$+ \dots + W_\lambda^s(x_{m-1}, x_m, x_m) \quad (43)$$

$$\leq \beta^nW_\lambda^s(x_0, x_1, x_1) + \beta^{n+1}W_\lambda^s(x_0, x_1, x_1) + \beta^{n+2}W_\lambda^s(x_0, x_1, x_1) + \dots \quad (44)$$

$$\leq (\beta^n + \beta^{n+1} + \beta^{n+2} + \dots)W_\lambda^s(x_0, x_1, x_1) \quad (45)$$

$$W_\lambda^s(x_n, x_m, x_m) \leq \left(\frac{\beta^n}{1-\beta}\right)W_\lambda^s(x_0, x_1, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty \quad (46)$$

This proves the sequence  $\{x_n\}$  is  $W_\lambda^s$ -Cauchy. Hence,  $\{x_n\}$  is  $W_\lambda^s$ -complete.

If the limit of the sequence  $\{x_n\}$  be defined as  $p$  then the contraction becomes.  $W_\lambda^s(x_{n+1}, Tp, Tp) \leq aW_\lambda^s(x_{n-1}, Tp, Tp) + bW_\lambda^s(x_n, Tp, Tp) + aW_\lambda^s(x_{n-1}, Tp, Tp)$  Taking the limit of the sequence at  $n \rightarrow \infty$ , and with the fact that  $W_\lambda^s$  continuous. Then,

$$W_\lambda^s(p, Tp, Tp) \leq aW_\lambda^s(p, Tp, Tp) + bW_\lambda^s(p, Tp, Tp) + aW_\lambda^s(p, Tp, Tp) \quad (47)$$

$$W_\lambda^s(p, Tp, Tp) \leq \beta W_\lambda^s(p, Tp, Tp) \quad (48)$$

since  $0 \leq \beta < 1$  Hence,

$$W_\lambda^s(p, Tp, Tp) \leq \beta W_\lambda^s(p, Tp, Tp) = W_\lambda^s(p, Tp, Tp) \quad (49)$$

$$Tp = p. \quad (50)$$

Since  $0 \leq \beta < 1$  Hence,  $p = Tp$ .

Suppose  $q$  is also a fixed point of  $T$  and  $p \neq q$ . Then the contraction becomes

$$W_\lambda^s(p, q, q) \leq aW_\lambda^s(p, q, q) + bW_\lambda^s(p, q, q) + cW_\lambda^s(p, q, q) \quad (51)$$

$$W_\lambda^s(p, q, q) \leq \beta W_\lambda^s(p, q, q) < W_\lambda^s(p, q, q) \quad (52)$$

Since  $0 \leq \beta < 1$  implies  $p = q$ .

therefore,  $T$  has a unique fixed point  $p$  in  $X$ .  $\square$

Here, we consider the coincidence point of a fixed point of modular S-metric space.

**Theorem 3** Let  $(X, W_\lambda^s)$  be a modular S-metric space and  $T, g : X \rightarrow X$  be two self-mapping of  $W_\lambda^s$ -contraction were there exists a real number  $\alpha, \beta$  where  $0 \leq \alpha + \beta < 1$  such that for each  $x, y, z \in X; \lambda > 0$  we have

$$W_\lambda^s(Tx, Ty, Tz) \leq \alpha \frac{W_\lambda^s(x, Tx, Tx)W_\lambda^s(y, Ty, Ty)}{W_\lambda^s(x, y, z)} + \beta W_\lambda^s(x, y, z) \quad (53)$$

Assume that the following conditions are fulfilled:

- (1)  $(X, W_\lambda^s)$  is complete;
- (2)  $T(X) \subseteq g(X)$ ; and
- (3)  $g$  is  $W_\lambda^s$  is continuous and commutes with  $T$ .

Then  $T$  and  $g$  have a unique coincidence point. That is a unique point  $p \in X$  such that  $gp = Tp = p$  exists.

*Proof.* Let  $x_0$  be an arbitrary element of  $X$  constructing a sequence  $\{x_n\}$  defined as follows  
 $gx_{n+1} = Tx_n$  for all  $n \in \mathbb{N}$ .

Here, we wish show that the sequence  $\{x_n\}$  is  $W_\lambda^s$ -convergent.

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) = (Tgx_{n-1}, Tgx_n, Tgx_n) \quad (54)$$

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) \leq \alpha \frac{W_\lambda^s(gx_{n-1}, gx_n, gx_n)W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1})}{W_\lambda^s(gx_{n-1}, gx_n, gx_n)} \quad (55)$$

$$+ \beta W_\lambda^s(gx_{n-1}, gx_n, gx_n) \quad (56)$$

$$\leq \alpha W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) + \beta W_\lambda^s(gx_{n-1}, gx_n, gx_n) \quad (57)$$

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) - \alpha W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) \quad (58)$$

$$\leq \beta W_\lambda^s(gx_{n-1}, gx_n, gx_n) \quad (59)$$

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) \leq \left(\frac{\beta}{1-\beta}\right) W_\lambda^s(gx_{n-1}, gx_n, gx_n) \quad (60)$$

$$= \left(\frac{\beta}{1-\alpha}\right) W_\lambda^s(Tgx_{n-2}, Tgx_{n-1}, Tgx_{n-1}) \quad (61)$$

$$\leq \left(\frac{\beta}{1-\alpha}\right)^2 W_\lambda^s(gx_{n-1}, gx_n, gx_n) \quad (62)$$

iteratively, we have

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) \leq \left(\frac{\beta}{1-\alpha}\right)^n W_\lambda^s(gx_0, gx_1, gx_1) \quad (63)$$

let

$$\frac{\beta}{1-\alpha} = k < 1$$

$$W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1}) \leq k^n W_\lambda^s(gx_0, gx_1, gx_1). \quad (64)$$

we conclude that,  $W_\lambda^s(gx_n, gx_{n+1}, gx_{n+1})$  converges to  $W_\lambda^s(gx_0, gx_1, gx_1)$  as  $n \rightarrow \infty$ . Then, the sequence  $\{x_n\}$  is  $W_\lambda^s$ -convergent.

Now, we shall show that the sequence  $\{x_n\}$  is  $W_\lambda^s$ -Cauchy.

$$W_\lambda^s(x_n, x_m, x_m) \leq W_\lambda^s(x_n, x_{n+1}, x_{n+1}) + W_\lambda^s(x_{n+1}, x_{n+2}, x_{n+2}) + W_\lambda^s(x_{n+2}, x_{n+3}, x_{n+3}) \quad (65)$$

$$+ \dots + W_\lambda^s(x_{m-1}, x_m, x_m) \quad (66)$$

$$\leq k^n W_\lambda^s(x_0, x_1, x_1) + k^{n+1} W_\lambda^s(x_0, x_1, x_1) + k^{n+2} W_\lambda^s(x_0, x_1, x_1) + \dots \quad (67)$$

$$\leq (k^n + k^{n+1} + k^{n+2} + \dots) W_\lambda^s(x_0, x_1, x_1) \quad (68)$$

$$W_\lambda^s(x_n, x_m, x_m) \leq \left(\frac{k^n}{1-k}\right) W_\lambda^s(x_0, x_1, x_1) \rightarrow 0 \text{ as } n \rightarrow \infty \quad (69)$$

This, confirm the sequence  $\{x_n\}$  is  $W_\lambda^s$ -Cauchy.

Here, we show that  $p$  is a coincidence point of  $T$  and  $g$ .

$$W_\lambda^s(gx_{n+1}, gp, gp) = (Tgx_n, Tgp, Tgp) \quad (70)$$

$$\leq \alpha \frac{W_\lambda^s(gx_n, gp, gp) W_\lambda^s(gp, gp, gp)}{W_\lambda^s(gx_n, gp, gp)} + \beta W_\lambda^s(gx_{n+1}, gp, gp) \quad (71)$$

Hence,

$$W_\lambda^s(gx_{n+1}, gp, gp) \leq \beta W_\lambda^s(gx_{n+1}, gp, gp) \quad (72)$$

Since  $g$  is  $W_\lambda^s$ -continuous,  $\{gx_n\} \rightarrow gp$  also  $g$  and  $T$  commute, we have  $gx_{n+1} = Tgx_n = gTx_n$  for all  $n \geq 0$ .

letting  $n \rightarrow \infty$  and the fact that  $T$  is continuous we have,

$$W_\lambda^s(gp, Tp, Tp) \leq \beta W_\lambda^s(gp, gp, gp) = 0$$

Hence,  $gp = Tp$

We now show the uniqueness of the coincidence point of  $T$  and  $g$ .

Suppose, there exists another coincidence point  $q \in X$  with  $q \neq p$ .

from our contraction we have;

$$W_\lambda^s(p, q, q) \leq \beta W_\lambda^s(Tp, Tq, Tq) = 0 \quad (73)$$

which is a contradiction since  $\beta < 1$ .

Hence, the coincidence point of  $T$  and  $g$  is unique.  $\square$

Results in modular S-metric have been established, using some generalized contractions defined in modular space, existence, uniqueness and coincidence point of a fixed point results were also established.

#### NOMENCLATURE

$W_\lambda^s$  Modular S-metric

$\lambda$  Time taking to cover a distance between two points

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