Original Research Article

Probabilistic Population Modeling with Interactions between Species

Abstract:

The probabilistic population model withinteractions between species will be developed. We use predator-prey interaction to come up with a system of probabilistic differential equations. By accepting a series of assumptions that may govern the environment,one can reach a variety of different models. In a special case we develop a system with predator's rate of change instead of the natural birth and death processes. The mean probability of the population as a mathematical expectationis computed as a solution of the system using random parameters.

Keywords: Probabilistic Model, Interactive Species.Expectation, stochastic parameters, Phase diagram, mutually exclusive events, Conditional probability, Probabilistic differential equations.

Sec. 1: Introductionto Deterministic Interactions

Let us assume $x(t)$ and $y(t)$ are the measures of two interactive objects or population of two species of time t. Suppose there is interactions between these two species, for example one is predator and the other is prey.

Lotka (1950) and Voltera (1931) were the first to produce a system of nonlinear model to study a deterministic behavior of the populations of predator and prey.

In a modeling approach, we may assume x(t) and y(t) follow the logistical growth in the absence predator-prey interaction. The mathematical model of the population in continuous deterministic case would be in the following form:

$$\begin{cases} \frac{dx}{dt} = (\mathrm{a} - b.x)\mathrm{x} - \mathrm{m.xy}, & \mathrm{x}(0) = x_0 \\ \frac{dy}{dt} = (\mathrm{d} - \mathrm{c.y})\mathrm{y} + \mathrm{n.xy} & \mathrm{y}(0) = y_0 \end{cases} \qquad (1)$$

where$K_1 = \frac{a}{b}$ and $K_2 = \frac{d}{c}$are two carrying capacities of two species. This is a deterministic interacting population model. We would like to study the probabilistic interactive model between two species.

In the following stochastic modeling, we do not consider predator y(t) using only the natural resources with the parameters c and d. In fact, prey population is the only resources.

Some stochastic model of predator-prey was developed by Leslie and Gower 1960 (see [11]). Random parameters used in this formulation.

A similar work published in 2002 by Swift Randall, using probability generating functions (see [18]).

The latest stochastic model can be observed in Singh A (see [16]) where random parameters was in modeling of predator-prey interactions.

In the following approach we will use probabilistic modeling for interactions between species.

Section 1- Conditions, Assumptions, and Parameters for Probabilistic Model:

Suppose $that\ x(t)$ and $y(t)$ are the populations of two species at time t, and to find the probability distributions of these populations at time $(t, t + \Delta t]$will accept the following axioms:

Axiom 1: The probability that the incidence of species x(t) will be killed by the predator y(t) in a very short time interval is directly proportional to:

i) the length of the intervalΔt;

ii) ii) the predator – prey population densities.

Let alpha αbe a constant of proportionality, then the probability of reproduction of m-individual is

$$p[one\ birth\ in\ \ \Delta t\ for\ x(t) = m\ ] = \alpha . m. \Delta t$$

and $p[\ one\ reproduction\ of\ \ y(t) = n\ individuals] = \beta . n. \Delta t$

Notice that: α is the growthrate of one individual of x in a unit of time and β is the rate on one reproduction of predator y in a unit of time.

Axiom 2: The probability that there is exactly one **kill-contact** between predator and prey in a very short period of time is proportional to the number of the predators and preys at time t and the length of the interval, $p[one\ contact\ during\ (t, t + \Delta t)] = \gamma . p_m . p_n . \Delta t$

Axiom 3: Prey uses natural resources to grow, and prey is the only resource of food available for predators.When the **kill-contact** between prey and predator happens, the prey population will be reduced from $m\ to\ m - 1$.

Axiom 4: We will take time increment (Δt)sufficiently small, so that no individual can have more than one event like incidence to kill or reproduce one individual during that time interval $(t, t + \Delta t]$.

Axiom 5: The probability of more thanone **kill-contact,** or one contact and one birth, is negligible.

Notes:

1. By axiom 1, the probability of no birth prey during $(t, t +\ \Delta t]$is

$$p[no\ birth\ for\ prey\ x(t) = m] = 1 - \alpha . m. \Delta t$$

By the same reason for the probability of no offspring during the time interval $(t, t + \Delta t]$ for predator is $p[no\ offspring\ for\ predator\ y(t) = n] = 1 - \beta . n. \Delta t$

2. Ina more complicated model one may assume that the probability of one birth of predator is proportional to the probability density of prey population at time t, that is in Axiom 1 there can be

   $p[one\ reproduction\ of\ predator] = \beta. P_m(t). n. \Delta t$

3. Probability of kill-incidence in small time interval is $=\gamma. p_m(t). p_n. \Delta t$
4. Probability of no kill-incidence in small time interval is $=1 - \gamma. p_m(t). p_n. \Delta t$

Parameters:

$\alpha = reproduction\ rate\ of\ prey$= natural birth rate -natural death rate

$\beta = reproduction\ rate\ for\ predator$

$\gamma = Rate\ of\ prey\ killed\ by\ predator$

$\delta = natural\ death\ rate\ for\ predator$

Part(I): Modeling the probabilistic equation for prey:

By this interactive axiom, the following (mutually exclusive) disjoint events can be assumed:

Event A1: At time t, there were $(m-1)$prey individuals and one birth occurring during $(t, t + \Delta t]$

Event B1: There were $(m+1)$ individuals at time t and one contact leading to one prey being consumed by predator (prey-death) occurring during $(t, t + \Delta t]$

Event C1: There were m-individuals at time t, no birth, and no contact occurring during

$$(t, t + \Delta t].$$

Notice: We will consider natural birth processes for both predator-prey and the only death processes for prey killing-incidence by predators. For predators in this modeling one can study both with and without natural death processes.

By these modeling axioms, these events are mutually exclusive. Thus

$$p(A1 \cup B1 \cup C1) = p(A1) + p(B1) + p(C1),$$

Since we assume that events are countably infinite,$p[\cup_{i=0}^{\infty} E_i] = \sum_{i=0}^{\infty} P(E_i)$.

In the following part of formulation, we are assuming that the rate of incidence between predator – prey is not the same as the rate of killing and consuming one prey by a predator. Gamma $\gamma$is the rate of killing incidence and beta is the birth rate of predators. So,$\gamma \neq \beta$ means if the predator is chasing a prey to capture and kill, it may not be successful in all tries.

Also,using two independent event, prob [no birth & no incidence]=prob[no birth] .prob[no incidence]

$p_m\ [t + \Delta t] = p[A1] + p[B1] + \ p[C1]$

$$= p[x(t) = (m-1) \cap \text{ one birth in } \Delta t] + p[x(t) = (m+1) \cap \text{ one contact in } \Delta t] + p[x(t) = m \cap \text{ one zero birth in } \Delta t] + p[x(t) = m \cap \text{zero contact in } \Delta t \bigcap \text{birth}]$$

$$= p[x(t) = m-1] . p[\text{one birth } | x(t) = m-1] + p_{m+1}(t). p[\text{one kill contact } | x(t) = m+1] + p[x(t) = m] . p[\text{no birth } \cap \text{ no contact} | x(t) = m]$$

$$= p_{m-1}(t). \alpha . \Delta t . (m-1) + p_{m+1}(t). \gamma (m+1). n . \Delta t + p_m (t). [1 - \alpha \cdot m . \Delta t] . [1 - \gamma (m)(n) . \Delta t]$$

where the vertical line “|” represents the conditional probability.Now we simplify this relation as follows:

$$p_m[t + \Delta t] = p_{m-1}(t). \alpha . \Delta t . (m-1) + p_{m+1}(t). \gamma (m+1). n . \Delta t + p_m(t) - p_m(t)[\alpha \cdot m + \gamma (m)(n) .]\Delta t + p_m(t)[\alpha \cdot m . \Delta t] . [\gamma (m)(n) . \Delta t].$$

Since(Δt^2)is negligible this relation will be described by the following

$$p_m(t + \Delta t) - p_m(t) = \{ p_{m-1}(t) \alpha (m-1) + p_{m+1}(t) n(m+1) \gamma - p_m(t)[\gamma mn + \alpha . m]\} \Delta t$$

$$\frac{p_m(t + \Delta t) - p_m(t)}{\Delta t} = p_{m+1}(t) n. (m+1) \gamma - p_m(t). [n \gamma + \alpha] + p_{m-1}(t) \alpha (m-1)$$

Let’s take the limit when delta $t \to 0$, then:

$$\frac{dp_m(t)}{dt} = p_{m+1}(t) n. (m+1) \gamma - p_m(t) m(n \gamma + \alpha) + p_{m-1}(t) \alpha. (m-1) \qquad (2)$$

This is the probabilistic differential equation for prey x(t)=m.

Part (II): Probabilistic Differential Equations for Predator y(t)=n:

For driving the probabilistic equation for predator y(t), we will assume that

$p_n(t) = prob[y(t) = n]$ where $y(t)$ is the population of predator at t.

We also consider the following mutually exclusive events:

Event A2: At time t, there are $y(t) = n - 1$ predators and one birth occurring for predator y(t) on the interval $(t, t + \Delta t]$.

Event B2: At time t, there are $y(t) = n$ predators and no birth or death occurring during $(t, t + \Delta t]$.

Event C2: There were y(t)=n+1 predators at time t, and one death occurring for predator during $(t, t + \Delta t]$.

In the previous section we introduced a parameter gamma,γ, representing the death rate for prey due to interactions between two species. The death rate for predators is the natural death rate that we can call delta δ. Thus, the probability of this event in small time interval is not directly proportional to the number of conflicts.

Since all events A2, B2, and C2 are mutually exclusive,

$p(t+\ \Delta t] = \ p[A2 \cup \ B2 \cup C2] = \ p(A2) + \ p(B2) + p(C2).$

Also using two independent events, prob [no birth & no incidence]=prob[no birth] .prob[no incidence]

$$p_n[t+\Delta t] = \ p[y(t) = (n-1) \cap \ one\ birth\ in\ \Delta t]$$
$$+ \ p[y(t) = (n+1) \cap \ one\ predator\ death\ in\ \Delta t]$$
$$+ \ p[y(t) = n \cap \ no\ predator\ birth\ in\ \Delta t]$$
$$= p[y(t) \ = \ n \ - \ 1] \,.\, p[one\ birth\ |y\ (t) \ = \ n \ -1] \ + \ p_{n+1}(t). p[one\ predator\ death\ |y(t)$$
$$= \ n+1] \ + \ p[y(t) \ = \ n] \ p[no\ birth\ for\ predator\ |y(t) \ = n]$$
$$= p_{n-1}(t). \beta \,.\, \Delta t \,.\, (n-1) + \ p_{n+1}(t). \delta \ \ (n+1). \Delta t + p_n\ (t). [1- \ \beta \cdot n \,.\, \Delta t] \,.\, [1- \ \delta \ \ (n) \,.\, \Delta t]$$

where the vertical line “|” represents the conditional probability.

Notice that the birth rate for predator is β and death rate for predator is δ . Now we simplify this relation as follows:

$$p_n[t+\Delta t] = p_{n-1}(t). \beta \,.\, \Delta t \,.\, (n-1) + \ p_{n+1}(t). \delta \ \ (n+1). \ \Delta t + p_n(t). [\ 1- \ \beta \cdot n. \Delta t]. [1- \ \delta \ \ . n \,.\, \Delta t]$$

$$p_n[t+\Delta t] - p_n(t)$$
$$= p_{n-1}(t). \beta \,.\, \Delta t \,.\, (n-1) + \ p_{n+1}(t). \delta \ \ (n+1). \ \Delta t - p_n(t). [\ \beta \cdot n \,.\, \Delta t]$$
$$- \ [p_n\ (t). \ \delta \ \ (n) \,.\, \Delta t]$$

We divide each side by Δt and notice that Δt^2 is negligible.

$$\frac{p_n[t+\Delta t] - p_n(t)}{\Delta t} = p_{n-1}(t). \beta \,. \ (n-1) + \ p_{n+1}(t). \delta \ \ (n+1) - \ p_n(t). [\ \beta \cdot n\ ] - \ [p_n\ (t). \ \delta \ \ n\ ]$$

Pass the limit as $\Delta t \to 0$:

$$\frac{dp_n(t)}{dt} = p_{n-1}(t). \beta \,. \ (n-1) + \ p_{n+1}(t). \delta \ \ (n+1) - \ p_n(t). [\beta \cdot n + \ \ \delta\ n\ ] \qquad (3)$$

This a probabilistic differential equation that can be used as a model for predator. Equations (2) and (3) together will be a system of nonlinear probabilistic differential equation:

$$\begin{cases} \frac{dp_m(t)}{dt} = p_{m+1}(t)\ n. (m+1)\ \gamma \ - p_m\ (t)\ m(n\,\gamma \ + \ \alpha) + \ p_{m-1}(t)\ \alpha. (m-1) \\ \frac{dp_n(t)}{dt} = p_{n+1}(t). (n+1)\delta - \ p_n(t). [\beta + \ \ \delta\ ]n + p_{n-1}(t). \beta \,. \ (n-1)\ ] \end{cases} \qquad (4)$$

Note (1): The first term of the second equation $p_{n+1}(t). (n+1)\delta$ is independent from n. But the similar term in the first equation $p_{m+1}(t)\ n. (m+1)\ \gamma$ has a factor n. This is based on the two parameters gamma and delta. The parameter γ is the death rate caused by killing incidents between two species and the parameter δ represents the natural death rate for predators.

Note (2): In this modeling of system of equations (4) we considered that the reproduction rate for prey alpha α is equal to the natural birth and death of prey. But for the predator the natural birth is beta β and death is deltaδ.

For simplicity one may assume that beta $\beta$is a reproduction factor of predators. By excluding the validity practice for $\delta = 0$ where the system (4) will be reduced to a system (6).

Note (3): The vector form of the system (4) can be described by the following relation

$$\begin{bmatrix} p_m \\ p_n \end{bmatrix}' = [n(m+1)\gamma \quad (n+1)\delta].\begin{bmatrix} p_{m+1} \\ p_{n+1} \end{bmatrix} + [-m(n\gamma+\alpha) \quad -(\beta+\delta)n].\begin{bmatrix} p_m \\ p_n \end{bmatrix} + [\alpha(m-1) \quad \beta(n-1)].\begin{bmatrix} p_{m-1} \\ p_{n-1} \end{bmatrix}$$

Part (III): Predator with a reproduction Parameter

For driving the probabilistic equation for predator y(t), we will assume that

$p_n(t) = prob[y(t) = n]$ where $y(t)$ is the population of predator at t.

We also consider the following mutually exclusive events:

Event A3: At time t, there are $y(t) = n - 1$ predators and one birth occurringfor predator y(t) on the interval $(t, t + \Delta t]$.

Event B3: At time t, there are$y(t) = n$ predators and no birth or death occurring during $(t, t + \Delta t]$.

Event C3: There were y(t)=n+1 predators at time t, and one death occurringfor predator during $(t, t + \Delta t]$.

Since all events A3, B3, and C3 are mutually exclusive, then

$p(t + \Delta t] = p[A3 \cup B3 \cup C3] = p(A3) + p(B3) + p(C3)$. As a result,

$$p_m(t + \Delta t] = prob[\, y(t) = (n-1) \cap \text{ one birth in } \Delta t] + prob[\, y(t) = n \cap \text{ no birth} - \text{death in } \Delta t]$$

$$= p[y(t) = (n-1)].prob[\text{ one birth in } \Delta t|\, y(t) = (n-1)] +$$

$$+p[\, y(t) = n].p[\text{ zero birth in } \Delta t \mid y(t) = n].$$

$$= p_{n-1}(t).\beta.(n-1)\,\Delta t + p_n(t).[1 - \beta\, n\, \Delta t]$$

Therefore:

$$p_n(t + \Delta t) - p_n(t) = [\beta\,(n-1)\,p_{n-1}(t) - \beta\, n\, p_n(t)]\,\Delta t$$

$$\frac{p_n(t + \Delta t) - p_n(t)}{\Delta t} = \beta\,[(n-1)\,p_{n-1}(t) - n\,p_n(t)]$$

$$\frac{dp_n(t)}{dt} = \beta\,[(n-1)\,p_{n-1}(t) - n\,p_n(t)] \tag{5}$$

As a result of both equations (2) and (5), the following is a system of probabilistic prey-predator model.

$$\begin{cases} \dfrac{dp_m(t)}{dt} = n\,(m+1)\,\gamma\, p_{m+1}(t) - m(n\gamma + \alpha)\, p_m(t) + \alpha\,(m-1)\, p_{m-1}(t) \\ \dfrac{dp_n(t)}{dt} = \beta\,[(n-1)\, p_{n-1}(t) - n\, p_n(t)] \end{cases} \qquad (6)$$

This system of differential equations can be modified and expressed in the following form,

$$\begin{cases} \dfrac{dp_m(t)}{dt} = \alpha\,[(m-1)p_{m-1}(t) - m\, p_m(t)] + n\,\gamma\,[(m+1)\, p_{m+1}(t) - m\, p_m\ t)] \\ \dfrac{dp_n(t)}{dt} = \beta[(n-1)\, p_{n-1}(t) - n\, p_n(t)] \end{cases} \qquad (7)$$

The solution to the system probabilistic differential equations (7) for m-prey and n-predator is:

$< p_m(t), p_n(t) >$ $at\ t\ with\ the$ initial conditions $at\ t = t_0$ which will be $< p_m(0), p_n(0) >$.

Part (IV): Solution to the Probabilistic Differential Equations

Taking sigma over the integers m and n will produce the following expectations, that is

$E[x(t) = m] = \sum_{m=1}^{\infty} m\, p_m(t)$ and $E[y(t) = n] = \sum_{n=1}^{\infty} n\, p_n(t)$

$$\begin{cases} \sum \dfrac{dp_m(t)}{dt} = \alpha\,[E_{m-1}(t) - E_m(t)] + n.\gamma\,[E_{m+1}(t) - E_m(t)] \\ \sum \dfrac{dp_m(t)}{dt} = \beta\,[E_{n-1}(t) - E_n(t)] \end{cases} \qquad (8)$$

One interpretations of m in axiom(1) is the probability of one birth of the predator in a different environment is proportional to the population of prey at that time when x(t)=m. That is

MODEL (1): $probability\ density\ of\ the\ predator\ we\ replace\ m\ by\ prob[x = m]$:

$$p[one\ birth\ of\ predator\ in\ t + \Delta t] = \beta\, .n.m.\Delta t$$

The second kind of interpretation can be describe by the following equation (model (2)).

That is the probability of one birth of the predator in a different environment is proportional to the density of population of prey at that time when prob[x(t)=m].

Intuitively the population change of preys in the absence of predators can follow the logistic growth.

Model (2): $p[one\ birth\ of\ predator\ in\ t + \Delta t] = \beta.n.p_m(t).\Delta t$

As a result, the second equation for predator will be in the following form

$$\frac{dp_n(t)}{dt} = \beta p_m(t)\,[(n-1)\, p_{n-1}(t) - n\, p_n(t)] \qquad (9)$$

And the first equation in the system of equation (4) and (5) will be in the following

$$\frac{dp_m(t)}{dt} = \alpha\,[(m-1)\, p_{m-1}(t) - m\, p_m(t)] + \gamma p_n(t)[(m+1)\, p_{m+1}(t) - mp_m(t)] \qquad (10)$$

To simplify the new system we substitute the following:

$mp_m\ (t)\ =\ U_m(t)\ and\ n\ p_n(t)\ =\ V_n(t)$, where:

$$p_m(t) = U_m(t)/m,\ and p_n(t)\ =\ V_n(t)/\text{n} \tag{11}$$

$$\sum mp_m\ (t)\ =\ E_m\ (t)\ and \sum np_n(t)\ =\ E_n(t)$$

$$\text{m}\frac{dp_m(\text{t})}{\text{dt}} = \frac{dU_m(\text{t})}{\text{dt}} \text{ and } n\frac{dp_n(\text{t})}{\text{dt}} = \frac{dV_n(\text{t})}{\text{dt}}$$

$$\frac{1}{m}.\frac{dU_m(t)}{dt} = \alpha\ [U_{m-1}(t) - U_m(t)]\ +\ n\,\gamma\ [U_{m+1}(t)\ -\ U_m(t)]$$

First Type: As a result, the first model will be

$$\begin{cases} \frac{dU_m(t)}{dt} = -m\,.\,\alpha\ [-U_{m-1}(t) + U_m(t)]\ +\ m\,n\,\gamma\ [U_{m+1}(t)\ -\ U_m(t)] \\ \frac{dV_n(t)}{dt} =\ \beta.n[V_{n-1}(t)\ -\ V_n(t)] \end{cases} \tag{12}$$

where $m =\ 1, 2, 3, \ldots.,$ and$n =\ 1, 2, 3, \ldots.$

Second Type Model: The second type model assumes in (9) where the second differential equation will be in the following form which can be verified in a similar approach.

$$\frac{dp_n(t)}{dt} = \beta p_m\ (t)\ [(n-1)\ p_{n-1}(t)\ -\ n\ p_n(t)] \tag{13}$$

$$\frac{dV_n(t)}{dt} =\ \beta.p_m(t)[V_{n-1}(t) - V_n(t)] <==> \beta.\frac{U_m(t)}{m}[V_{n-1}(t) - V_n(t)] \tag{14}$$

Third Type Model: The model (12) of this probabilistic model is in the following form:

$$\begin{cases} \frac{dU_m(\text{t})}{\text{dt}} =\ m.\alpha\ [U_{m-1}(t) - U_m(t)]\ +\ \gamma.M.\frac{V_n}{n}\ [U_{m+1}(t)\ -\ U_m(t)] \\ \frac{dV_n(\text{t})}{\text{dt}} =\ \beta.n.\frac{U_m(t)}{m}[V_{n-1}(t)\ -\ V_n(t)] \end{cases} \tag{15}$$

Solution to the second equation of the system (12): This equation can be written in the following form:

$$\frac{dV_n(\text{t})}{\text{dt}} = \beta.n.[V_{n-1}(t)\ -\ V_n(t)]\ , n\ = 1, 2, 3, \ldots.$$

Let us write this equation for n=1,2,3,.... thus

$$\frac{dV_1}{\text{dt}} + \frac{dV_2}{\text{dt}} + \ldots. + \frac{dV_n}{\text{dt}} + \ldots. = \beta\ [V_0\ +\ V_1\ +\ V_2 + \ldots + V_{n-1} + \ldots,]$$

$$\frac{d\,[\boldsymbol{V_0} + \boldsymbol{V_1} + \boldsymbol{V_2} + \ldots + \boldsymbol{V_{n-1}} + \ldots]}{dt} = \beta V_n(t)$$

Suppose the value of n is bounded to N, then define $V_1\ +\ V_2 + \ldots + V_N = \vec{V}_N$, then the initial value problem:

$$\frac{dV_N}{dt} = \beta V_N(t),\ \ V_N(0) = V_0 \xrightarrow{yields} V_N\ (\text{t}) = V_0(\text{t})\ e^{\beta \text{t}} \qquad for\ n\ = 1, 2, 3, \ldots. \tag{16}$$

Solution of model (I): To solve the system of probabilistic differential equations$V$ model (1), first we calculate the mean of the function values of $< p_m(t), p_n(t) >$. The computation for standard deviation needs more investigation which is beyond this preliminary article.

Denote the initial size of population at time $t = 0$ *by* x_0 and y_0 then:

$p_{x_0}(0)$=$p[x(0) = x_0]$ =1 *and* $p_{y_0}(0)$=$p[y(0) = y_0] = 1$

And $p_{m\neq x_0}(0) = p_{n\neq y_0}(0) = 0$

These equations are for the population size of m and n, that is p_m(t), p_n(t).

Expected value and variance: By the definition of expectation

$$E[x(t)] = \sum_{x=1 \ to \ m} x(t)p_x(t), \quad and \quad E[y(t)] = \sum_{y=1 \ to \ n} y(t)\, p_y(t)$$

Suppose $y(t) = m(t)$ and $x(t) = n(t)$ has a discrete value then

$$E[x(t)] = \sum_{m=1}^{\infty} m\,.\,p_m(t) \quad and \quad E[y(t)] = \sum_{n=1}^{\infty} n\,.\,p_n(t)$$

Since m and n are positive integers 1,2,3… , take the derivative for specified values of M and N, of these relation with respect to t;

$$\frac{dE_y}{dt} = \sum_{n=1}^{\infty} N\,.\frac{dP_N(t)}{dt} \quad and \quad \frac{dE_x}{dt} = \sum_{m=1}^{\infty} M\,.\frac{dP_M(t)}{dt} \qquad (17)$$

If we substitute $\frac{dp_N(t)}{dt}$ and $\frac{dp_M(t)}{dt}$ from model,1, 2, *and* 3, then we will find the expectation of the population of any of these models.

Expectation of the probability density function of the predator y(t): Now let us substitute in $\frac{dE_y}{dt}$, using model one:

$$\frac{dE_y}{dt} = \sum_{n=1}^{\infty} n\{\beta\,[(n-1)\,p_{n-1}(t) - np_n(t)]\} = \sum_{n=1}^{\infty} \beta\,[(n-1)\,p_{n-1}(t) - n^2 p_n(t)]\}$$

$$= \beta \sum_{n=1}^{\infty} [n\,(n-1)\,p_{n-1}(t) - n^2 p_n(t)]$$

$$\frac{dE_y}{dt} = \beta \sum_{N=1}^{\infty} [n\,(n-1)\,p_{n-1}(t) - \beta \sum_{n=1}^{\infty} n^2 p_n(t)$$

$$= \beta.o.p_{-1} + \beta \sum_{N=2}^{\infty} [n\,(n-1)\,p_{n-1}(t) + \beta \sum_{n=1}^{\infty} n^2 p_n(t) = \beta \sum_{i=1}^{\infty} ip_i(t) + i^2 p_i(t) - i^2 p_i.$$

$$\frac{dE_y}{dt} = \beta \sum_{i=1}^{\infty} ip_i(t) = \beta\, E(y)$$

$\frac{dE_y}{dt} = \beta\, E(y), E_y(t_0)$=$E_{y_0}$ implies that $E_y(t)$ = $E_y(t_0)\, e^{\beta t}$.

$$\frac{dE_y}{dt} = \beta\, E(y), E_y(t_0) = E_{y_0} <==> E_y(t) = E_{y0} \cdot e^{\beta t} \quad (18)$$

Solving the First Probabilistic Differential Equation for Prey: The probabilistic differential equation (5) for Pm(t) is the following:

$$\frac{dP_m(t)}{dt} = \alpha\, [(m-1)\, p_{m-1}(t) - m\, p_m\, (t)] + n\,\gamma\, [(m+1)\, p_{m+1}(t) - m\, p_m(t)]$$

Givenall fixed parameters$\alpha, \gamma, and\ n$.Substitute expectations E(x(t)) using relations(17) x(t):

$$\frac{dE_x}{dt} = \sum_{m=1}^{\infty} m\, \frac{dP_m(t)}{dt} = \sum_{m=1}^{\infty} \alpha[m\,(m-1)p_{m-1}(t) - m^2 p_m\,(t) + \sum_{m=1}^{\infty} n\,\gamma[m\,(m+1)p_{m+1}(t) - m^2 p_m(t)]$$

$$= \alpha \sum_{m=1}^{\infty} \alpha[m\,(m-1)p_{m-1}(t) - \alpha \sum_{m=1}^{\infty} m^2 p_m(t) + n\gamma \sum_{m=1}^{\infty} m\,(m+1)p_{m+1}(t) - n\gamma \sum_{m=1}^{\infty} m^2\, p_m(t)$$

$$= \alpha \sum_{j=1}^{\infty} (j+1)j\, p_j(t) - (\alpha + n\gamma) \sum_{m=1}^{\infty} m^2 p_m\,(t) + n\gamma \sum_{m=2}^{\infty} (K-1)Kp_K(t)$$

Assume and substitute j+1=m then j=m-1, m+1=k, then m=k-1.

$$\frac{dE_x}{dt} = \alpha.2p_1(t) - (\alpha + n\gamma)\, p_1(t) + \sum_{m=2}^{\infty} \{\alpha\,(m+1)m\, p_m(t) - (\alpha + n\gamma)m^2 p_m(t) + n\gamma\,(m-1)mp_m(t)\}$$

$$\frac{dE_x}{dt} = p_1[2\alpha - \alpha - n\gamma] + \textstyle\sum_{m=2}^{\infty}[m\,(m+1)\alpha - m^2(\alpha + n\gamma) + n\gamma.m(m-1)]p_m(t) +$$

$$\sum_{m=2}^{\infty} +m\alpha - n\gamma.m^2 + n\gamma.m^2 - n\gamma.m]p_m(t)$$

$$= p_1(t).[\alpha - n\gamma] + [\alpha - n\gamma] \textstyle\sum_2^{\infty} m\, p_m(t) = = p_1(t).[\alpha - n\gamma] + [\alpha - n\gamma] \sum_2^{\infty} m\, p_m(t)$$

$$= [\alpha - n\gamma].\{1.p_1(t)\} + \sum_{2}^{\infty} m\, p_m(t)$$

$$= [\alpha - n\gamma] \sum_{1}^{\infty} m\, p_m(t) \xrightarrow{yields} = [\alpha - n\gamma]\, E_x(t)$$

Thus, we have the following differential equation

$$\frac{dE_x}{dt} = [\alpha - n\gamma]E_x(t) \xrightarrow{yields} \frac{dE_x}{E_x} = [\alpha - n\gamma]\, dt$$

$$ln\, E_x = [\alpha - n\gamma]t + C \xrightarrow{yields} E_x(t) = e^{[\alpha - n\gamma]t}.\, C$$

$$E_x(t) = E_x(t_0)\,.\, e^{[\alpha - n\,\gamma]\, t}$$

Conclusion:

The probabilistic model of the predator and prey by Differential Equation approach can be presented in the following system:Where α, n, and γ are constant parameters.

$$\begin{cases} \frac{dE_x}{dt} = [\alpha - n\gamma]E_x(t), \quad E_x(t_0) = E_{x0} \\ \frac{dE_y}{dt} = \beta\, E(y), \qquad E_y(t_0) = E_{y_0} \end{cases} \tag{19}$$

And the expectation solution also can be described by the following:

$$E_x(t) = E_x(t_0).\, e^{[\alpha - n\,\gamma]t} \; and \quad E_y(t) = E_{y0} \cdot e^{\beta t} \tag{20}$$

where $E_x(t_0)$ is the initial mean population of the prey and $E_y(t_0)$ is the initial mean population of predator. The natural birth rates for prey and predator are $\alpha\ and\ \beta$. The death rate for prey is γ which is not the natural death, and it is the rate of capture and kill by predator.

This is an interaction modeling between species to create a nonlinear stochastic system. A predator prey model with some special assumptions were used as a vehiclein modeling. Some of these assumptions are considered for convenient computation and understanding the behaviors of the system. For prey, we considered only the natural birth process and ignored the natural death process due to the predator consuming prey as the only food available.Since beta in (18) is a positive growth rate, the predator is growing exponentially. Predators will consume more and more of the prey leading to the extinction of one species. As a result, the power $[\alpha\ -\ n\,\gamma]$ can be either positive or negative. For certain values of n:

$$[\alpha\ -\ n\,\gamma] > 0\ \ implies\, n < \frac{\alpha}{\gamma} \tag{20}$$

This ratio indicates how long this exploitation of prey by the predator can continue? The integer part of this fraction indicates that if we can teach the wolf to be careful on consumption of rabbits in closed environmentsbecause it is gravely dangerous to grow that fast and the ceiling population number will be $n < \left\lfloor \frac{\alpha}{\gamma} \right\rfloor$.

This approach will assure us to validate the conclusion of the model process in comparison to deterministic model (1).

Our next step indeveloping this research is to assume that all interactions between two species with natural birth-death process. The ideal will be the interaction case of both species with logistic growth. The variance of this probabilistic distribution also needs to be calculated.

The final step will be interesting to compare two models in deterministic and stochastic cases.

> with(DEtools);

> DEplot(diff(X(t), t) = r*X(t), X(t), t = 0 .. 15, {[0, 1], [0, 5], [0, 10]}, color = green);

> DEplot(diff(Y(t), t) = b*Y(t), Y(t), t = 0 .. 15, {[0, 1], [0, 5], [0, 10]}, color = blue);

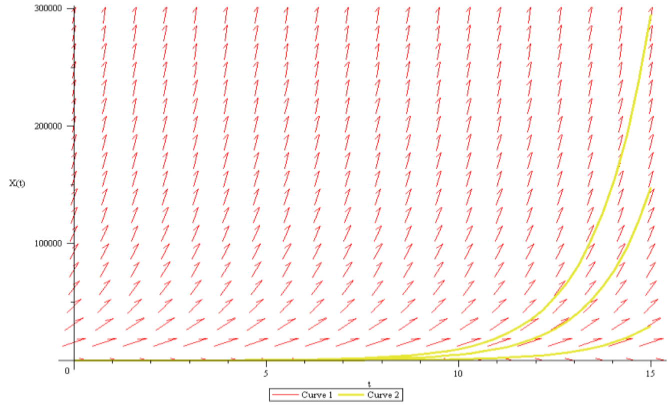

Fig.1 (a) Phase diagram for the prey growth with the values of the given parameters:

$$\alpha = .75, \beta = .45, \gamma = .002, n = 24$$

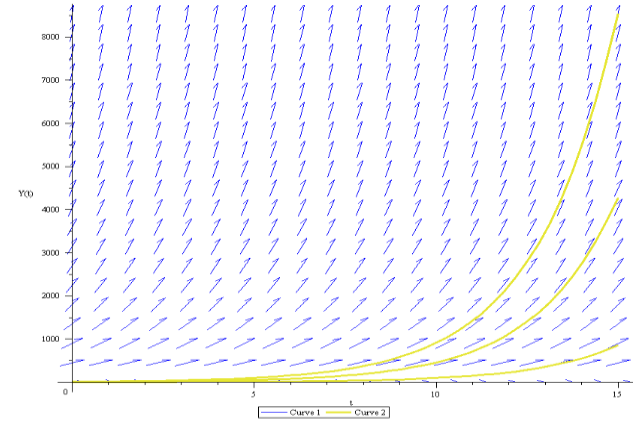

Fig.1 (b) Phase diagram for the predator growth with the given values of the parameters:

$$\alpha = .75, \beta = .45, \gamma = .002, n = 24$$

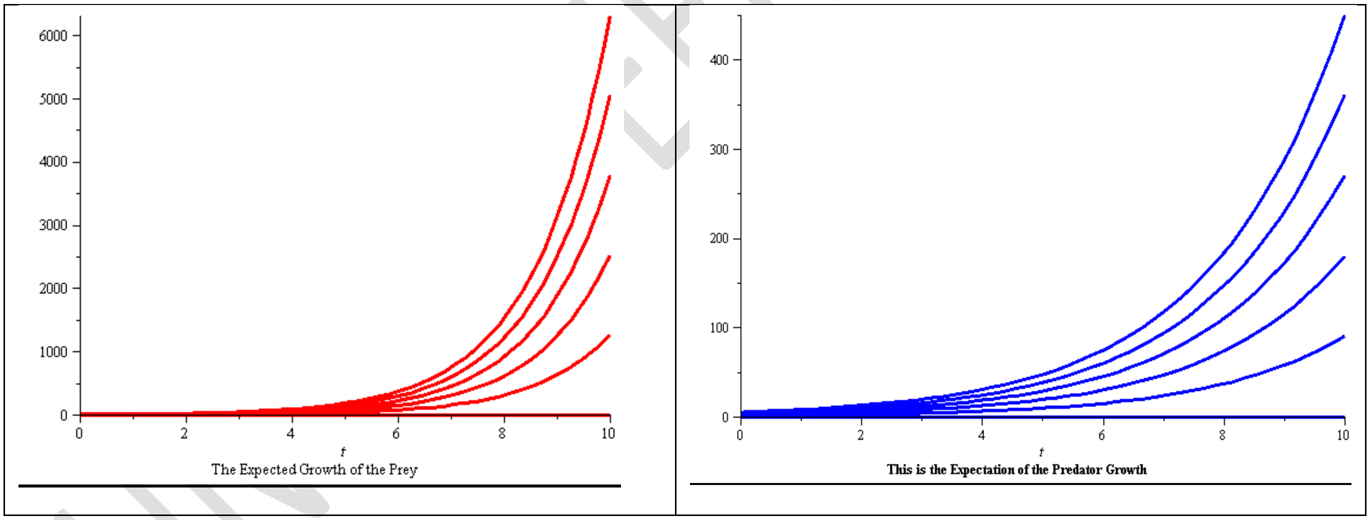

Fig.2: The expected values of Predator - Prey with coexisting.

In the next Maple program, we will demonstrate randomly selected parameters to show the behavior of the expected values of the predator-prey population model.

>with(DEtools);

> with(plots); a := (1/100)*(rand(10 .. 100))(); b := (1/100)*(rand(10 .. 80))(); c := (1/100)*(rand(1 .. 10))(); n := (rand(1 .. 5))();

```
>r := a-n*c;
>X(0) := X0 = (rand(1 .. 100))();      # for random initial values
> Y(0) := Y0 = (rand(1 .. 100))();
> xde := diff(x(t), t) = r*x(t);
> yde := diff(y(t), t) = b*y(t);
> dsolve({xde, yde}, {x(t), y(t)});
> soln := dsolve({xde, yde, x(0) = 10, y(0) = 500}, {x(t), y(t)});
> soln1 := dsolve({xde, yde, x(0) = 50, y(0) = 60}, {x(t), y(t)}, type = numeric,
output = listprocedure, abserr = 0.1e-2);
> seq(soln1(t), t = 0 .. 5);
> with(plots);
> odeplot(soln1, [[t, x(t)], [t, y(t)]], t = 0 .. 10, color = blue, thickness = 3);
```

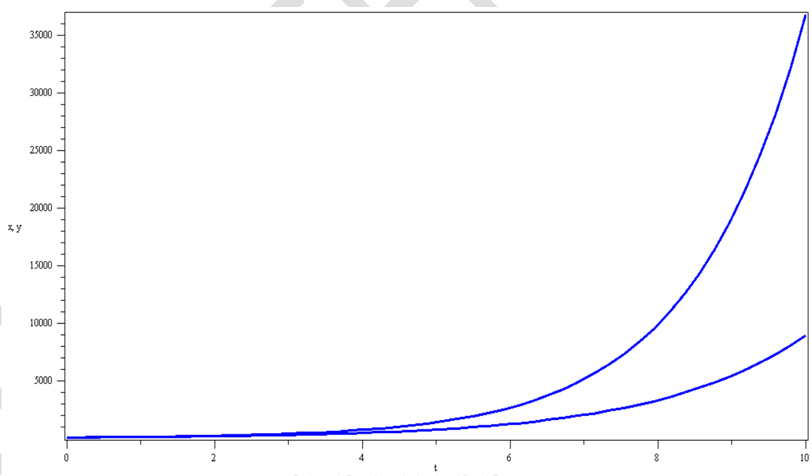

Fig.3: The graph of the expected values of x(t) and y(t) with randomly selected parameters in the same coordinate system.

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