On the Numerical Approximation of Higher Order Differential Equation $\gamma^{iv} = f(z, \gamma, \gamma', \gamma'', \gamma''')$

Abstract

This research examines the general $k-step$ block algorithm for solving higher order Initial Value Problems (IVPs) using Linear Block Algorithm (LBA). Some distinct fourth order IVPs with different values of h were directly solved using the new method and the results obtained were compared with those in literature. The accuracy of the newly derived scheme proved to be better as it outperformed those of existing methods. One of the advantages of the new method is that it does not require much computational burden and it is also self-starting. Basic properties of the method were also analyzed and the result of analysis showed that the method is consistent, zero-stable and convergent.

Keywords: Accuracy, algorithm, convergence, computational burden, higher order IVPs, time constraint.

1. Introduction

The fourth order scheme considered in this research will be used to solve the initial value problems (IVPs)

$$\left.\begin{aligned} &\gamma^{iv} = f(z, \gamma, \gamma', \gamma'', \gamma'''), \\ &\gamma(z_0) = \gamma_a, \gamma'(z_0) = \gamma'_b, \gamma''(z_0) = \gamma''_c, \gamma'''(z_0) = \gamma'''_d \end{aligned}\right\} \quad (1)$$

Some authors observed that solving (1) is a common mathematical designed tool due to what other researchers observed when solving higher order IVPs [1]. The most common mathematical designed tools by some researchers includes numerical methods such as Euler method, Runge-Kutta (RK) method, Trapezoidal rule and Taylor series method, which are used to solve the first order IVPs. These numerical methods are also used to solve the higher order IVPs indirectly by reducing it to the first order system of equations [2-6]. However, this process is easy to implement but it will increase the number of equations as well as increase the cost for the process [7-9].

Block hybrid method is also among the direct methods used to solve (1). Block method is capable of finding numerical solutions at more than a point at a time. [2, 10] discourse the computational burden and zero-stability barrier in hybrid block method. The use of block hybrid method to approximate IVPs (1) directly is considered by some researcher in literature such as [3, 7, 10-14]. [10] developed the four step hybrid block method for the direct solution of fourth order ordinary differential equations, where he adopt power series as a basic function and

UNDER PEER REVIEW

directly applied it on fourth order IVPs. The process adopted in the mathematical formulation of the method is the approach coined as LBA.

2. Mathematical Formulation of the Higher-order Block Algorithm

2.1 The $k-step$ Generalized algorithm

The mathematical approach adopted was that of block method formulations for solving IVPs of the form (1) where $Y_{n+k}=(y_{n+a}, y_{n+b},\cdots, y_{n+k})$ and $Y_{n+k}^{(j)}=(y_{n+a}^{(j)}, y_{n+b}^{(j)},\cdots, y_{n+k}^{(j)})$. In order to obtain the unknown values that the generalized algorithm are given by

$$y_{n+\zeta}=\sum_{j=0}^{3}\frac{(\zeta h)^j}{j!}y_n^{(j)}+\sum_{j=0}^{k}(\nu_{i\zeta} f_{n+j}), \quad \zeta=a, b, \cdots, k \tag{2}$$

and its higher derivatives

$$y_{n+\zeta}^{q}=\sum_{j=0}^{4-(q+1)}\frac{(\zeta h)^j}{j!}y_n^{(j+q)}+\sum_{j=0}^{k}(\kappa_{\zeta jq} f_{n+j}), \quad q=1_{(\zeta=a, b, \cdots, k)}, q=2_{(\zeta=a, b, \cdots, k)}, q=3_{(\zeta=a, b, \cdots, k)} \tag{3}$$

$\nu_{\zeta j}=X^{-1}Y$ and $\kappa_{\zeta jq}=X^{-1}Z$ where

$$X=\begin{pmatrix} 1 & 1 & 1 & \cdots & k \\ 0 & \frac{(ah)^1}{1!} & \frac{(bh)^1}{1!} & \cdots & \frac{(kh)^1}{1!} \\ 0 & \frac{(ah)^2}{2!} & \frac{(bh)^2}{2!} & \cdots & \frac{(kh)^2}{2!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{(ah)^n}{n!} & \frac{(bh)^n}{n!} & \cdots & \frac{(kh)^n}{n!} \end{pmatrix}, Y=\begin{pmatrix} \frac{(\tau h)^4}{4!} \\ \frac{(\tau h)^5}{5!} \\ \frac{(\tau h)^6}{6!} \\ \vdots \\ \frac{(\tau h)^{(4+n)}}{(4+n)!} \end{pmatrix}, Z=\begin{pmatrix} \frac{(\tau h)^{4-q}}{(4-q)!} \\ \frac{(\tau h)^{(5-q)+a}}{((5-q)+a)!} \\ \frac{(\tau h)^{(6-q)+b}}{((6-q)+b)!} \\ \vdots \\ \frac{(\tau h)^{(4-q)+n}}{((4-q)+n)!} \end{pmatrix}$$

Consider the following Lemma:

UNDER PEER REVIEW

Lemma 1

The general linear multistep method adopts only one block form for every $k - step$ block method. This lemma seen below is generalized algorithm which will be adopted to develop the fourth order scheme from linear block algorithm.

This algorithm can be verified with the help of the lemma 2.1 on (2) and (3) as we develop the block method using the generalized algorithm with $(-b, -r, 0, m, p, u, d)$ point of evaluations and then compare the output to the $k = 1$ block method derived with those in literature.

3. Analyses of the Block Scheme

The analysis of the block method is carried out in this section.

3.1. Order of the Method

The linear operator $\ell[y(x_n); h]$ related to the newly derived method is established in this subsection.

Proposition 1

The local truncation error of the newly derived scheme is $C_{11} h^{11} y^{11}(x_n) + 0(h^{12})$.

Proof

According to [5], the linear difference operators associated with the hybrid method (5) to (8) are given by

$$\left.\begin{array}{l}
\ell[y(x_n); h] = y(x_n - dh) - \begin{pmatrix} \alpha_m(x_n + mh) + \alpha_p(x_n + ph) + \alpha_u(x_n + uh) + \\ \alpha_d(x_n + dh) + h^4 \sum\limits_{j=0}^{k} \left(\beta_j(x) f_{n+j} + \beta_k(x) f_{n+k}\right) \end{pmatrix}, k = -b, -r, 0, m, p, u, d \\
\ell[y(x_n); h] = y(x_n - rh) - \begin{pmatrix} \alpha_m(x_n + mh) + \alpha_p(x_n + ph) + \alpha_u(x_n + uh) + \\ \alpha_d(x_n + dh) + h^4 \sum\limits_{j=0}^{k} \left(\beta_j(x) f_{n+j} + \beta_k(x) f_{n+k}\right) \end{pmatrix}, k = -b, -r, 0, m, p, u, d \\
\ell[y(x_n); h] = y(x_n + mh) - \begin{pmatrix} \alpha_m(x_n + mh) + \alpha_p(x_n + ph) + \alpha_u(x_n + uh) + \\ \alpha_d(x_n + dh) + h^4 \sum\limits_{j=0}^{k} \left(\beta_j(x) f_{n+j} + \beta_k(x) f_{n+k}\right) \end{pmatrix}, k = -b, -r, 0, m, p, u, d \\
\ell[y(x_n); h] = y(x_n + ph) - \begin{pmatrix} \alpha_m(x_n + mh) + \alpha_p(x_n + ph) + \alpha_u(x_n + uh) + \\ \alpha_d(x_n + dh) + h^4 \sum\limits_{j=0}^{k} \left(\beta_j(x) f_{n+j} + \beta_k(x) f_{n+k}\right) \end{pmatrix}, k = -b, -r, 0, m, p, u, d \\
\ell[y(x_n); h] = y(x_n + uh) - \begin{pmatrix} \alpha_m(x_n + mh) + \alpha_p(x_n + ph) + \alpha_u(x_n + uh) + \\ \alpha_d(x_n + dh) + h^4 \sum\limits_{j=0}^{k} \left(\beta_j(x) f_{n+j} + \beta_k(x) f_{n+k}\right) \end{pmatrix}, k = -b, -r, 0, m, p, u, d \\
\ell[y(x_n); h] = y(x_n + dh) - \begin{pmatrix} \alpha_m(x_n + mh) + \alpha_p(x_n + ph) + \alpha_u(x_n + uh) + \\ \alpha_d(x_n + dh) + h^4 \sum\limits_{j=0}^{k} \left(\beta_j(x) f_{n+j} + \beta_k(x) f_{n+k}\right) \end{pmatrix}, k = -b, -r, 0, m, p, u, d
\end{array}\right\} \quad (9)$$

UNDER PEER REVIEW

Supposing $y(x)$ is sufficiently differentiable; expanding equation (9) in power of h with the aid of Taylor series. It is important to state that the first non-zero term of each formula in Equation (9) is $C_{11}h^{11}y^{11}(x_n)+0(h^{12})$. The same procedure applies to the derivative scheme in Equation (6) to (8).

Definition 1. [5]

A linear multistep method for a fourth order problem is of order p if it satisfies the condition $c_0 = c_1 = c_2 = c_3 = \cdots = c_p = c_{p+1} = 0,\ c_{p+2} \neq 0,$ where

$$\left.\begin{aligned} c_0 &= \sum_{j=0}^{k} \alpha_j \\ c_1 &= \sum_{j=0}^{k} \left(j\alpha_j - \beta_j \right) \\ &. \\ &. \\ &. \\ c_p &= \sum_{j=0}^{k} \left[\frac{1}{p!} j^p \alpha_j - \frac{1}{(p-1)!} \left(j^{p-1} \beta_j \right) \right],\ p = 2, 3, \cdots, q+1 \end{aligned}\right\} \tag{10}$$

The parameter $c_{p+2} \neq 0$ is referred to as the error constant with the local truncation error defined as $x_{n+k} = c_{p+2} h^{p+2} y^{(p+2)}(x_n) + c_{p+3} h^{p+3} y^{(p+3)}(x_n) + c_{p+4} h^{p+4} y^{(p+4)}(x_n) + 0(h^{p+5})^{\cdots}$

$$\begin{bmatrix}
\sum_{j=0}^{\infty} \frac{\left(-\frac{1}{4}h\right)^j}{j!} y_n^j - y_n + \frac{1}{4} h y'_n - \frac{1}{32} h^2 y''_n + \frac{1}{384} h^3 y'''_n + \frac{16039}{92897280} h^4 y''''_n - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
\left\{ \frac{15271}{928972800}\left(-\frac{1}{4}\right)^j - \frac{397}{2167603200}\left(-\frac{3}{4}\right)^j - \frac{7327}{185794560}\left(\frac{1}{4}\right)^j + \frac{4127}{232243200}\left(\frac{1}{2}\right)^j - \frac{137}{26542080}\left(\frac{3}{4}\right)^j - (1)^j \frac{2209}{3251404800} \right\} \\
\sum_{j=0}^{\infty} \frac{\left(-\frac{3}{4}h\right)^j}{j!} y_n^j - y_n + \frac{3}{4} h y'_n - \frac{9}{32} h^2 y''_n + \frac{9}{128} h^3 y'''_n - \frac{5931}{1146880} h^4 y''''_n - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
\left\{ \frac{87237}{11468800}\left(-\frac{1}{4}\right)^j + \frac{6579}{80281600}\left(-\frac{3}{4}\right)^j + \frac{2187}{2293760}\left(\frac{1}{4}\right)^j - \frac{2673}{2867200}\left(\frac{1}{2}\right)^j + \frac{117}{327680}\left(\frac{3}{4}\right)^j - \frac{2187}{40140800}(1)^j \right\} \\
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{4}h\right)^j}{j!} y_n^j - y_n - \frac{1}{4} h y'_n - \frac{1}{32} h^2 y''_n - \frac{1}{384} h^3 y'''_n + \frac{12203}{92897280} h^4 y''''_n - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
\left\{ -\frac{4783}{928972800}\left(-\frac{1}{4}\right)^j + \frac{41}{433520640}\left(-\frac{3}{4}\right)^j + \frac{8959}{185794560}\left(\frac{1}{4}\right)^j - \frac{709}{46448640}\left(\frac{1}{2}\right)^j + \frac{743}{185794560}\left(\frac{3}{4}\right)^j - \frac{1627}{3251404800}(1)^j \right\} \\
\sum_{j=0}^{\infty} \frac{\left(\frac{1}{2}h\right)^j}{j!} y_n^j - y_n - \frac{1}{2} h y'_n - \frac{1}{8} h^2 y''_n - \frac{1}{48} h^3 y'''_n - \frac{571}{362880} h^4 y''''_n - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
\left\{ -\frac{293}{3628800}\left(-\frac{1}{4}\right)^j + \frac{13}{8467200}\left(-\frac{3}{4}\right)^j + \frac{947}{725760}\left(\frac{1}{4}\right)^j - \frac{461}{1814400}\left(\frac{1}{2}\right)^j + \frac{7}{103680}\left(\frac{3}{4}\right)^j - \frac{107}{12700800}(1)^j \right\} \\
\sum_{j=0}^{\infty} \frac{\left(\frac{3}{4}h\right)^j}{j!} y_n^j - y_n - \frac{3}{4} h y'_n - \frac{9}{32} h^2 y''_n - \frac{9}{128} h^3 y'''_n - \frac{6903}{1146880} h^4 y''''_n - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
\left\{ -\frac{729}{2293760}\left(-\frac{1}{4}\right)^j + \frac{477}{80281600}\left(-\frac{3}{4}\right)^j + \frac{17253}{2293760}\left(\frac{1}{4}\right)^j - \frac{729}{2867200}\left(\frac{1}{2}\right)^j + \frac{549}{2293760}\left(\frac{3}{4}\right)^j - \frac{243}{8028160}(1)^j \right\} \\
\sum_{j=0}^{\infty} \frac{(h)^j}{j!} y_n^j - y_n - h y'_n - \frac{1}{2} h^2 y''_n - \frac{1}{6} h^3 y'''_n - \frac{43}{2835} h^4 y''''_n - \sum_{j=0}^{\infty} \frac{h^{j+2}}{j!} y_n^{j+2} \\
\left\{ \frac{23}{28350}\left(-\frac{1}{4}\right)^j + \frac{1}{66150}\left(-\frac{3}{4}\right)^j + \frac{131}{5670}\left(\frac{1}{4}\right)^j + \frac{43}{14175}\left(\frac{1}{2}\right)^j + \frac{1}{810}\left(\frac{3}{4}\right)^j - \frac{61}{793800}(1)^j \right\}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

UNDER PEER REVIEW

Corollary 1 [5]. The local truncation error of the newly derived scheme is given by

$$\left.\begin{array}{l}\left(-2.9822\times10^{-11}\right)h^{11}y^{(11)}(x_n)+0\left(h^{12}\right)\\ \left(3.9916\times10^{-09}\right)h^{11}y^{(11)}(x_n)+0\left(h^{12}\right)\\ \left(1.8897\times10^{-11}\right)h^{11}y^{(11)}(x_n)+0\left(h^{12}\right)\\ \left(3.1338\times10^{-10}\right)h^{11}y^{(11)}(x_n)+0\left(h^{12}\right)\\ \left(1.1812\times10^{-09}\right)h^{11}y^{(11)}(x_n)+0\left(h^{12}\right)\\ \left(3.0104\times10^{-09}\right)h^{11}y^{(11)}(x_n)+0\left(h^{12}\right)\end{array}\right\} \tag{11}$$

Therefore, the newly derived scheme is of uniform order seven as well as error constant is given by

$$C_{07}=\begin{pmatrix}-2.9822\times10^{-11}\\ 3.9916\times10^{-09}\\ 1.8897\times10^{-11}\\ 3.1338\times10^{-10}\\ 1.1812\times10^{-09}\\ 3.0104\times10^{-09}\end{pmatrix}$$

3.2. Consistency

The newly derived scheme is consistent according to [3] if all the following conditions are fulfilled

i. The order of the scheme must be greater than or equal to one i.e. $(p\geq 1)$

ii. The linear multistep method $\sum_{j=0}^{k}\alpha_j=0$ and $\alpha_j=0$

iii. $p(z)=p'(z)=0$ for $z=1$

iv. $p''(z)=2!\sigma(z)$ for $z=1$

v. $p'''(z)=3!\sigma(z)$ for $z=1$

vi. $p''''(z)=4!\sigma(z)$ for $z=1$

Therefore, the newly derived scheme is consistent.

3.3. Zero-stability of the Method

Definition 2. [5]

A method is zero-stable, if the first characteristic polynomial $\rho(z)$ with roots $z_s=1,2,\cdots,k$ given by $\rho(z)=\det\left[zA^0-A'\right]=0$ satisfies $|z_s|\leq 1$ and every root satisfying $|z_s|=1$ has multiplicity not more than the order of the differential equation.

The first characteristic polynomial of the new hybrid method is given mathematically as

$$\det\left[zA^0-A'\right]=\left\|\begin{bmatrix}z&0&0&0&0&0\\0&z&0&0&0&0\\0&0&z&0&0&0\\0&0&0&z&0&0\\0&0&0&0&z&0\\0&0&0&0&0&z\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\right\|=0 \tag{12}$$

UNDER PEER REVIEW

Evaluating Equation (12), we obtain

$$z^5(z-1)=0 \tag{13}$$

Solving (13) for z gives $z_1 = z_2 = z_3 = z_4 = z_5 = 0$ and $z_\eta = 1$. Hence, the hybrid method is zero-stable.

3.4. Convergence

Theorem 1. [5]

Consistent and zero-stable are necessary and sufficient conditions for a linear multistep method to be convergent.

Therefore, the newly derived scheme is convergent since it is consistent and zero-stable.

3.5. Linear Stability

Definition 3. [5]

The region of absolute stability of a numerical method is the set of complex values λh for which all solutions of the test problem $y'''' = -\lambda^4 y$ will remain bounded as $n \to \infty$.

The concept of A-stability according to [8] is discussed by applying the test equation

$$y^{(k)} = \lambda^{(k)} y \tag{9}$$

To yield

$$Y_m = \mu(z) Y_{m-1},\ z = \lambda h \tag{10}$$

Where $\mu(z)$ is the amplification matrix of the form

where [illegible] is the amplification matrix given by

$$\mu(z) = \left(\xi^0 - z\eta^{(0)} - z^4\eta^{(0)}\right)^{-1}\left(\xi^1 - z\eta^{(1)} - z^4\eta^{(1)}\right) \tag{11}$$

The matrix $\mu(z)$ has Eigen values $(0, 0, \cdots, \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function for of the method is given by

$$\zeta = -\frac{\begin{pmatrix} 806361831z^6 - 27\,481556844z^5 + 180\,584636052z^4 + 588\,452573440z^3 \\ -9494\,005155840z^2 + 43347\,438489600z - 98322\,481152000 \end{pmatrix}}{\begin{array}{c} -114307200z^6 + 1505044800z^5 + 5245430400z^4 - 212\,653728000z^3 \\ + 688\,586572800^2 + 8625\,367296000z - 49161\,240576000 \end{array}}$$

The boundary locus method is adopted in generating the stability polynomial of the hybrid method. The polynomial is

$$\left.\begin{array}{l} \bar{h}(w) = \left(-\frac{1}{24576}w^5 + \frac{1}{286720}w^6\right)h^6 + \left(-\frac{3}{143360}w^6 - \frac{1019}{1290240}w^5\right)h^5 + \left(-\frac{1849}{215040}w^5 - \frac{71}{215040}w^6\right)h^4 \\ + \left(\frac{1}{224}w^5 - \frac{41}{14}w^6\right)h^3 + \left(-\frac{95}{336}w^5 - \frac{1}{336}w^6\right)h^2 + \left(-\frac{3}{14}w^5 - \frac{11}{14}w^6\right)h + w^5 + w^6 \end{array}\right\} \tag{12}$$

UNDER PEER REVIEW

The polynomial is used to plot the region as

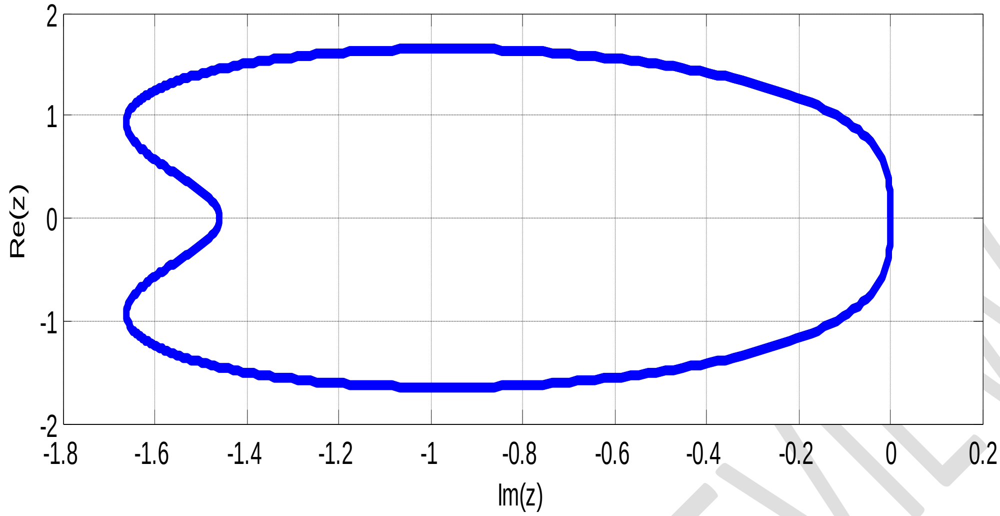

Fig. 1: Regions of absolute stability of the method

The region of absolute stability of the method is a region in the complex z plane. The numerical solution of (9) satisfies $y_j \to 0 \; as \; j \to \infty$ for any initial condition [8].

The stability region obtained in Figure 1 is $A-stable$ according [9].

4. Numerical Problems

To validate the accuracy and convergence of the new method, some fourth order IVPs are solved directly due to the experienced in conventional process when solving higher order IVPs (1).

The results obtained from the new method are compared with the existing methods with different step-size h as shown below.

The following acronyms are used in the tables and graphs below

ES: Exact Solution

CS: Computed Solution

ENM: Error in New Method

EEM: Error in Existing Method

EKEQ[15]: Error in Kuboye, Elusakin & Quadri [15]

EAA[7]: Error in Atabo & Adee [7]

EAO[11]: Error in Adeyeye & Omar [11]

UNDER PEER REVIEW

EAO[14]: Error in Adoghe & Omole [14]

EAC[16]: Error in Ahamad & Charan [16]

EUOA[13]: Error in Ukpebor, Omole & Adoghe [13]

EFO[17]: Error in Familua & Omole [17].

ENM1: Error in New Method at $h = 0.01$

ENM2: Error in New Method at $h = 0.025$

ENM3: Error in New Method at $h = 0.05$

ENM4: Error in New Method at $h = 0.1$

ENM5: Error in New Method at $h = 0.103125$

ENM6: Error in New Method at $h = 0.003125$

EEM1: EAO[11] at $h = 0.01$

EEM2: EAO[11] at $h = 0.025$

EEM3: EAO[11] at $h = 0.05$

EEM4: EAO[11] at $h = 0.1$

EEM5: EAO[14] at $h = 0.103125$

EEM6: EKEQ[15] at $h = 0.1$

EEM7: EAC[16] at $h = 0.1$

EEM8: EUOA[13] at $h = 0.1$

EEM9: EAA[7] at $h = 0.1$

EEM10: EAO[11] at $h = 0.103125$

EEM11: EFO[17] block method at $h = 0.103125$

EEM12: EFO[17] PC method at $h = 0.103125$

UNDER PEER REVIEW

Problem I:

Consider the IVP

$$\gamma^{iv} = -\gamma'', \gamma(0) = 0, \gamma'(0) = \frac{-1.1}{72 - 50pi}, \gamma''(0) = \frac{1}{144 - 50pi}, \gamma'''(0) = \frac{1.2}{144 - 100pi},$$

with the exact solution,

$$\gamma(z) = \frac{1 - z - \cos z - 1.2\sin z}{144 - 100pi}$$

where $z \in \left[0, \frac{pi}{2}\right]$. This problem was also solved by [11, 17].

Problem II:

Consider the IVP

$$\gamma^{iv} = z, \gamma(0) = 0, \gamma'(0) = 1, \gamma''(0) = 1, \gamma'''(0) = 0,$$

with the exact solution

$$\gamma(z) = \frac{z^5}{120zpi} + z$$

This problem was also solved by [15, 7, 11, 13, 16].

Problem III:

Consider the IVP

$$\gamma^{iv} = \sin z + \cos z, \gamma'''(0) = 7, \gamma''(0) = 0, \gamma'(0) = -1, \gamma(0) = 0,$$

with the exact solution

$$\gamma(z) = -\sin z + \cos z + z^3 - 1$$

This problem was also solved by [11, 17]

UNDER PEER REVIEW

5. Results and Discussions

Table I: Results of new method for solving Problem I

z	ES	CS	ENM				
	$h = 0.025$		$h = 0.025$	$h = 0.01$	$h = 0.1$	$h = 0.05$	$h = 0.103125$
1	0.00032113317081669604	0.00032113317081669604	0.0000(+00)	0.0000(+00)	0.0000(+00)	0.0000(+00)	0.0000(+00)
2	0.00063848728577052155	0.00063848728577052155	0.0000(+00)	0.0000(+00)	1.0000(-20)	1.0000(-20)	1.0000(-20)
3	0.00095195576164134882	0.00095195576164134882	0.0000(+00)	0.0000(+00)	0.0000(+00)	1.0000(-20)	2.0000(-20)
4	0.00126143444360699400	0.00126143444360699400	0.0000(+00)	0.0000(+00)	4.0000(-20)	1.0000(-20)	8.0000(-20)
5	0.00156682167033659058	0.00156682167033659058	0.0000(+00)	0.0000(+00)	1.4000(-19)	1.0000(-20)	2.1000(-19)
6	0.00186801833752561197	0.00186801833752561197	0.0000(+00)	0.0000(+00)	3.2000(-19)	1.0000(-20)	4.7000(-19)
7	0.00216492795983283584	0.00216492795983283584	0.0000(+00)	0.0000(+00)	6.3000(-19)	2.0000(-20)	9.0000(-19)
8	0.00245745673118054163	0.00245745673118054164	1.0000(-20)	1.0000(-20)	1.1000(-18)	1.0000(-20)	1.5600(-18)
9	0.00274551358338025566	0.00274551358338025566	0.0000(+00)	1.0000(-20)	1.7900(-18)	1.0000(-20)	2.5100(-18)
10	0.00302901024304740444	0.00302901024304740445	1.0000(-20)	2.0000(-20)	2.7300(-18)	1.0000(-20)	3.8200(-18)

Table II: Comparison of results for solving Problem I with [11, 14].

z	ENM					EEM				
						EAO[11]				EAO[14]
	$h = 0.01$	$h = 0.025$	$h = 0.05$	$h = 0.1$	$h = 0.103125$	$h = 0.01$	$h = 0.025$	$h = 0.05$	$h = 0.1$	$h = 0.103125$
1	0.0000(+00)	0.0000(+00)	0.0000(+00)	0.0000(+00)	0.0000(+00)	6.5021(-19)	6.5052(-19)	5.4210(-18)	4.6295(-16)	2.1149(-18)
2	0.0000(+00)	0.0000(+00)	1.0000(-20)	1.0000(-20)	1.0000(-20)	8.6736(-19)	9.3368(-19)	6.2884(-17)	6.9966(-15)	1.0576(-17)
3	0.0000(+00)	0.0000(+00)	1.0000(-20)	1.0000(-20)	2.0000(-20)	8.6736(-19)	1.7347(-18)	2.3289(-16)	2.8003(-14)	1.2683(-17)
4	0.0000(+00)	0.0000(+00)	1.0000(-20)	1.0000(-20)	8.0000(-20)	2.6021(-18)	6.9389(-18)	6.3578(-16)	7.1500(-14)	2.1963(-17)
5	0.0000(+00)	0.0000(+00)	1.0000(-20)	1.0000(-20)	2.1000(-19)	2.6021(-18)	1.9082(-17)	1.4554(-15)	1.4503(-13)	2.5623(-17)
6	0.0000(+00)	0.0000(+00)	1.0000(-20)	1.0000(-20)	4.7000(-19)	2.6021(-18)	3.9899(-17)	2.8970(-15)	2.5942(-13)	3.7297(-17)
7	0.0000(+00)	0.0000(+00)	2.0000(-20)	2.0000(-20)	9.0000(-19)	3.4695(-18)	7.3726(-17)	5.2623(-15)	4.3649(-13)	4.4098(-17)
8	1.0000(-20)	1.0000(-20)	1.0000(-20)	1.0000(-20)	1.5600(-18)	1.7347(-18)	1.2837(-16)	8.8714(-15)	8.8714(-13)	5.9762(-17)
9	1.0000(-20)	0.0000(+00)	1.0000(-20)	1.0000(-20)	2.5100(-18)	1.7347(-18)	2.0817(-16)	1.4107(-14)	1.0825(-12)	7.1313(-17)
10	2.0000(-20)	1.0000(-20)	1.0000(-20)	1.0000(-20)	3.8200(-18)	1.7347(-18)	3.1919(-16)	2.1415(-14)	1.5981(-12)	9.2590(-17)

UNDER PEER REVIEW

Table III: Results of new method for solving Problem II

z	ES	CS	ENM			
	$h = 0.003125$		$h = 0.003125$	$h = 0.01$	$h = 0.1$	$h = 0.103125$
1	0.003125000000000248353	0.003125000000000248353	0.0000(+00)	0.0000(+00)	0.0000(+00)	1.0000(-20)
2	0.006250000000007947286	0.006250000000007947286	0.0000(+00)	1.0000(-20)	2.0000(-20)	1.0000(-20)
3	0.009375000000060349703	0.009375000000060349703	0.0000(+00)	1.0000(-20)	2.0000(-20)	3.0000(-20)
4	0.012500000000254313151	0.012500000000254313151	0.0000(+00)	1.0000(-20)	3.0000(-20)	4.0000(-20)
5	0.015625000000776102146	0.015625000000776102146	0.0000(+00)	2.0000(-20)	5.0000(-20)	6.0000(-20)
6	0.018750000001931190491	0.018750000001931190491	0.0000(+00)	2.0000(-20)	5.0000(-20)	6.0000(-20)
7	0.021875000004174063603	0.021875000004174063603	0.0000(+00)	2.0000(-20)	6.0000(-20)	5.0000(-20)
8	0.025000000008138020833	0.025000000008138020833	0.0000(+00)	3.0000(-20)	8.0000(-20)	6.0000(-20)
9	0.028125000014664977789	0.028125000014664977789	0.0000(+00)	3.0000(-20)	8.0000(-20)	5.0000(-20)
10	0.031250000024835268656	0.031250000024835268656	0.0000(+00)	3.0000(-20)	8.0000(-20)	6.0000(-20)

Table IV: Comparison of results for solving Problem II with [15, 7, 11, 13, 16].

z	ENM				EEM				
	$h = 0.003125$	$h = 0.01$	$h = 0.1$	$h = 0.103125$	EKEQ[15]	EAC[16]	EA[11]	EUAO[13]	EAA[7]
					$h = 0.1$	$h = 0.1$	$h = 0.1$	$h = 0.1$	$h = 0.1$
1	0.0000(+00)	0.0000(+00)	0.0000(+00)	1.0000(-20)	0.0000(+00)	2.9976(-15)	0.0000(+00)	0.0000(+00)	0.0000(+00)
2	0.0000(+00)	1.0000(-20)	2.0000(-20)	1.0000(-20)	0.0000(+00)	2.9976(-15)	2.7756(-17)	0.0000(+00)	0.0000(+00)
3	0.0000(+00)	1.0000(-20)	2.0000(-20)	3.0000(-20)	0.0000(+00)	1.8225(-05)	5.5511(-17)	0.0000(+00)	0.0000(+00)
4	0.0000(+00)	1.0000(-20)	3.0000(-20)	4.0000(-20)	5.5511(-17)	7.6800(-05)	0.0000(+00)	0.0000(+00)	0.0000(+00)
5	0.0000(+00)	2.0000(-20)	5.0000(-20)	6.0000(-20)	1.1102(-16)	6.9944(-15)	0.0000(+00)	0.0000(+00)	0.0000(+00)
6	0.0000(+00)	2.0000(-20)	5.0000(-20)	6.0000(-20)	1.1102(-16)	0.0000(+00)	0.0000(+00)	0.0000(+00)	0.0000(+00)
7	0.0000(+00)	2.0000(-20)	6.0000(-20)	5.0000(-20)	2.2205(-16)	2.9976(-15)	0.0000(+00)	0.0000(+00)	0.0000(+00)
8	0.0000(+00)	3.0000(-20)	8.0000(-20)	6.0000(-20)	0.0000(+00)	2.9976(-15)	0.0000(+00)	2.0000(-18)	0.0000(+00)
9	0.0000(+00)	3.0000(-20)	8.0000(-20)	5.0000(-20)	1.1102(-16)	0.0000(+00)	0.0000(+00)	2.0000(-18)	1.1100 (-16)
10	0.0000(+00)	3.0000(-20)	8.0000(-20)	6.0000(-20)	2.2205(-16)	7.5000(-03)	0.0000(+00)	1.0000(-17)	0.0000(+00)

UNDER PEER REVIEW

Table V: Result of new method for solving Problem III

z	ES	CS	ENM		
	$h = 0.003125$		$h = 0.003125$	$h = 0.01$	$h = 0.103125$
1	- 0.00312984720468769600	- 0.00312984720468769600	0.0000(+00)	0.0000(+00)	4.0000(-20)
2	- 0.00626924635577210114	- 0.00626924635577210114	0.0000(+00)	1.0000(-20)	6.0000(-20)
3	- 0.00941798368752841945	- 0.00941798368752841945	0.0000(+00)	1.0000(-20)	2.3900(-18)
4	- 0.01257584533946248273	- 0.01257584533946248273	0.0000(+00)	1.0000(-20)	1.1200(-17)
5	- 0.01574261735661109244	- 0.01574261735661109244	0.0000(+00)	2.0000(-20)	3.3380(-17)
6	- 0.01891808568984328399	- 0.01891808568984328399	0.0000(+00)	1.0000(-20)	7.8390(-17)
7	- 0.02210203619616251069	- 0.02210203619616251069	0.0000(+00)	2.0000(-20)	1.5845(-16)
8	- 0.02529425463900974441	- 0.02529425463900974441	0.0000(+00)	2.0000(-20)	2.8864(-16)
9	- 0.02849452668856748983	- 0.02849452668856748983	0.0000(+00)	2.0000(-20)	4.8688(-16)
10	- 0.03170263792206470950	- 0.03170263792206470950	0.0000(+00)	4.0000(-20)	7.7405(-16)

Table VI: Comparison of results for solving Problem III with [11, 17].

z	ENM			EEM		
	$h = 0.003125$	$h = 0.01$	$h = 0.103125$	EAO[11]	EFO[17]	
					block method	PC method
				$h = 0.103125$	$h = 0.103125$	$h = 0.103125$
1	0.0000(+00)	0.0000(+00)	4.0000(-20)	5.8350(-18)	5.8350(-18)	9.9903(-13)
2	0.0000(+00)	1.0000(-20)	6.0000(-20)	4.6708(-17)	4.6712(-17)	2.0098(-12)
3	0.0000(+00)	1.0000(-20)	2.3900(-18)	5.2467(-17)	4.3748(-16)	1.2049(-11)
4	0.0000(+00)	1.0000(-20)	1.1200(-17)	9.3430(-17)	2.3340(-16)	3.0118(-11)
5	0.0000(+00)	2.0000(-20)	3.3380(-17)	9.9220(-17)	2.3920(-16)	6.3035(-11)
6	0.0000(+00)	1.0000(-20)	7.8390(-17)	1.4019(-16)	2.8020(-16)	1.2285(-10)
7	0.0000(+00)	2.0000(-20)	1.5845(-16)	1.4613(-16)	6.7177(-16)	2.2020(-10)
8	0.0000(+00)	2.0000(-20)	2.8864(-16)	1.8712(-16)	4.6706(-16)	3.5856(-10)
9	0.0000(+00)	2.0000(-20)	4.8688(-16)	1.9324(-16)	5.1408(-16)	6.6871(-10)
10	0.0000(+00)	4.0000(-20)	7.7405(-16)	5.8350(-18)	5.8350(-18)	9.99039-13)

UNDER PEER REVIEW

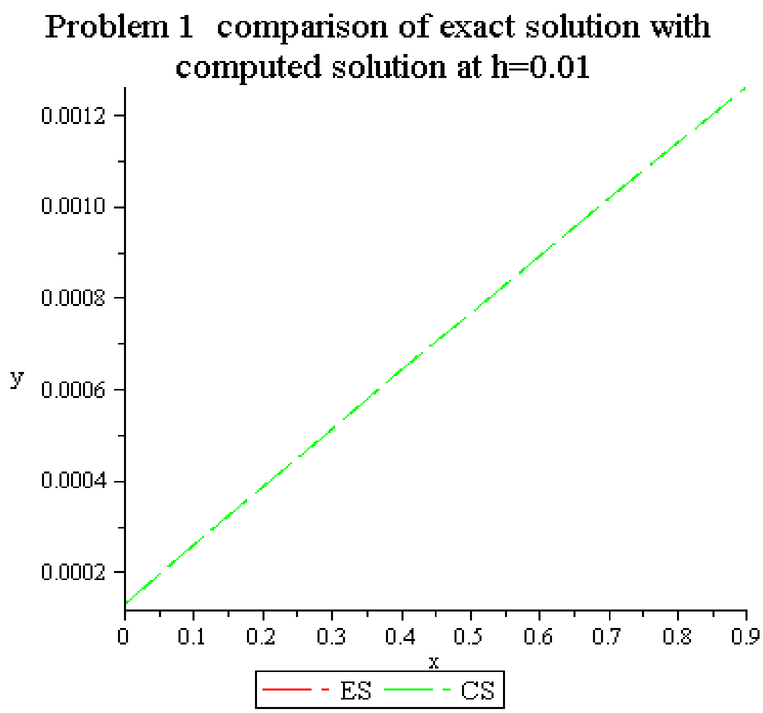

Figure 2: graphical curves for Problem I

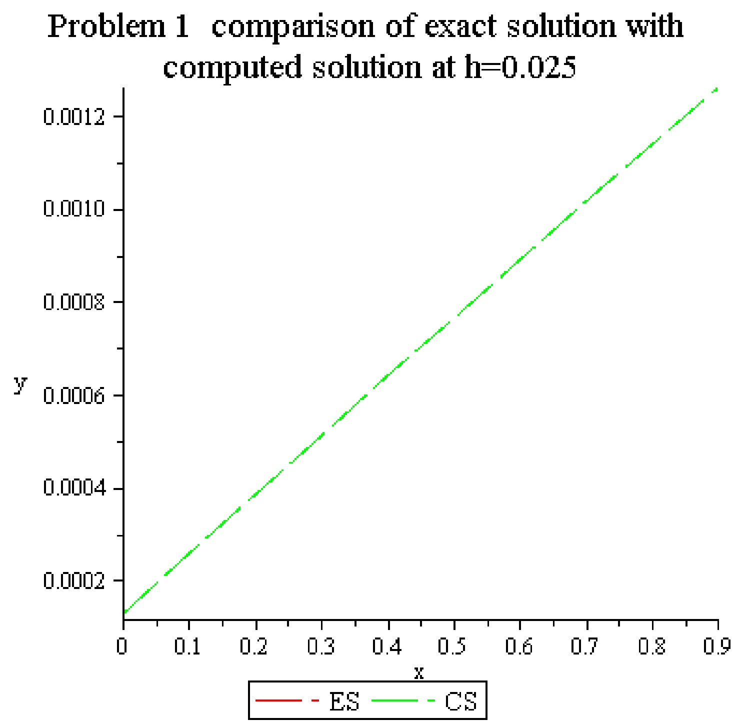

Figure 3: graphical curves for Problem I

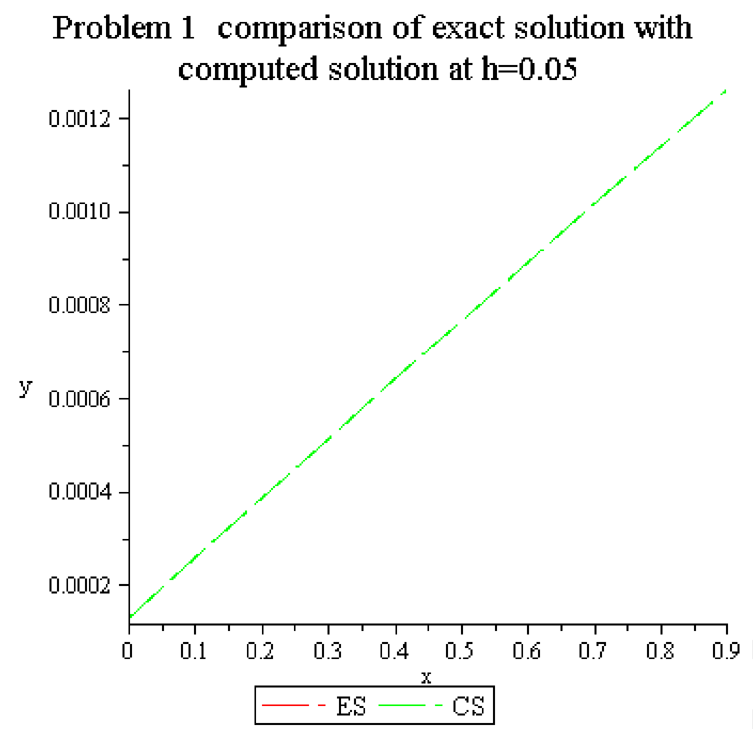

Figure 4: graphical curves for Problem I

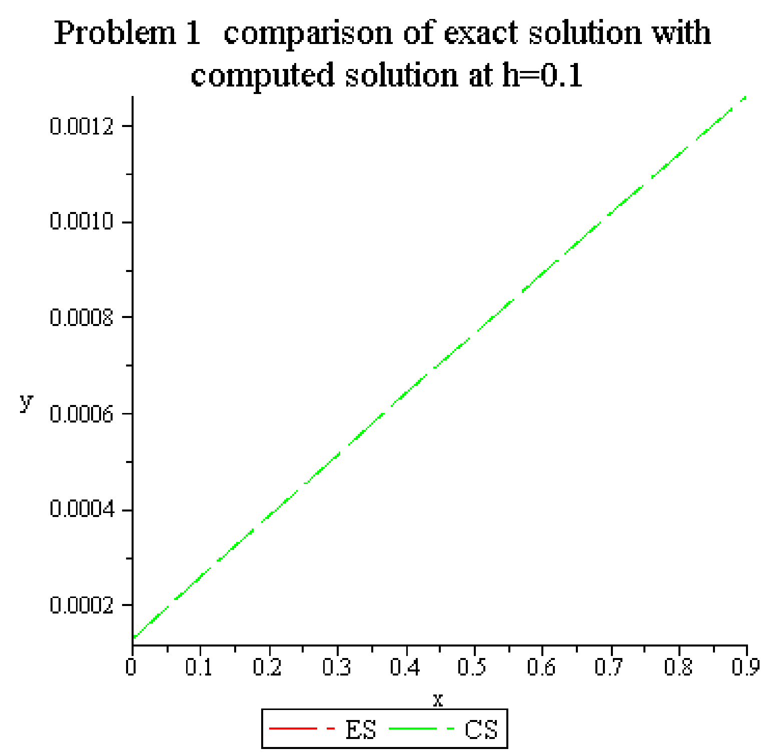

Figure 5: graphical curves for Problem I

UNDER REVIEW

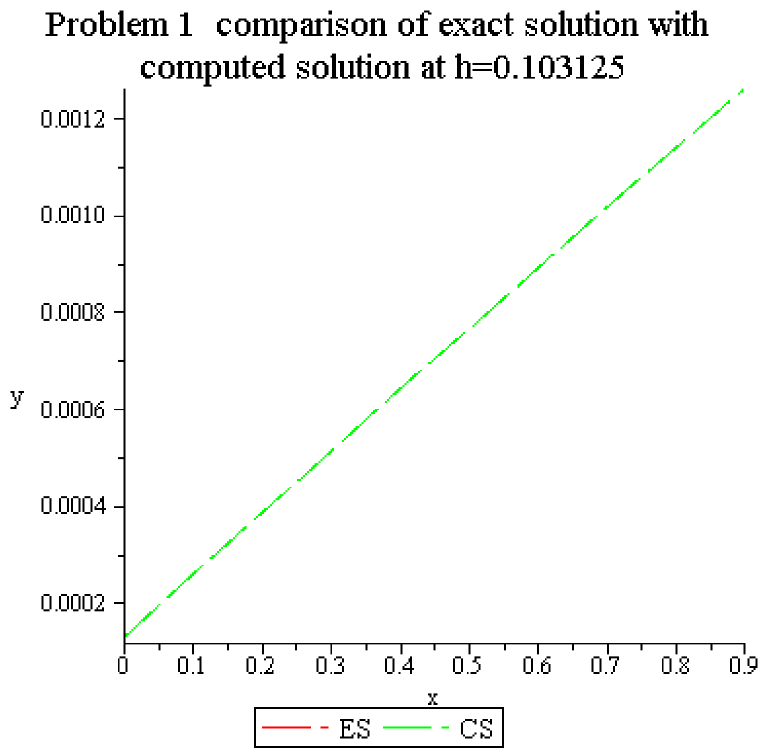

Figure 6: graphical curves for Problem I

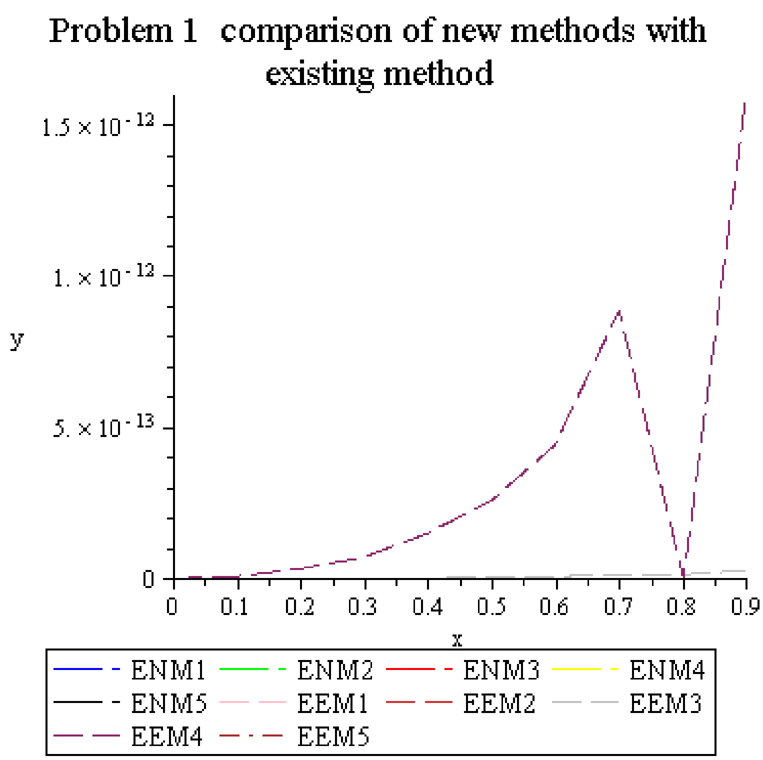

Figure 7: graphical curves for Problem I

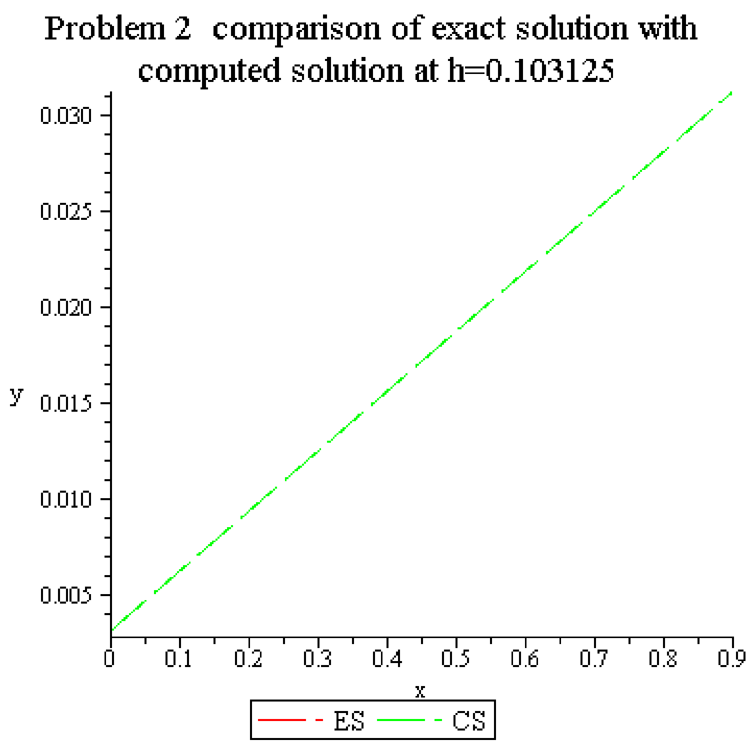

Figure 8: graphical curves for Problem II

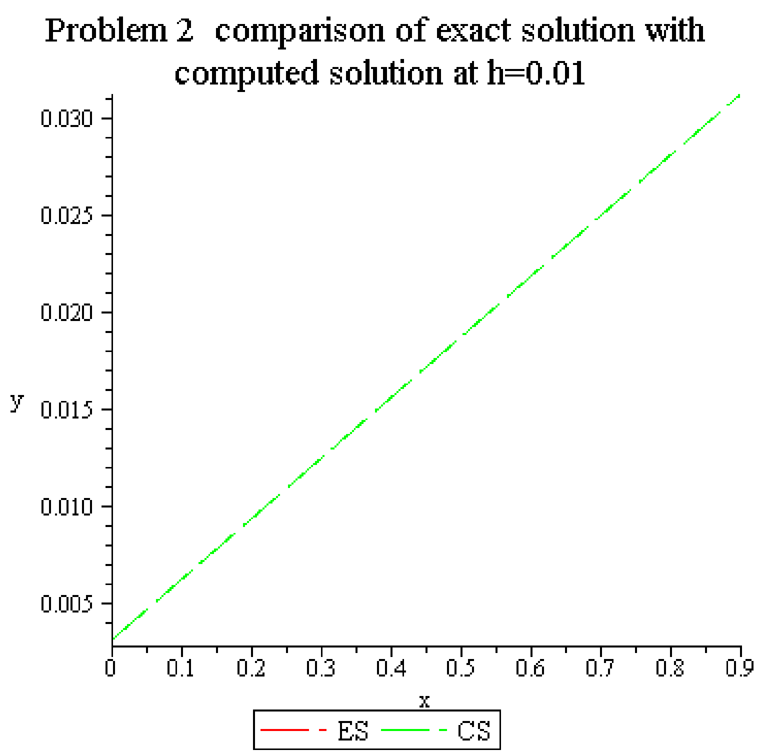

Figure 9: graphical curves for Problem II

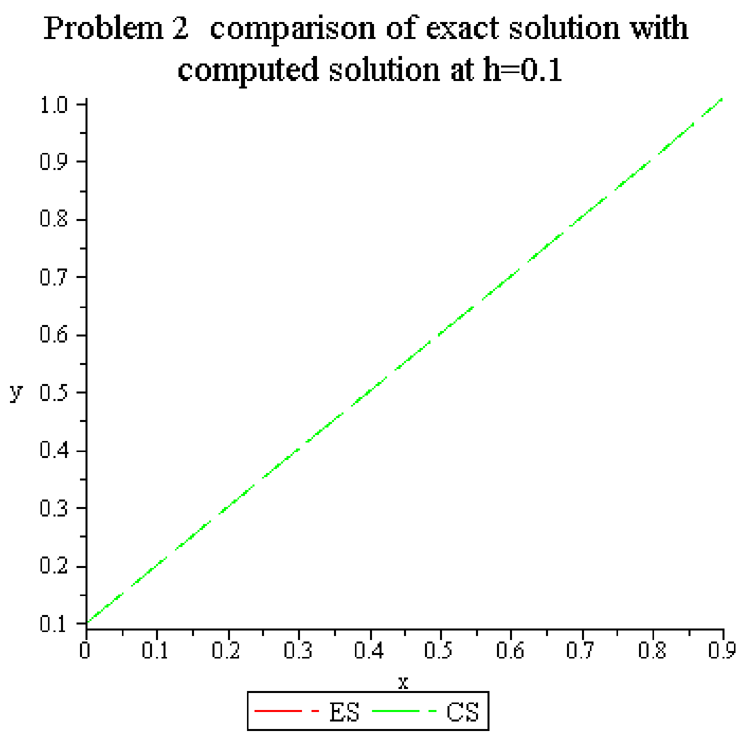

Figure 10: graphical curves for Problem II

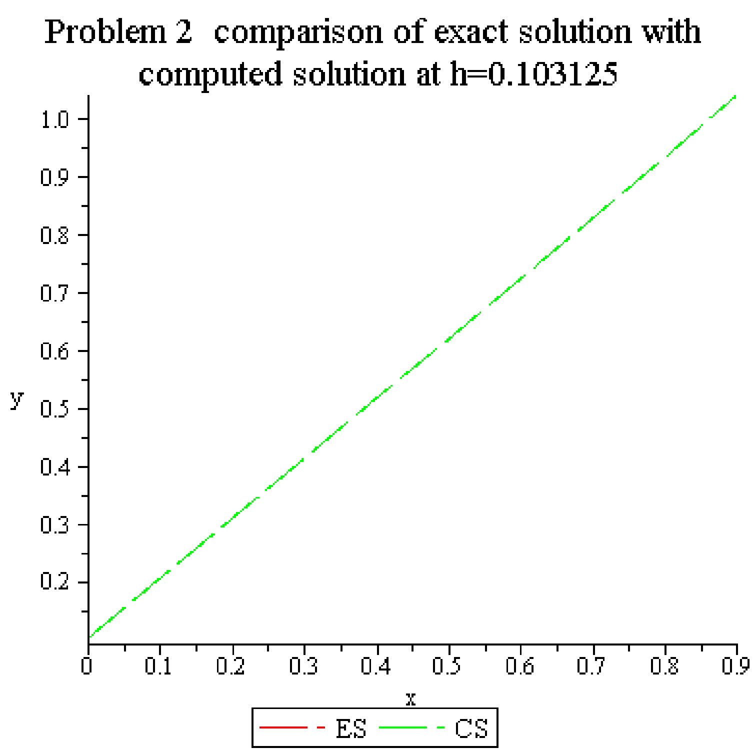

Figure 11: graphical curves for Problem II

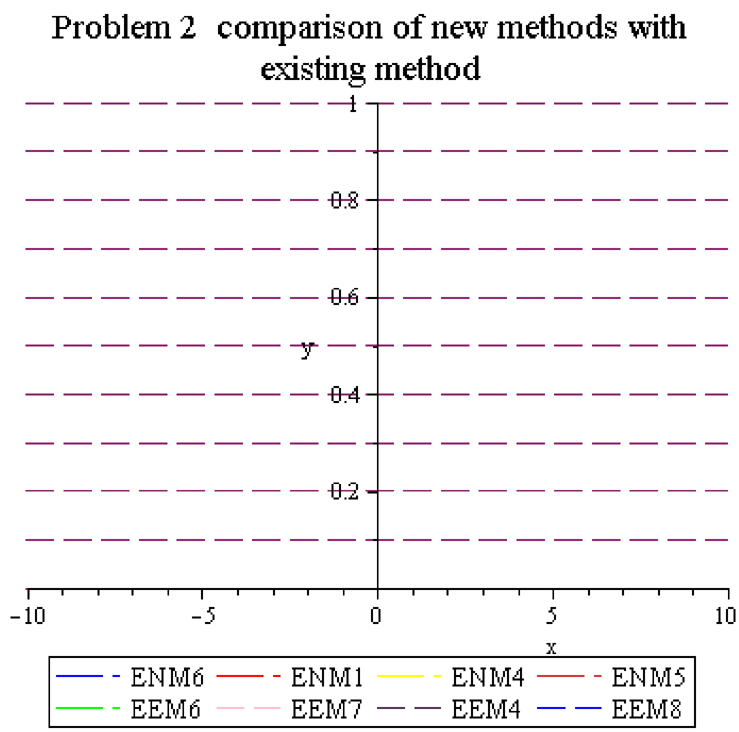

Figure 12: graphical curves for Problem II

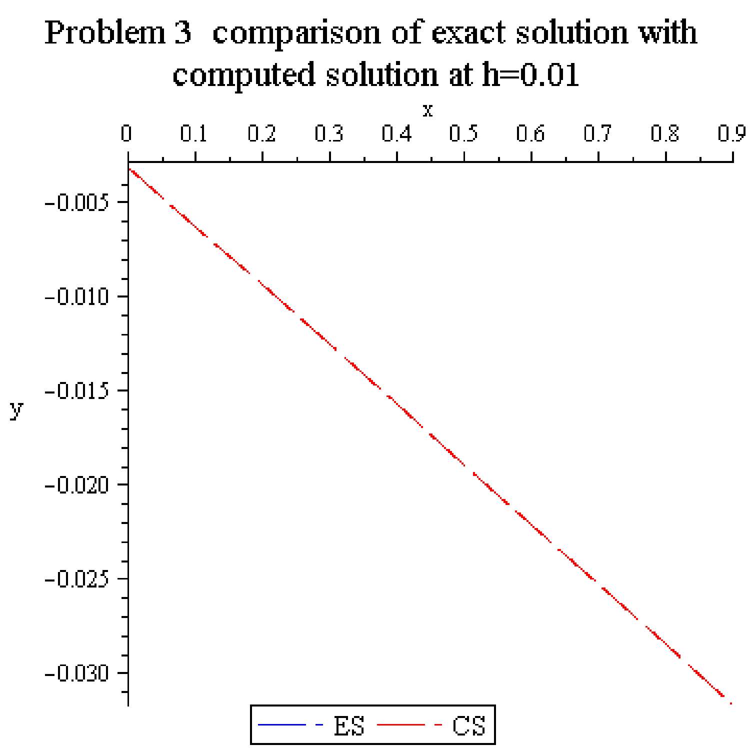

Figure 13: graphical curves for Problem III

UNDER REVIEW

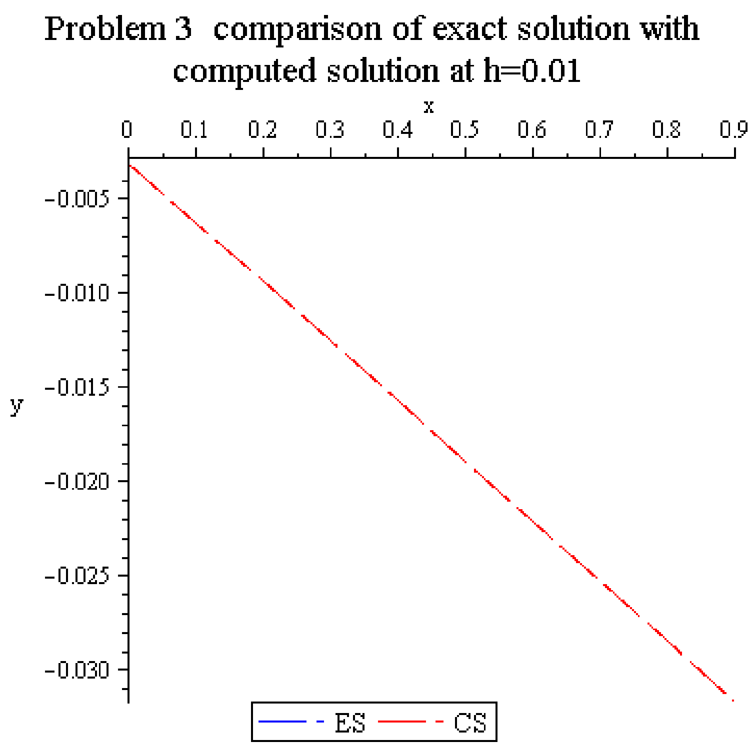

Figure 14: graphical curves for Problem III

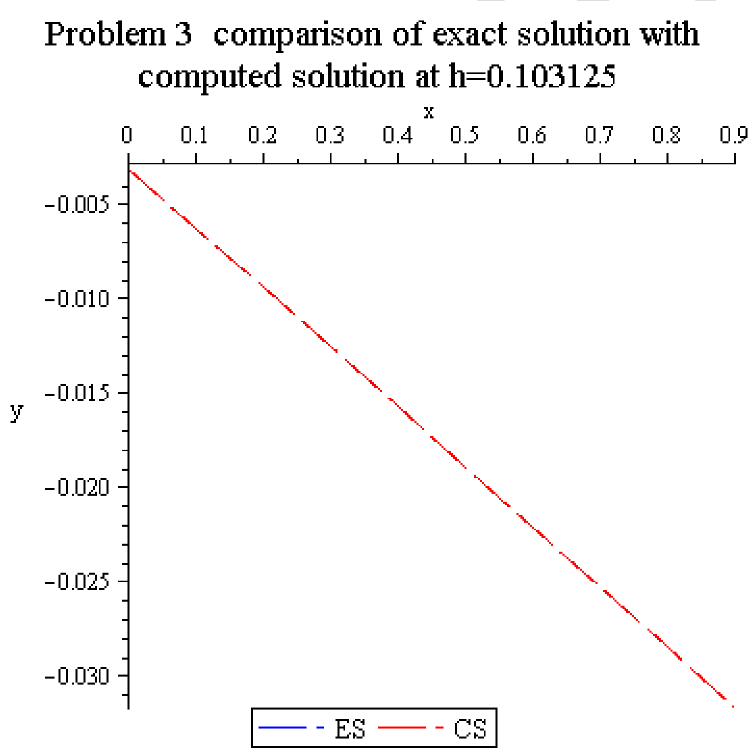

Figure 15: graphical curves for Problem III

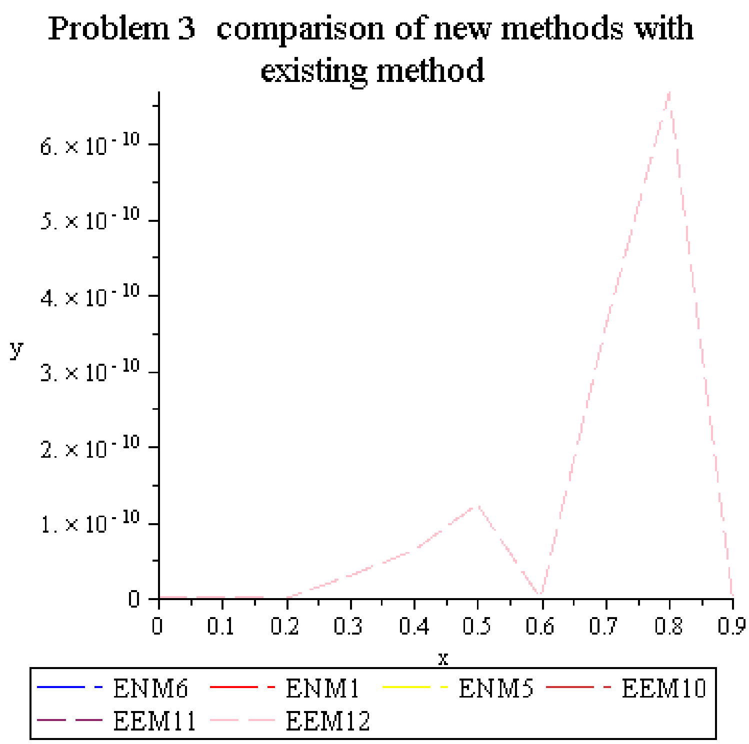

Figure 16: graphical curves for Problem III

The new method developed using the linear block algorithm was applied to solve three special fourth order IVPs (1). Problem I was solved using the step size $h = 0.1, 0.05, 0.025, 0.01, 0.103125$ and compared with [11], who developed an implicit five-step block method for solving (1) using the step size $h = 0.1, 0.05, 0.025, 0.01$ and [14] using $h = 0.103125$. It is clear from table II that, the new method performs better and graphically shown. Figure 2 to 6 shown the exact solution and computed solution when solving problem I, where h varies. Figure 7 is the textual comparison of new method with existing methods. The new method was also applied on Problem II using $h = 0.003125, 0.01, 0.1, 0.103125$, while [15, 7, 11, 13, 16] solved the same problem using $h = 0.1$. From the table IV and figure 8 to 12, the new method is accurate. Finally, Problem III was solved with $h = 0.003125, 0.01, 0.103125$, while [11, 17] solved using $h = 0.103125$. The performance of the result is shown in table VI and figure 13 to 16. Hence, from the results presented in the tables above, the new method has shown better accuracy and faster convergence.

The textual curve of each problem is graphically shown; the ES converge to the CS. Therefore, the ES moves parallel to the CS on each problem.

5. Conclusion

The fourth order scheme is derived from general $k-step$ block algorithm using the LBA which is quite straight forward to adopt. The properties of the scheme were analyzed. Some special fourth order IVPs were directly implemented and compared with existing fourth order schemes.

UNDER PEER REVIEW

The new method proved better accuracy and faster convergence than the existing methods considered in this research. The graphical solution of the problems consider were textual shown.

Appendix

$$X=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -bh & -rh & 0 & mh & ph & uh & dh \\ \frac{(-bh)^2}{2!} & \frac{(-rh)^2}{2!} & 0 & \frac{(mh)^2}{2!} & \frac{(ph)^2}{2!} & \frac{(uh)^2}{2!} & \frac{(dh)^2}{2!} \\ \frac{(-bh)^3}{3!} & \frac{(-rh)^3}{3!} & 0 & \frac{(mh)^3}{3!} & \frac{(ph)^3}{3!} & \frac{(uh)^3}{3!} & \frac{(dh)^3}{3!} \\ \frac{(-bh)^4}{4!} & \frac{(-rh)^4}{4!} & 0 & \frac{(mh)^4}{4!} & \frac{(ph)^4}{4!} & \frac{(uh)^4}{4!} & \frac{(dh)^4}{4!} \\ \frac{(-bh)^5}{5!} & \frac{(-rh)^5}{5!} & 0 & \frac{(mh)^5}{5!} & \frac{(ph)^5}{5!} & \frac{(uh)^5}{5!} & \frac{(dh)^5}{5!} \\ \frac{(-bh)^6}{6!} & \frac{(-rh)^6}{6!} & 0 & \frac{(mh)^6}{6!} & \frac{(ph)^6}{6!} & \frac{(uh)^6}{6!} & \frac{(dh)^6}{6!} \end{pmatrix}, Y=\begin{pmatrix} \frac{(\tau h)^4}{4!} \\ \frac{(\tau h)^5}{5!} \\ \frac{(\tau h)^6}{6!} \\ \frac{(\tau h)^5}{5!} \\ \frac{(\tau h)^7}{7!} \\ \frac{(\tau h)^8}{8!} \\ \frac{(\tau h)^9}{9!} \end{pmatrix}, Z=\begin{pmatrix} \frac{(\tau h)^{4-q}}{(4-q)!} \\ \frac{(\tau h)^{5-q}}{(5-q)!} \\ \frac{(\tau h)^{6-q}}{(6-q)!} \\ \frac{(\tau h)^{7-q}}{(7-q)!} \\ \frac{(\tau h)^{8-q}}{(8-q)!} \\ \frac{(\tau h)^{9-q}}{(9-q)!} \\ \frac{(\tau h)^{10-q}}{(10-q)!} \end{pmatrix}$$

Computing the above equations to obtain the coefficients of the polynomial $\rho_\eta, \eta = -b, -r, 0, m, p, u, d$

Substituting $x = x_\eta + th$, the polynomial in (2) takes the form

$$p(x_\eta + th) = \alpha_m y_{\eta+m} + \alpha_p y_{\eta+p} + \alpha_u y_{\eta+u} + \alpha_d y_{\eta+d} + h^4\left(\beta_{-b} g_{\eta-b} + \beta_{-r} g_{\eta-r} + \beta_0 g_\eta + \beta_m g_{\eta+m} + \beta_p g_{\eta+p} + \beta_u g_{\eta+u} + \beta_d g_{\eta+d}\right) \qquad (4)$$

where

$\rho_0 = 1$

$\rho_m = \zeta$

$\rho_p = \frac{1}{2}\zeta^2$

$\rho_u = \frac{1}{6}\zeta^3$

$$\varpi_{-b} = \frac{h^4\zeta^5\begin{pmatrix} 3\zeta^5 - 5d\zeta^4 - 5m\zeta^4 - 5p\zeta^4 + 5r\zeta^4 - 5u\zeta^4 + 9dm\zeta^3 + 9dp\zeta^3 - 9dr\zeta^3 \\ +9du\zeta^3 + 9mp\zeta^3 - 9mr\zeta^3 + 9mu\zeta^3 - 9pr\zeta^3 + 9pu\zeta^3 - 9ru\zeta^3 - \\ 18dmp\zeta^2 + 18dmr\zeta^2 - 18dmu\zeta^2 + 18dpr\zeta^2 - 18dpu\zeta^2 + 18dru\zeta^2 + \\ 18mpr\zeta^2 - 18mpu\zeta^2 + 18mru\zeta^2 + 18pru\zeta^2 - 42dmpr\zeta + 42dmpu\zeta \\ -42dmru\zeta - 42dpru\zeta - 42mpru\zeta + 126dmpru \end{pmatrix}}{15120(b(b-r)(b+p)(b+m)(b+d)(b+u))}$$

UNDER PEER REVIEW

$$\varpi_{-r}=-\frac{h^4\zeta^5\left(\begin{array}{l}3\zeta^5+5b\zeta^4-5d\zeta^4-5m\zeta^4-5p\zeta^4-5u\zeta^4-9bd\zeta^3-9bm\zeta^3+9dm\zeta^3\\-9bp\zeta^3+9dp\zeta^3-9bu\zeta^3+9du\zeta^3+9mp\zeta^3+9mu\zeta^3+9pu\zeta^3+\\18bdm\zeta^2+18bdp\zeta^2+18bdu\zeta^2+18bmp\zeta^2-18dmp\zeta^2+18bmu\zeta^2-\\18dmu\zeta^2+18bpu\zeta^2-18dpu\zeta^2-18mpu\zeta^2-42bdmp\zeta-42bdmu\zeta\\-42bdpu\zeta-42bmpu\zeta+42dmpu\zeta+126bdmpu\end{array}\right)}{15120\left(r(r+u)(p+r)(m+r)(d+r)(b-r)\right)}$$

$$\varpi_0=\frac{h^4\zeta^4\left(\begin{array}{l}3\zeta^6+5b\zeta^5-5d\zeta^5-5m\zeta^5-5p\zeta^5+5r\zeta^5-5u\zeta^5-9bd\zeta^4-9bm\zeta^4+9dm\zeta^4\\-9bp\zeta^4+9br\zeta^4+9dp\zeta^4-9dr\zeta^4-9bu\zeta^4+9du\zeta^4+9mp\zeta^4-9mr\zeta^4+9mu\zeta^4\\-9pr\zeta^4+9pu\zeta^4-9ru\zeta^4+18bdm\zeta^3+18bdp\zeta^3-18bdr\zeta^3+18bdu\zeta^3+18bmp\zeta^3\\-18bmr\zeta^3-18dmp\zeta^3+18dmr\zeta^3+18bmu\zeta^3-18bpr\zeta^3-18dmu\zeta^3+18dpr\zeta^3\\+18bpu\zeta^3-18bru\zeta^3-18dpu\zeta^3+18dru\zeta^3+18mpr\zeta^3-18mpu\zeta^3+18mru\zeta^3\\+18pru\zeta^3-42bdmp\zeta^2+42bdmr\zeta^2-42bdmu\zeta^2+42bdpr\zeta^2-42bdpu\zeta^2+\\42bdru\zeta^2+42bmpr\zeta^2-42dmpr\zeta^2+42bmpu\zeta^2+42bmru\zeta^2+42dmpu\zeta^2\\-42dmru\zeta^2+42bpru\zeta^2-42dpru\zeta^2-42mpru\zeta^2-126bdmpr\zeta+126bdmpu\zeta\\-126bdmru\zeta-126bmpru\zeta+126dmpru\zeta+630bdmpru\end{array}\right)}{15120bdmpru}$$

$$\varpi_m=\frac{h^4\zeta^5\left(\begin{array}{l}-3\zeta^5-5b\zeta^4+5d\zeta^4+5p\zeta^4-5r\zeta^4+5u\zeta^4+9bd\zeta^3+9bp\zeta^3-9br\zeta^3\\-9dp\zeta^3+9dr\zeta^3+9bu\zeta^3-9du\zeta^3+9pr\zeta^3-9pu\zeta^3+9ru\zeta^3-\\18dbp\zeta^2+18bdr\zeta^2-18bdu\zeta^2+18bpr\zeta^2-18dpr\zeta^2-18bpu\zeta^2+\\18bru\zeta^2+18dpu\zeta^2-18dru\zeta^2-18pru\zeta^2-42dbpr\zeta+42bdpu\zeta\\-42bdru\zeta-42bpru\zeta+42dpru\zeta+126bdpru\end{array}\right)}{15120\left(m(m-u)(m+r)(m-p)(d-m)(b+m)\right)}$$

$$\varpi_p=-\frac{h^4\zeta^5\left(\begin{array}{l}-3\zeta^5-5b\zeta^4+5d\zeta^4+5m\zeta^4-5r\zeta^4+5u\zeta^4+9bd\zeta^3+9bm\zeta^3-9dm\zeta^3\\-9br\zeta^3+9dr\zeta^3+9bu\zeta^3-9du\zeta^3+9mr\zeta^3-9mu\zeta^3+9ru\zeta^3-\\18bdm\zeta^2+18bdr\zeta^2-18bdu\zeta^2+18bmr\zeta^2-18dmr\zeta^2-18bmu\zeta^2+\\18dmu\zeta^2+18bru\zeta^2-18dru\zeta^2-18mru\zeta^2-42bdmr\zeta+42bdmu\zeta\\-42bdru\zeta-42bmru\zeta+42dmru\zeta+126bdru\end{array}\right)}{15120\left(p(p-u)(p+r)(m-p)(d-p)(b+p)\right)}$$

UNDER PEER REVIEW

$$\varpi_u = \frac{h^4\zeta^5\begin{pmatrix} -3\zeta^5 - 5b\zeta^4 + 5d\zeta^4 + 5m\zeta^4 + 5p\zeta^4 + 5r\zeta^4 + 9bd\zeta^3 + 9bm\zeta^3 - 9dm\zeta^3 \\ +9bp\zeta^3 - 9br\zeta^3 - 9dp\zeta^3 + 9dr\zeta^3 - 9mp\zeta^3 + 9mr\zeta^3 + 9pr\zeta^3 - \\ 18bdm\zeta^2 + 18bdp\zeta^2 + 18bdr\zeta^2 - 18bmp\zeta^2 - 18bmr\zeta^2 + 18bmp\zeta^2 - \\ 18dmr\zeta^2 + 18bpr\zeta^2 - 18dpr\zeta^2 - 18mpr\zeta^2 - 42bdmp\zeta + 42bdmr\zeta \\ -42bdpr\zeta - 42bmpr\zeta + 42dmpr\zeta + 126bdru \end{pmatrix}}{15120(u(r+u)(p-u)(m-u)(d-u)(b+u))}$$

$$\varpi_d = -\frac{h^4\zeta^5\begin{pmatrix} -3\zeta^5 - 5b\zeta^4 + 5m\zeta^4 + 5p\zeta^4 + 5r\zeta^4 + 5u\zeta^4 + 9bm\zeta^3 + 9bp\zeta^3 - 9br\zeta^3 \\ +9bu\zeta^3 - 9mp\zeta^3 - 9mr\zeta^3 - 9mu\zeta^3 - 9pr\zeta^3 + 9pu\zeta^3 + 9ru\zeta^3 - \\ 18bdp\zeta^2 + 18bmr\zeta^2 - 18bmu\zeta^2 + 18bpr\zeta^2 - 18bpu\zeta^2 + 18bru\zeta^2 - \\ 18mpr\zeta^2 + 18mpu\zeta^2 - 18mru\zeta^2 - 18pru\zeta^2 - 42bdpr\zeta + 42bmpu\zeta \\ -42bmru\zeta - 42bpru\zeta + 42mpru\zeta + 126bmpru \end{pmatrix}}{15120(d(d-u)(d+r)(d-p)(d-m)(b+d))}$$

Now the higher derivatives of (3) is given by

$$\zeta_{-b} = -\frac{2h^4\zeta^5\begin{pmatrix} 3(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)!+ \\ 220\zeta^2(-q+5)!(-q+6)!(-q+8)!(-q+9)!(-q+10)!+ \\ 320\zeta^3(-q+5)!(-q+6)!(-q+7)!(-q+9)!(-q+10)!- \\ 8960\zeta^4(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+10)!+ \\ 30720\zeta^5(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!- \\ 42\zeta(-q+5)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! \end{pmatrix}}{5(h\zeta)^q((-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)!)}$$

$$\zeta_{-r} = \frac{2h^4\zeta^5\begin{pmatrix} 3(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)!+ \\ 180\zeta^2(-q+5)!(-q+6)!(-q+8)!(-q+9)!(-q+10)!+ \\ 4800\zeta^3(-q+5)!(-q+6)!(-q+7)!(-q+9)!(-q+10)!- \\ 34560\zeta^4(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+10)!+ \\ 92160\zeta^5(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!- \\ 26\zeta(-q+5)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! \end{pmatrix}}{315(h\zeta)^q((-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)!)}$$

UNDER PEER REVIEW

$$\zeta_0 = \frac{h^4\zeta^5 \left(\begin{array}{l} 9(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! - \\ 27\zeta(-q+4)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! - \\ 284\zeta^2(-q+4)!(-q+5)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! + \\ 2880\zeta^3(-q+4)!(-q+5)!(-q+6)!(-q+8)!(-q+9)!(-q+10)! - \\ 1536\zeta^4(-q+4)!(-q+5)!(-q+6)!(-q+7)!(-q+9)!(-q+10)! - \\ 92160\zeta^5(-q+4)!(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+10)! + \\ 368640\zeta^6(-q+4)!(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)! \end{array} \right)}{9(h\zeta)^q\left((-q+4)!(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)!\right)}$$

$$\zeta_m = -\frac{2h^4\zeta^5 \left(\begin{array}{l} -3(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! + \\ 212\zeta^2(-q+5)!(-q+6)!(-q+8)!(-q+9)!(-q+10)! - \\ 448\zeta^3(-q+5)!(-q+6)!(-q+7)!(-q+9)!(-q+10)! - \\ 6400\zeta^4(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+10)! + \\ 30720\zeta^5(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)! - \\ 6\zeta(-q+5)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! \end{array} \right)}{(h\zeta)^q\left((-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)!\right)}$$

$$\zeta_p = \frac{4h^4\zeta^5 \left(\begin{array}{l} -3(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! + \\ 320\zeta^2(-q+5)!(-q+6)!(-q+8)!(-q+9)!(-q+10)! - \\ 1280\zeta^3(-q+5)!(-q+6)!(-q+7)!(-q+9)!(-q+10)! - \\ 10240\zeta^4(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+10)! + \\ 61440\zeta^5(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)! + \\ 6\zeta(-q+5)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! \end{array} \right)}{5(h\zeta)^q\left((-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)!\right)}$$

$$\zeta_u = -\frac{2h^4\zeta^5 \left(\begin{array}{l} -3(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! + \\ 324\zeta^2(-q+5)!(-q+6)!(-q+8)!(-q+9)!(-q+10)! - \\ 2112\zeta^3(-q+5)!(-q+6)!(-q+7)!(-q+9)!(-q+10)! + \\ 11520\zeta^4(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+10)! + \\ 92160\zeta^5(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)! + \\ 10\zeta(-q+5)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! \end{array} \right)}{9(h\zeta)^q\left((-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)!\right)}$$

$$\zeta_d = -\frac{h^4\zeta^5 \left(\begin{array}{l} -3(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! + \\ 320\zeta^2(-q+5)!(-q+6)!(-q+8)!(-q+9)!(-q+10)! - \\ 2560\zeta^3(-q+5)!(-q+6)!(-q+7)!(-q+9)!(-q+10)! - \\ 10240\zeta^4(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+10)! + \\ 122880\zeta^5(-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)! + \\ 12\zeta(-q+5)!(-q+7)!(-q+8)!(-q+9)!(-q+10)! \end{array} \right)}{35(h\zeta)^q\left((-q+5)!(-q+6)!(-q+7)!(-q+8)!(-q+9)!(-q+10)!\right)}$$

UNDER PEER REVIEW

The first algorithm (2) is expanded together with the expression for the fourth derivatives as

$$\left.\begin{array}{l}
y_{n-b}=y_n-bhy'_n+\dfrac{(-bh)^2}{2!}y''_n+\dfrac{(-bh)^3}{3!}y'''_n+h^4\begin{pmatrix}\nu_{01}f_{n-b}+\nu_{02}f_{n-r}+\nu_{03}f_n+\nu_{04}f_{n+m}\\+\nu_{05}f_{n+p}+\nu_{06}f_{n+u}+\nu_{07}f_{n+d}\end{pmatrix}\\
y_{n-r}=y_n-rhy'_n+\dfrac{(-rh)^2}{2!}y''_n+\dfrac{(-rh)^3}{3!}y'''_n+h^4\begin{pmatrix}\nu_{01}f_{n-b}+\nu_{02}f_{n-r}+\nu_{03}f_n+\nu_{04}f_{n+m}\\+\nu_{05}f_{n+p}+\nu_{06}f_{n+u}+\nu_{07}f_{n+d}\end{pmatrix}\\
y_{n+m}=y_n+mhy'_n+\dfrac{(mh)^2}{2!}y''_n+\dfrac{(mh)^3}{3!}y'''_n+h^4\begin{pmatrix}\nu_{01}f_{n-b}+\nu_{02}f_{n-r}+\nu_{03}f_n+\nu_{04}f_{n+m}\\+\nu_{05}f_{n+p}+\nu_{06}f_{n+u}+\nu_{07}f_{n+d}\end{pmatrix}\\
y_{n+p}=y_n+phy'_n+\dfrac{(ph)^2}{2!}y''_n+\dfrac{(ph)^3}{3!}y'''_n+h^4\begin{pmatrix}\nu_{01}f_{n-b}+\nu_{02}f_{n-r}+\nu_{03}f_n+\nu_{04}f_{n+m}\\+\nu_{05}f_{n+p}+\nu_{06}f_{n+u}+\nu_{07}f_{n+d}\end{pmatrix}\\
y_{n+u}=y_n+uhy'_n+\dfrac{(uh)^2}{2!}y''_n+\dfrac{(uh)^3}{3!}y'''_n+h^4\begin{pmatrix}\nu_{01}f_{n-b}+\nu_{02}f_{n-r}+\nu_{03}f_n+\nu_{04}f_{n+m}\\+\nu_{05}f_{n+p}+\nu_{06}f_{n+u}+\nu_{07}f_{n+d}\end{pmatrix}\\
y_{n+d}=y_n+dhy'_n+\dfrac{(dh)^2}{2!}y''_n+\dfrac{(dh)^3}{3!}y'''_n+h^4\begin{pmatrix}\nu_{01}f_{n-b}+\nu_{02}f_{n-r}+\nu_{03}f_n+\nu_{04}f_{n+m}\\+\nu_{05}f_{n+p}+\nu_{06}f_{n+u}+\nu_{07}f_{n+d}\end{pmatrix}
\end{array}\right\}\qquad(5)$$

The second algorithm (3) is expanded to yield the higher derivatives as

$$\left.\begin{array}{l}
y'_{n-b}=y'_n-bh\,y''_n+\dfrac{(-bh)^2}{2!}y'''_n+h^3\begin{pmatrix}\kappa_{11}f_{n-b}+\kappa_{12}f_{n-r}+\kappa_{13}f_n+\kappa_{14}f_{n+m}\\+\kappa_{15}f_{n+p}+\kappa_{16}f_{n+u}+\kappa_{17}f_{n+d}\end{pmatrix}\\
y'_{n-r}=y'_n-rh\,y''_n+\dfrac{(-rh)^2}{2!}y'''_n+h^3\begin{pmatrix}\kappa_{11}f_{n-b}+\kappa_{12}f_{n-r}+\kappa_{13}f_n+\kappa_{14}f_{n+m}\\+\kappa_{15}f_{n+p}+\kappa_{16}f_{n+u}+\kappa_{17}f_{n+d}\end{pmatrix}\\
y'_{n+m}=y'_n+mh\,y''_n+\dfrac{(mh)^2}{2!}y'''_n+h^3\begin{pmatrix}\kappa_{11}f_{n-b}+\kappa_{12}f_{n-r}+\kappa_{13}f_n+\kappa_{14}f_{n+m}\\+\kappa_{15}f_{n+p}+\kappa_{16}f_{n+u}+\kappa_{17}f_{n+d}\end{pmatrix}\\
y'_{n+p}=y'_n-+ph\,y''_n+\dfrac{(ph)^2}{2!}y'''_n+h^3\begin{pmatrix}\kappa_{11}f_{n-b}+\kappa_{12}f_{n-r}+\kappa_{13}f_n+\kappa_{14}f_{n+m}\\+\kappa_{15}f_{n+p}+\kappa_{16}f_{n+u}+\kappa_{17}f_{n+d}\end{pmatrix}\\
y'_{n+u}=y'_n+uh\,y''_n+\dfrac{(uh)^2}{2!}y'''_n+h^3\begin{pmatrix}\kappa_{11}f_{n-b}+\kappa_{12}f_{n-r}+\kappa_{13}f_n+\kappa_{14}f_{n+m}\\+\kappa_{15}f_{n+p}+\kappa_{16}f_{n+u}+\kappa_{17}f_{n+d}\end{pmatrix}\\
y'_{n+d}=y'_n+dh\,y''_n+\dfrac{(dh)^2}{2!}y'''_n+h^3\begin{pmatrix}\kappa_{11}f_{n-b}+\kappa_{12}f_{n-r}+\kappa_{13}f_n+\kappa_{14}f_{n+m}\\+\kappa_{15}f_{n+p}+\kappa_{16}f_{n+u}+\kappa_{17}f_{n+d}\end{pmatrix}
\end{array}\right\}\qquad(6)$$

UNDER PEER REVIEW

$$\left.\begin{aligned}
y''_{n-b} &= y''_{n} - bh\,y'''_{n} + h^{2}\begin{pmatrix}\kappa_{21} f_{n-b} + \kappa_{22} f_{n-r} + \kappa_{23} f_{n} + \kappa_{24} f_{n+m} \\ + \kappa_{25} f_{n+p} + \kappa_{26} f_{n+u} + \kappa_{27} f_{n+d}\end{pmatrix} \\
y''_{n-r} &= y''_{n} - rh\,y'''_{n} + h^{2}\begin{pmatrix}\kappa_{21} f_{n-b} + \kappa_{22} f_{n-r} + \kappa_{23} f_{n} + \kappa_{24} f_{n+m} \\ + \kappa_{25} f_{n+p} + \kappa_{26} f_{n+u} + \kappa_{27} f_{n+d}\end{pmatrix} \\
y''_{n+m} &= y''_{n} + mh\,y'''_{n} + h^{2}\begin{pmatrix}\kappa_{21} f_{n-b} + \kappa_{22} f_{n-r} + \kappa_{23} f_{n} + \kappa_{24} f_{n+m} \\ + \kappa_{25} f_{n+p} + \kappa_{26} f_{n+u} + \kappa_{27} f_{n+d}\end{pmatrix} \\
y''_{n+p} &= y''_{n} + ph\,y'''_{n} + h^{2}\begin{pmatrix}\kappa_{21} f_{n-b} + \kappa_{22} f_{n-r} + \kappa_{23} f_{n} + \kappa_{24} f_{n+m} \\ + \kappa_{25} f_{n+p} + \kappa_{26} f_{n+u} + \kappa_{27} f_{n+d}\end{pmatrix} \\
y''_{n+u} &= y''_{n} + uh\,y'''_{n} + h^{2}\begin{pmatrix}\kappa_{21} f_{n-b} + \kappa_{22} f_{n-r} + \kappa_{23} f_{n} + \kappa_{24} f_{n+m} \\ + \kappa_{25} f_{n+p} + \kappa_{26} f_{n+u} + \kappa_{27} f_{n+d}\end{pmatrix} \\
y''_{n+d} &= y''_{n} + dh\,y'''_{n} + h^{2}\begin{pmatrix}\kappa_{21} f_{n-b} + \kappa_{22} f_{n-r} + \kappa_{23} f_{n} + \kappa_{24} f_{n+m} \\ + \kappa_{25} f_{n+p} + \kappa_{26} f_{n+u} + \kappa_{27} f_{n+d}\end{pmatrix}
\end{aligned}\right\} \tag{7}$$

$$\left.\begin{aligned}
y'''_{n-b} &= y'''_{n} + h\begin{pmatrix}\kappa_{31} f_{n-b} + \kappa_{32} f_{n-r} + \kappa_{33} f_{n} + \kappa_{34} f_{n+m} \\ + \kappa_{35} f_{n+p} + \kappa_{36} f_{n+u} + \kappa_{37} f_{n+d}\end{pmatrix} \\
y'''_{n-r} &= y'''_{n} + h\begin{pmatrix}\kappa_{31} f_{n-b} + \kappa_{32} f_{n-r} + \kappa_{33} f_{n} + \kappa_{34} f_{n+m} \\ + \kappa_{35} f_{n+p} + \kappa_{36} f_{n+u} + \kappa_{37} f_{n+d}\end{pmatrix} \\
y'''_{n+m} &= y'''_{n} + h\begin{pmatrix}\kappa_{31} f_{n-b} + \kappa_{32} f_{n-r} + \kappa_{33} f_{n} + \kappa_{34} f_{n+m} \\ + \kappa_{35} f_{n+p} + \kappa_{36} f_{n+u} + \kappa_{37} f_{n+d}\end{pmatrix} \\
y'''_{n+p} &= y'''_{n} + h\begin{pmatrix}\kappa_{31} f_{n-b} + \kappa_{32} f_{n-r} + \kappa_{33} f_{n} + \kappa_{34} f_{n+m} \\ + \kappa_{35} f_{n+p} + \kappa_{36} f_{n+u} + \kappa_{37} f_{n+d}\end{pmatrix} \\
y'''_{n+u} &= y'''_{n} + h\begin{pmatrix}\kappa_{31} f_{n-b} + \kappa_{32} f_{n-r} + \kappa_{33} f_{n} + \kappa_{34} f_{n+m} \\ + \kappa_{35} f_{n+p} + \kappa_{36} f_{n+u} + \kappa_{37} f_{n+d}\end{pmatrix} \\
y'''_{n+d} &= y'''_{n} + h\begin{pmatrix}\kappa_{31} f_{n-b} + \kappa_{32} f_{n-r} + \kappa_{33} f_{n} + \kappa_{34} f_{n+m} \\ + \kappa_{35} f_{n+p} + \kappa_{36} f_{n+u} + \kappa_{37} f_{n+d}\end{pmatrix}
\end{aligned}\right\} \tag{8}$$

The unknown coefficients of ν is given by $\nu_{\zeta j} = X^{-1}Y$ where

UNDER PEER REVIEW

$$y_{n-\frac{1}{4}},\begin{pmatrix}\nu_{01}\\\nu_{02}\\\nu_{03}\\\nu_{04}\\\nu_{05}\\\nu_{06}\\\nu_{07}\end{pmatrix}=\begin{pmatrix}\frac{15271}{928972800}\\-\frac{397}{2167603200}\\\frac{16039}{92897280}\\-\frac{7327}{185794560}\\\frac{4127}{232243200}\\-\frac{137}{26542080}\\\frac{2209}{3251404800}\end{pmatrix};\ y_{n-\frac{3}{4}},\begin{pmatrix}\nu_{01}\\\nu_{02}\\\nu_{03}\\\nu_{04}\\\nu_{05}\\\nu_{06}\\\nu_{07}\end{pmatrix}=\begin{pmatrix}\frac{87237}{11468800}\\\frac{6579}{80281600}\\\frac{5931}{1146880}\\\frac{2187}{2293760}\\-\frac{2673}{2867200}\\\frac{117}{327680}\\-\frac{2187}{40140800}\end{pmatrix};\ y_{n+\frac{1}{4}},\begin{pmatrix}\nu_{01}\\\nu_{02}\\\nu_{03}\\\nu_{04}\\\nu_{05}\\\nu_{06}\\\nu_{07}\end{pmatrix}=\begin{pmatrix}-\frac{4783}{928972800}\\\frac{41}{433520640}\\\frac{12203}{92897280}\\\frac{8959}{185794560}\\-\frac{709}{46448640}\\\frac{743}{185794560}\\-\frac{1627}{3251404800}\end{pmatrix};$$

$$y_{n+\frac{1}{2}},\begin{pmatrix}\nu_{01}\\\nu_{02}\\\nu_{03}\\\nu_{04}\\\nu_{05}\\\nu_{06}\\\nu_{07}\end{pmatrix}=\begin{pmatrix}-\frac{293}{3628800}\\\frac{13}{8467200}\\\frac{571}{362880}\\\frac{947}{725760}\\-\frac{461}{1814400}\\\frac{7}{103680}\\-\frac{107}{12700800}\end{pmatrix};\ y_{n+\frac{3}{4}},\begin{pmatrix}\nu_{01}\\\nu_{02}\\\nu_{03}\\\nu_{04}\\\nu_{05}\\\nu_{06}\\\nu_{07}\end{pmatrix}=\begin{pmatrix}-\frac{729}{2293760}\\\frac{477}{80281600}\\\frac{6903}{1146880}\\\frac{17253}{2293760}\\-\frac{729}{2867200}\\\frac{549}{2293760}\\-\frac{243}{8028160}\end{pmatrix};\ y_{n-1},\begin{pmatrix}\nu_{01}\\\nu_{02}\\\nu_{03}\\\nu_{04}\\\nu_{05}\\\nu_{06}\\\nu_{07}\end{pmatrix}=\begin{pmatrix}\frac{23}{28350}\\\frac{1}{66150}\\\frac{43}{2835}\\\frac{131}{5670}\\\frac{43}{14175}\\\frac{1}{810}\\-\frac{61}{793800}\end{pmatrix}$$

More so, the unknown coefficients of κ is given by $\kappa_{\zeta\, jq} = X^{-1}Z$ where

$$y'_{n-\frac{1}{4}},\begin{pmatrix}\kappa_{11}\\\kappa_{12}\\\kappa_{13}\\\kappa_{14}\\\kappa_{15}\\\kappa_{16}\\\kappa_{17}\end{pmatrix}=\begin{pmatrix}-\frac{27779}{77414400}\\\frac{5849}{1625702400}\\-\frac{15847}{5806080}\\\frac{11299}{15482880}\\-\frac{3233}{9676800}\\\frac{4537}{46448640}\\-\frac{437}{33868800}\end{pmatrix};\ y'_{n-\frac{3}{4}},\begin{pmatrix}\kappa_{11}\\\kappa_{12}\\\kappa_{13}\\\kappa_{14}\\\kappa_{15}\\\kappa_{16}\\\kappa_{17}\end{pmatrix}=\begin{pmatrix}-\frac{22923}{409600}\\-\frac{23643}{20070400}\\\frac{81}{35840}\\-\frac{2349}{81920}\\\frac{6723}{358400}\\-\frac{3699}{573440}\\\frac{2349}{2508800}\end{pmatrix};\ y'_{n+\frac{1}{4}},\begin{pmatrix}\kappa_{11}\\\kappa_{12}\\\kappa_{13}\\\kappa_{14}\\\kappa_{15}\\\kappa_{16}\\\kappa_{17}\end{pmatrix}=\begin{pmatrix}-\frac{6871}{77414400}\\\frac{2701}{1625702400}\\\frac{89}{45360}\\\frac{14639}{15482880}\\-\frac{2707}{9676800}\\\frac{673}{9289728}\\-\frac{611}{67737600}\end{pmatrix};$$

$$y'_{n+\frac{1}{2}},\begin{pmatrix}\kappa_{11}\\\kappa_{12}\\\kappa_{13}\\\kappa_{14}\\\kappa_{15}\\\kappa_{16}\\\kappa_{17}\end{pmatrix}=\begin{pmatrix}-\frac{349}{604800}\\\frac{139}{12700800}\\\frac{3887}{362880}\\\frac{1433}{120960}\\-\frac{59}{37800}\\\frac{167}{362880}\\-\frac{247}{4233600}\end{pmatrix};\ y'_{n+\frac{3}{4}},\begin{pmatrix}\kappa_{11}\\\kappa_{12}\\\kappa_{13}\\\kappa_{14}\\\kappa_{15}\\\kappa_{16}\\\kappa_{17}\end{pmatrix}=\begin{pmatrix}-\frac{567}{409600}\\\frac{513}{20070400}\\\frac{1863}{71680}\\\frac{23409}{573440}\\\frac{1377}{358400}\\\frac{657}{573440}\\-\frac{81}{627200}\end{pmatrix};\ y'_{n+1},\begin{pmatrix}\kappa_{11}\\\kappa_{12}\\\kappa_{13}\\\kappa_{14}\\\kappa_{15}\\\kappa_{16}\\\kappa_{17}\end{pmatrix}=\begin{pmatrix}-\frac{7}{2700}\\\frac{19}{396900}\\\frac{1093}{22680}\\\frac{47}{540}\\\frac{233}{9450}\\\frac{107}{11340}\\-\frac{29}{264600}\end{pmatrix}$$

UNDER PEER REVIEW

$$y''_{n-\frac{1}{4}},\begin{pmatrix}\kappa_{21}\\ \kappa_{22}\\ \kappa_{23}\\ \kappa_{24}\\ \kappa_{25}\\ \kappa_{26}\\ \kappa_{27}\end{pmatrix}=\begin{pmatrix}\frac{4183}{645120}\\ -\frac{731}{13547520}\\ \frac{30389}{967680}\\ -\frac{2171}{215040}\\ \frac{761}{161280}\\ -\frac{2693}{1935360}\\ \frac{139}{752640}\end{pmatrix};\ y''_{n-\frac{3}{4}},\begin{pmatrix}\kappa_{21}\\ \kappa_{22}\\ \kappa_{23}\\ \kappa_{24}\\ \kappa_{25}\\ \kappa_{26}\\ \kappa_{27}\end{pmatrix}=\begin{pmatrix}\frac{23139}{71680}\\ \frac{1359}{100352}\\ -\frac{8097}{35840}\\ \frac{21843}{71680}\\ -\frac{675}{3584}\\ \frac{651}{10240}\\ -\frac{459}{50176}\end{pmatrix};\ y''_{n+\frac{1}{4}},\begin{pmatrix}\kappa_{21}\\ \kappa_{22}\\ \kappa_{23}\\ \kappa_{24}\\ \kappa_{25}\\ \kappa_{26}\\ \kappa_{27}\end{pmatrix}=\begin{pmatrix}-\frac{239}{215040}\\ \frac{289}{13547520}\\ \frac{2867}{138240}\\ \frac{1879}{129024}\\ -\frac{41}{10752}\\ \frac{1879}{1935360}\\ -\frac{271}{2257920}\end{pmatrix};$$

$$y''_{n+\frac{1}{2}},\begin{pmatrix}\kappa_{21}\\ \kappa_{22}\\ \kappa_{23}\\ \kappa_{24}\\ \kappa_{25}\\ \kappa_{26}\\ \kappa_{27}\end{pmatrix}=\begin{pmatrix}-\frac{13}{5040}\\ \frac{1}{21168}\\ \frac{1453}{30240}\\ \frac{19}{240}\\ \frac{1}{1008}\\ \frac{23}{15120}\\ -\frac{1}{4704}\end{pmatrix};\ y''_{n+\frac{3}{4}},\begin{pmatrix}\kappa_{21}\\ \kappa_{22}\\ \kappa_{23}\\ \kappa_{24}\\ \kappa_{25}\\ \kappa_{26}\\ \kappa_{27}\end{pmatrix}=\begin{pmatrix}-\frac{297}{71680}\\ \frac{39}{501760}\\ \frac{2703}{35840}\\ \frac{10719}{71680}\\ \frac{27}{512}\\ \frac{117}{14336}\\ -\frac{27}{50176}\end{pmatrix};\ y''_{n+1},\begin{pmatrix}\kappa_{21}\\ \kappa_{22}\\ \kappa_{23}\\ \kappa_{24}\\ \kappa_{25}\\ \kappa_{26}\\ \kappa_{27}\end{pmatrix}=\begin{pmatrix}-\frac{1}{210}\\ \frac{1}{13230}\\ \frac{187}{1890}\\ \frac{143}{630}\\ \frac{11}{105}\\ \frac{19}{270}\\ \frac{8}{2205}\end{pmatrix}$$

$$y'''_{n-\frac{1}{4}},\begin{pmatrix}\kappa_{31}\\ \kappa_{32}\\ \kappa_{33}\\ \kappa_{34}\\ \kappa_{35}\\ \kappa_{36}\\ \kappa_{37}\end{pmatrix}=\begin{pmatrix}-\frac{7513}{80640}\\ \frac{863}{1693440}\\ -\frac{6413}{30240}\\ \frac{6857}{80640}\\ -\frac{103}{2520}\\ \frac{2951}{241920}\\ -\frac{23}{14112}\end{pmatrix};\ y'''_{n-\frac{3}{4}},\begin{pmatrix}\kappa_{31}\\ \kappa_{32}\\ \kappa_{33}\\ \kappa_{34}\\ \kappa_{35}\\ \kappa_{36}\\ \kappa_{37}\end{pmatrix}=\begin{pmatrix}-\frac{10881}{8960}\\ -\frac{1607}{12544}\\ \frac{1753}{1120}\\ -\frac{15471}{8960}\\ \frac{297}{280}\\ -\frac{3211}{8960}\\ \frac{81}{1568}\end{pmatrix};\ y'''_{n+\frac{1}{4}},\begin{pmatrix}\kappa_{31}\\ \kappa_{32}\\ \kappa_{33}\\ \kappa_{34}\\ \kappa_{35}\\ \kappa_{36}\\ \kappa_{37}\end{pmatrix}=\begin{pmatrix}-\frac{649}{80640}\\ \frac{271}{1693440}\\ \frac{3683}{30240}\\ \frac{12953}{80640}\\ -\frac{79}{2520}\\ \frac{1879}{241920}\\ \frac{67}{70560}\end{pmatrix};$$

$$y'''_{n+\frac{1}{2}},\begin{pmatrix}\kappa_{31}\\ \kappa_{32}\\ \kappa_{33}\\ \kappa_{34}\\ \kappa_{35}\\ \kappa_{36}\\ \kappa_{37}\end{pmatrix}=\begin{pmatrix}-\frac{19}{5040}\\ \frac{1}{21168}\\ \frac{739}{7560}\\ \frac{1571}{5040}\\ \frac{31}{315}\\ -\frac{67}{15120}\\ \frac{1}{3528}\end{pmatrix};\ y'''_{n+\frac{3}{4}},\begin{pmatrix}\kappa_{31}\\ \kappa_{32}\\ \kappa_{33}\\ \kappa_{34}\\ \kappa_{35}\\ \kappa_{36}\\ \kappa_{37}\end{pmatrix}=\begin{pmatrix}-\frac{81}{8960}\\ \frac{13}{62720}\\ \frac{137}{1120}\\ \frac{2241}{8960}\\ \frac{81}{280}\\ \frac{901}{8960}\\ -\frac{27}{7840}\end{pmatrix};\ y'''_{n+\frac{1}{4}},\begin{pmatrix}\kappa_{31}\\ \kappa_{32}\\ \kappa_{33}\\ \kappa_{34}\\ \kappa_{35}\\ \kappa_{36}\\ \kappa_{37}\end{pmatrix}=\begin{pmatrix}\frac{2}{315}\\ -\frac{2}{6615}\\ \frac{107}{1890}\\ \frac{122}{315}\\ \frac{34}{315}\\ \frac{346}{945}\\ \frac{67}{882}\end{pmatrix}$$

UNDER PEER REVIEW

References

[1] A. Adeyeye & Z. Omar, "Solving fourth order linear initial and boundary value problems using an implicit block method", Proceeding of the Third International Conference on Computing, Mathematics and Statistics (2019) 168.

http://10.1007/978-981-13-7279_21.

[2] D. Raymond, T. Y. Kyagya, J. Sabo & A. Lydia, "Numerical application of higher order linear block scheme on testing some modeled problems of fourth order problem", Journal of Nigerian Society of Physical (African Scientific Reports) **2** (2023) 2.

DOI:10.46481/asr.2022.2.1.67.

[3] J. Sabo, Y. Skwame & J. Z. Donald, "On the simulation of higher order linear block algorithm for modelling fourth order initial value problems", Asian Research Journal of Mathematics **18** (2022) 23.

[4] O. O. Olaiya, M. I. Modebei & S. A. Bello, "A one-step block hybrid integrator for solving fifth order Korteweg-de vries equaions", Journal of Nigerian Society of Physical **4** (2022) 2. DOI:10.46481/jnsps.2022.797.

[5] L. J. Kwari, J. Sunday, J. N. Ndam, A. Shokri & Y. Wang, "On the simulations of second-order oscillatory problems with applications to physical systems", Axioms **12** (2023) 9, 10. https://doi.org/10.3390/axioms12030282.

[6] M. I. Modebei, O. O. Olaiya & I. P. Ngwongwo, "Computational study of some three step hybrid integrators for solution of third order ordinary differential equations", Journal of Nigerian Society of Physical Sciences **3** (2021) 459.

[7] V. O. Atabo and S. O. Adee, "A new special 15-step block method for solving general fourth ordinary differential equations", Journal of Nigerian Society of Physical Sciences **3** (2021) 308, 331. DOI:10.46481/jnsps.2021.337.

[8] M. Kida, S. Adamu, O. O. Aduroja & T. P. Pantuvo, "Numerical solution of stiff and oscillatory problems using third derivative trigonometrically fitted block method", Journal of Nigerian Society of Physical Sciences **4** (2022) 34.

[9] J. Sunday, G. M. Kumleng, N. M. Kamoh, J. A. Kwanamu, Y. Skwame & O. Sarjiyus, "Implicit four-point hybrid block integrator for the simulations of stiff models Journal of Nigerian Society of Physical Sciences **4** (2022) 287-296. DOI:10.46481/jnsps.2022.777.

UNDER PEER REVIEW

[10] R. Abdelrahim, “four step hybrid block method for the direct solution of fourth order ordinary differential equations”, Int. J. Nonlinear Anal. Appl. **12** (2021) 216.

[11] O. Adeyeye & Z. Omar, “Implicit five-step block method with generalised equidistant points for solving fourth order linear and non-linear initial value problems”, Ain Shams Engineering Journal **10** (2019) 1, 5, 6.

[12] O. Adeyeye & Z. Omar, “A new algorithm for developing block methods for solving fourth order ordinary differential equations”, Global Journal of Pure and Applied Mathematics, **12** (2016) 1466.

[13] L. A. Ukpebor, E.O. Omole & L.O. Adoghe, “An order four numerical scheme for fourth-order initial value problems using lucas polynomial with application in ship dynamics”, International Journal of Mathematical Research **9** (2020) 35.

[14] L. O. Adoghe, and E. O. Omole, “A two-step hybrid block method for the numerical integration of higher order intial value problem of ordinary differential equations”, World science news. An International Scientific Journal **118** (2019) 246.

[15] J. O. Kuboye, O. R. Elusakin, O. F. Quadri, “Numerical algorithm for the direct solution of fourth order ordinary differential equation”, Nigerian Society of Physical Sciences **2** (2022) 218. DOI:10.46481/jnsps.2020.100.

[16] N. Ahamad & S. Charan, “Study of numerical solution of fourth order ordinary differential equations by fifth order Runge-Kutta method”, Int. J. Sci. Res. Sci. Eng.Technol **6** (2019) 235, 236.

[17] A. D. Familua & E. O. Omole, “Five points mono hybrid point linear multistep method for solving nth order ordinary differential equations using power series function” Asian Research Journal of Mathematics **3** (2017) 13, 14.

UNDER PEER REVIEW