Original Research Article

A NEW LIFE-TIME MODEL WITH BATHTUB AND INVERTED BATHTUB FAILURE RATE FUNCTION.

ABSTRACT

In this paper, a three parameter model called Zubair- Kumaraswamy (Z-Kum) distribution is proposed. The extension was done using Zubair G-Family (2018) of continuous probability distribution to extend the well known Kumaraswamy distribution to make it more flexible in modeling and predicting real world phenomenon. Some basic structural properties of the new distributions like cdf, pdf, quantile functions, moments, moment generating functions, characteristics functions and order statistics was obtained. Survival function, hazard function, reversed hazard rate function and a cumulative hazard rate function was also obtained. Behavior of the hazard rate plot exhibit increase, decrease, Bathtub and inverted Bathtub shape. Maximum likelihood estimate was used to estimate the Z-Kum distribution parameters. Monte Carlo simulation was carried out to evaluate the performance of MLE method adopted. Result of the simulation studies indicates that MLE is good for the estimation of our distribution parameters. To compare the proposed model with the other fitted existing models, analytical measure of goodness of fit of some information criterion was considered using three real life data sets. From the results obtained, it is evident that our proposed model gives better fit than the other competing models and therefore, our proposed model provide greater flexibility in modeling real world phenomenon.

Key Words: Kumaraswamy, Bathtub, Quantile Function, Moments, Hazard rate

1. Introduction.

The quality of the procedures used in statistical analysis depends heavily on the assumed probability model or distribution that the random variable follows. Many lifetime data used for statistical analysis follow a particular probability distribution and therefore knowledge of the appropriate distribution that any phenomenon follows greatly improves the sensitivity, reliability and efficiency of the statistical analysis associated with it. Furthermore, it is true that several probability distributions exist for modeling lifetime data; however, some of these lifetime data do not follow any of the existing and well known standard probability distributions (models) or at least are inappropriately described by them. Due to change in the world population and rapid global development in science and technology, developing new distributions which could better describe some of these phenomena and provide greater flexibility and wider acceptability in the modeling of lifetime data become an inevitable.

Recently, attempts have been made to define new models that extend well known distributions and provide a greater flexibility in modeling real data and to improve the goodness-of-fit of the generated family. For instance, Eugene *et al*. (2002) introduced a new class of distributions generated from the beta distribution. Zografos and Balakrishnan (2009) proposed the gamma generated family. Bourguignon *et al*. (2014) presented the Weibull-generated (*W*-*G*) family of distributions with two extra parameters, Kummer beta generalized family of distributions by Pescim *et al.* (2012), geometric exponential-Poisson family of distributions by Nadarajah *et al*. (2013), exponentiated T-X family of distributions by Alzaghal et al.(2013), weibull generalized family of distributins by Bourguignion *et al.* (2014); modified beta generalized family distributions by Nadarajah *et al*. (2014), kumaraswamy Weibul by Cordeiro et al. (2010) kumaraswamy gumbel by Cordeiro *et al.* (2012), Kumaraswamy Birnbaum-Saunders by Saulo and Bourguibnon (2012), Kumaraswamy Pareto by Bourguibnon *et al.* (2013), Kumaraswamy generalized Rayleigh by Gomes *et al.* (2014). Kumaraswamy inverse by Rayleigh Roges et al. (2014), Kumaraswamy modified inverse Weibull by Aryal and Elbatal (2015), Kumaraswamy Laplace by Aryal (2015), Kumaraswamy exponential-Weibull by Cordeiro *et al.* (2016), Kumaraswamy exponentiated inverse Rayleigh by Haq (2016);

2.0 Some Existing Probability Distributions

2.1 Kumarasawamy Distribution

The pdf and cdf of Kumarasawamy distribution are as given in (1) and (2) respectively.

$$G(x;a,b) = 1-\left(1-x^{a}\right)^{b}, \qquad 0<x<1,\ a>0, b>0. \qquad , \qquad (1)$$

$$g(x;a,b) = abx^{a-1}\left(1-x^{a}\right)^{b-1} \quad , \quad 0<x<1,\ a>0, b>0. \qquad (2)$$

where a and b are two shape parameters.

2.2 Zubair G Family of distribution

A family of life distributions, called the Zubair-G family was introduced by Zubair (2018). The benefit of using this family is that its cdf has a closed form solution and capable of data modeling with monotonic and non-monotonic failure rates. The CDF and PDF of the new family defined by Zubair (2018) for random Variable X is given in (3) and (4) respectively.

$$F(x,\alpha\varphi)=\frac{exp\{\alpha G\left(x;\varphi\right)^{2}\}-1}{\exp(\alpha)-1}, \qquad \alpha>0,\ x\varepsilon\Re. \qquad (3)$$

$$f(x,\alpha,\varphi)=\frac{2\alpha g\left(x;\phi\right)G\left(x;\varphi\right)exp\{\alpha G\left(x;\varphi\right)^{2}\}}{\exp(\alpha)-1}, \qquad \alpha>0, x\varepsilon\Re, \qquad (4)$$

Where φ is vector of the baseline distribution parameter, α is the parameter of Zubair G-family $g\left(x;\phi\right)$ and $G\left(x;\varphi\right)$ are pdf and cdf of the baseline distribution respectively.

3.0 Proposed Distribution.

3.1 Zubair-Kumarasawamy (Z-Kum) Distribution

To obtain the CDF of Z-Kum distribution, we let $G(x)$ be cdf of the Kumarasawamy random variable given by $G(x;a,b)=1-\left(1-x^{a}\right)^{b}$ and substitute in (3). Then the CDF of Z-Kw distribution is obtained as in (4)

$g_{(x;a,b)}=abx^{a-1}\left(1-x^{a}\right)^{b-1}$ then the cdf of Zubair–Kumarasawamy (Z-Kum) distribution is given by

$$F(x;\alpha,a,b)_{Z-Kum}=\frac{\exp\left(\alpha(1-(1-x^{a})^{b})^{2}\right)-1}{\exp(\alpha)-1} \quad 0<x<1,\ a>0,b>0. \qquad (4)$$

The corresponding probability density function (pdf) of Zubair–Kumarasawamy (Z-Kum) distribution denoted by $f(x;\alpha,a,b)_{Z-Kw}$ is obtained by differentiating (4) with respect to x.

From the definition

$$f(x;\alpha,a,b)=\frac{dF(x;\alpha,a,b)}{dx} \qquad (5)$$

To obtain $\frac{dF(x;\alpha,a,b)_{Z-Bx}}{dx}$

let, $U=\left(1-(1-x^{a})^{b}\right)$ (6)

So, $F(x;\alpha,a,b)=\frac{\exp\left(\alpha U^{2})-1\right)}{\exp(\alpha)-1}$ (7)

$$\frac{du}{dx}=abx^{a-1}(1-x^{a})^{b-1} \quad (8)$$

Then,

$$\frac{dF(x;\alpha,a,b)}{dx}=\frac{du}{dx}.\frac{dF(x;\alpha,a,b)}{du} \quad (9)$$

$$\frac{dF(x;\alpha,a,b)}{du}=\frac{2u\exp\left(\alpha(1-(1-x^{a})^{b})^{2}\right)}{\exp(\alpha)-1} \quad (10)$$

Substitute (6), (8) and (10) in (9), $f(x;\alpha,a,b)_{Z-Kum}$ is obtained as in (11).

$$f(x;\alpha,a,b)_{Z-Kum}=\frac{2\alpha abx^{a-1}\left(\left(1-x^{a}\right)^{b-1}(1-(1-x^{a})^{b})\right)\exp\left(\alpha(1-(1-x^{a})^{b})^{2}\right)}{\exp(\alpha)-1} \quad (11)$$

$0<x<1,\ a>0,b>0.$

Theorem 3.1 $f(x;\alpha,a,b)_{Z-Kum}$ is a pdf. Then $\int_{0}^{1}f(x;\alpha,a,b)_{Z-Kum}dx=1$ (12)

Proof: Let $T_{2}=\int_{0}^{1}f(x;\alpha,a,b)_{Z-Kum}dx$, then

$$T_{2}=\int_{0}^{1}\frac{2\alpha abx^{a-1}\left(\left(1-x^{a}\right)^{b-1}(1-(1-x^{a})^{b})\right)\exp\left(\alpha(1-(1-x^{a})^{b})^{2}\right)}{\exp(\alpha)-1}dx \quad (13)$$

$$T_{2}=\frac{2\alpha ab}{-1+\exp(\alpha)}.\int_{0}^{1}\left((1-(1-x^{a})^{b})+(1-x^{a})^{-1+b}\right)\exp\left(\alpha\left(-(1-x^{a})^{b}+1\right)^{2}\right)x^{-1+a}dx \quad (14)$$

Applying u substitution method by let $u_{2}=1-x^{a},\ \frac{du_{2}}{dx}=-ax^{a-1},dx=\frac{du}{-ax^{a-1}}\ and\ 0<u<1$

$$T_{2}=\frac{2\alpha ab}{-1+\exp(\alpha)}\int_{0}^{1}-\frac{(u_{2}^{\ b-1}-u_{2}^{\ b})\exp\left(\alpha(-u_{2}^{\ b}+1)^{2}\right)}{a}du \quad (15)$$

$$By\,letting\ V=1-u_2^b, \frac{dv}{du_2}=-bu_2^{b-1},\ du_2=-bu_2^{b-1}dv\ \ and\ 0<v<1$$

$$T_2=\frac{2\alpha ab}{-1+\exp(\alpha)}\left(\frac{1}{a}.\int_0^1-\frac{\exp(\alpha v^2)v}{b}dv\right) \tag{16}$$

Let $w=\alpha v^2, \frac{dw}{dv}=2\alpha v, dv=\frac{dw}{2ab}$ *and* $0<w<\alpha$

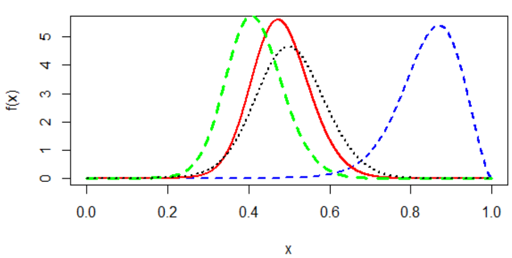
$$T_2=\frac{2\alpha ab}{-1+\exp(\alpha)}\left(\frac{1}{a}.\frac{1}{b}.\int_0^\alpha\frac{\exp(w)}{2\alpha}dw\right) \tag{17}$$

$$T_2=\frac{2\alpha ab}{-1+\exp(\alpha)}\left(\frac{1}{a}.\frac{1}{b}.\frac{1}{2\alpha}\left(\exp(w)\right)_0^\alpha\right) \tag{18}$$

$$=T_2=\frac{\left(\exp(w)\right)_0^\alpha}{-1+\exp(\alpha)}=1 \tag{19}$$

Hence, the proof.

Figures **1** and **2** below displayed the plots of the *pdf* and *cdf* of the Z-Bx distribution for some selected parameter values respectively.



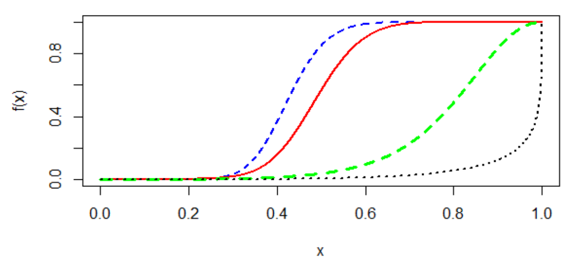


Figure 1: Plot of Z-Kw PDF

Figure 2: Plot of Z-Kw CDF

4.0 Properties of the Z-Kum distribution.

This section studies the statistical properties Z-Kum distribution such as the quintile function, order statistics and moments. Survival functions, Hazard rate function, Reversed Hazard rate function, Cumulative hazard rate functions are also discussed in details.

4.1 Quantile Function of Z-Kum Distribution.

The quantile function of Z-Kum distribution is obtained by inverting (4) as given in (20)

$$Q\left(U\right)=\left(1-\left(1-\left(\frac{1}{\alpha}\ln(u(\exp(\alpha)-1)+1)\right)^{\frac{1}{2}}\right)^{\frac{1}{b}}\right)^{\frac{1}{a}} \qquad (20)$$

where $u \sim U(0,1)$

To obtain the first quartile, the median, and the third quartile, we replace u with 0.25, 0.5 and 0.75 in (20) respectively. .

4.2 Order Statistics

The order statistics and their moments have great importance in many statistical problems and applications in reliability analysis and life testing. Suppose $X_1, X_2, ..., X_n$ is a random sample from a distribution with *pdf*, *f(x)*, and let $X_{1n}, X_{2n}, ..., X_{in}$ denote the corresponding order statistic obtained from this sample. The i^{th} order statistic of the proposed distributions can be using (21)

$$f_{in}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\left(1-F(x)\right)^{n-i} \qquad (21)$$

Using bionomial expansion,

$$\left(1-F(x)\right)^{n-i}=\sum_{k=0}^{n-i}(-1)^k\binom{n-1}{k}F(x)^k \qquad (22)$$

$$f_{in}(x)=\frac{n!}{(i-1)!(n-i)!}f(x)F(x)^{i-1}\sum_{k=0}^{n-i}(-1)^k\binom{n-1}{k}F(x)^k \qquad (23)$$

Inserting (3) and (11) in (21), the pdf of the i^{th} order statistics can be given as in (24)

$$f_{in}(x)=\frac{n!}{(i-1)!(n-i)!}\left(\frac{2\alpha abx^{a-1}\left(\left(1-x^a\right)^{b-1}-(1-x^a)^{2b-1}\right)\exp\left(\alpha(1-(1-x^a)^b)^2\right)}{\exp(\alpha)-1}\right)$$

$$\times\left(\frac{\exp\left(\alpha(1-(1-x^a)^b)^2\right)-1}{\exp(\alpha)-1}\right)^{i-1}\sum_{k=0}^{n-i}(-1)^k\binom{n-1}{k}\left(\frac{\exp\left(\alpha(1-(1-x^a)^b)^2\right)-1}{\exp(\alpha)-1}\right)^k \qquad (24)$$

4.3 Moments.

The moments of a random variable are important in statistical inference. They are used to investigate important characteristics of a distribution such as the measures of central tendency, measures of dispersion and measures of shapes. In this subsection, the r^{th} non-central moment of the Z-Kum random variable is derived.

By definition,

$$\mu_r^1 = E(X^r) = \int_{\infty}^{-\infty} X^r f(x,\phi)dx \qquad (25)$$

Using (25), we have

$$\mu_r^1 = E(X^r) = \int_{\infty}^{-\infty} X^r . \frac{2\alpha abx^{a-1}\left(\left(1-x^a\right)^{b-1}(1-(1-x^a)^b)\right)\exp\left(\alpha(1-(1-x^a)^b)^2\right)}{\exp(\alpha)-1}dx \qquad (26)$$

$$\mu_r^1 = E(X^r) = \frac{2\alpha ab}{\exp(\alpha)-1}\int_0^1 x^{a-1+r}\left(\left(1-x^a\right)^{b-1}(1-(1-x^a)^b)\right)\exp\left(\alpha(1-(1-x^a)^b)^2\right)dx \qquad (27)$$

Let $y=\left(1-x\right)^b$ $\quad \frac{dy}{dx} = -b\left(1-x^a\right)^{b-1}.ax^{a-1}$ $\quad dx = \frac{dy}{-b\left(1-x^a\right)^{b-1}.ax^{a-1}}$ $\quad x=\left(y^{1/b}+1\right)^a$

$$\mu_r^1 = E(X^r) = \frac{-2\alpha}{\exp(\alpha)-1}\int_0^1\left(\left(y^{1/b}+1\right)^a\right)^r \exp\left(\alpha(1-y)^2\right)dy \qquad (28)$$

Using power series

$$\left(\left(y^{1/b}+1\right)^a\right)^r = \sum_{j=0}^{r}\binom{r}{j}\left(y^{1/b}\right)^j\left(1^{r-j}\right)$$

$$\exp\left(\alpha(1-y)^2\right) = \sum_{j-0}^{\infty}\frac{\left(\alpha\left(1-y\right)^2\right)^j}{j!}$$

$$\mu_r^1 = E(X^r) = \frac{-2\alpha\, a^j}{j!\exp(\alpha)-1}\sum_{j=0}^{r}\sum_{j=0}^{\infty}\int_0^1 y^{(1/b+1)-1}\left(1-y\right)^{(2j+2)-1}dy \qquad (29)$$

Finally

$$\mu_r^1 = E(X^r) = \frac{-2\alpha\, a^j}{j!\exp(\alpha)-1}\sum_{j=0}^{r}\sum_{j=0}^{\infty}B\left(j/b+1, 2(j+1)\right) \qquad (30)$$

where

$$B\left(j/b+1,2(j+1)\right)=\int_0^1 y^{(1/b+1)-1}\,(1-y)^{(2j+2)-1}\,dy$$

Where Mean= $(x)=\mu_1^{/}, V(x)=\mu_2^{/}-\mu_1^{/2}$, Skewness $=\dfrac{\mu_3^{/}-3\mu_2^{/}\mu_1^{/}+4\mu_1^{/3}}{(\mu_2^{/}-\mu_1^{/2})^{\frac{3}{2}}}$,

Kurtosis $=\dfrac{\mu_4^{/}-4\mu_3^{/}\mu_1^{/}+6\mu_1^{/2}\mu_2^{/}-3\mu_1^{/4}}{(\mu_3^{/}-\mu_1^{/})^2}$

4.4 Moment generating function of Z-Kum distribution.

Theorem 4.1 Let X has a Z-kum distribution. Then the moment generating function is given by

$$M_X(t)=\frac{-2\alpha\, a^j}{j!\exp(\alpha)-1}\sum_{j=0}^{r}\sum_{j=0}^{\infty}\frac{t^r}{r!}B\left(j/b+1,2(j+1)\right) \quad (31)$$

Proof: By definition

$$M_X(t)=\mathrm{E}\left(e^{tx}\right)=\int_0^{\infty}e^{tx}f(x)\,dx \quad (32)$$

Using Taylor series expansion, the moment generating function can be given as

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}u_r \quad (33)$$

where u_r is the r^{th} non-central moment. Substituting the r^{th} non-central moment as in (30) gives the moment generating function of Z-kum distribution as

$$M_X(t)=\frac{-2\alpha\, a^j}{j!\exp(\alpha)-1}\sum_{j=0}^{r}\sum_{j=0}^{\infty}\frac{t^r}{r!}B\left(j/b+1,2(j+1)\right) \quad (34)$$

Hence the proof.

4.5 Characteristic Function of Z-Ikum distribution

Theorem 4.2 Let X has a Z-kum distribution. Then the characteristic function is given by

$$\phi_X(t)=\frac{-2\alpha\, a^j}{j!\exp(\alpha)-1}\sum_{j=0}^{r}\sum_{j=0}^{\infty}\frac{(it)^r}{r!}B\left(j/b+1,2(j+1)\right) \quad (35)$$

Proof: By definition

$$\phi_X(t)=\mathrm{E}\left(e^{itx}\right)=\int_0^{\infty}e^{itx}f(x)\,dx \quad (36)$$

Using Taylor series expansion, the moment generating function can be given as

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} u_r \tag{37}$$

where u_r is the r^{th} non-central moment. Substituting the r^{th} non-central moment as in (30) gives the characteristics function of Z-kum distribution as

$$\varphi_X(t) = \frac{-2\alpha\, a^j}{j!\exp(\alpha)-1} \sum_{j=0}^{r} \sum_{j=0}^{\infty} \frac{(it)^r}{r!} B\left(j/b+1, 2(j+1)\right) \tag{38}$$

Hence the proof.

4.6 Survival Function of Z-Kum distribution.

Using (39) the survival function of Z-Kum distribution is obtain as in (40)

$$S(x) = 1 - F(x) \tag{39}$$

$$S(x) = \frac{\exp(\alpha) - \exp\left(\alpha + \alpha\left(-x^a+1\right)^{2b} - 2\alpha\left(-x^a+1\right)^b\right)}{\exp(\alpha)-1} \tag{40}$$

4.7 Hazard Rate Function of Z-Kum distribution.

The hazard rate functions of Z-Kum distribution is obtain using (41) as given in (42)

$$h(x) = \frac{f(x)}{S(x)} \tag{41}$$

$$h(x) = \frac{2\alpha abx^{a-1}\left(\left(1-x^a\right)^{b-1}(1-(1-x^a)^b)\right)\exp\left(\alpha(1-(1-x^a)^b)^2\right)}{\exp(\alpha) - \exp\left(\alpha + \alpha\left(-x^a+1\right)^{2b} - 2\alpha\left(-x^a+1\right)^b\right)} \tag{43}$$

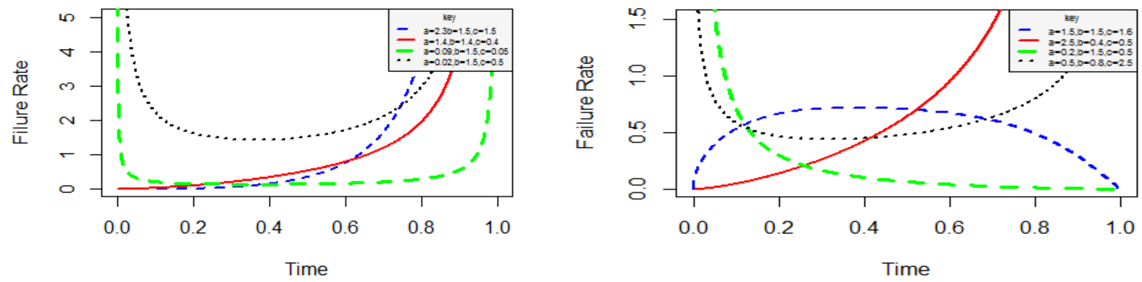


Figure 3: Plot of of Z-kum hr.

Plot of the hazard rate function of Z-Kum distribution for specific sets of parameter values is of Bath-Tub shape, inverted Bath-Tub shape, increasing and decreasing shape.

4.8 Reverse Hazard Rate Function of Z-Kum distribution.

The Reverses hazard rate functions of Z-Kum distribution is obtain using (43) as in (44)

$$r(x) = \frac{f(x)}{F(x)} \qquad (43)$$

$$r(x) = \frac{2\alpha abx^{a-1}\left(\left(1-x^{a}\right)^{b-1} - (1-x^{a})^{2b-1}\right)\exp\left(\alpha(1-(1-x^{a})^{b})^{2}\right)}{\exp\left(\alpha(1-(1-x^{a})^{b})^{2}\right)-1} \qquad (44)$$

4.9 Cumulative Hazard Rate Function of Z-Kum distribution.

The cumulative hazard rate function of Z-Kum distribution is obtain using (45) as given in (46)

$$H(x) = -\log\left(1-F(x)\right) \qquad (45)$$

$$H(x) = -\ln\left(\exp(\alpha) - \exp\left(\alpha + \alpha\left(-x^{a}+1\right)^{2b} - 2\alpha\left(-x^{a}+1\right)^{b}\right)\right) - \ln\left(\exp(\alpha)-1\right) \qquad (46)$$

5.0 Parameters Estimation and Simulation Studies of Z-Burr Distribution.

5.1 Parameter Estimation

To illustrate the applications of the developed distributions with regards to modeling real data sets, it is vital to develop estimators for estimating the parameters of the distributions. In this section, estimators are developed for estimating the parameters of the new distributions using the well known method of maximum likelihood estimate (MLE).

Let $X_1, X_2 ....X_n$ be a random sample from Z-Kum distribution with unknown parameter vector $\phi = (\alpha, a, b,)^T$, the likelihood function of the distribution is obtain using (47)

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta) \tag{47}$$

$$L(\phi) = \prod_{i=1}^{n}\left(\frac{2\alpha abx^{a-1}\left(\left(1-x^{a}\right)^{b-1}(1-x^{a})^{2b-1}\right)\exp\left(\alpha(1-(1-x^{a})^{b})^{2}\right)}{\exp(\alpha)-1}\right) \tag{48}$$

$$L(\phi) = \left(\frac{2^n\alpha^n a^n b^n \prod_{i=1}^{n} x_i^{a-1}\left(\prod_{i=1}^{n}\left(1-x^{a}\right)^{b-1}\prod_{i=1}^{n}(1-x^{a})^{2b-1}\right)\exp\left(\alpha^n\sum_{i=1}^{n}(1-(1-x^{a})^{b})^{2}\right)}{\left(\exp(\alpha)-1\right)^{n}}\right) \tag{49}$$

$$\log\left(L(\phi)\right) = \left(\begin{array}{l} n\ln 2 + n\ln\alpha + n\ln a + n\ln b + (a-1)\sum_{i=1}^{n}\ln x_i + (b-1)\left(\sum_{i=1}^{n}\ln\left(1-x^{a}\right)\right) + (2b-1)\left(\sum_{i=1}^{n}\ln(1-x^{a})\right) \\ + n\ln\alpha\sum\left(1-(1-x_i^{a})^{b}\right)^{2} \end{array}\right)$$

$$-n\ln\left(\exp(\alpha)-1\right)$$

.... (50)

$$\frac{\partial\left(\log(\phi)\right)}{\partial\alpha} = \frac{n}{2} + \sum_{i=1}^{n}\left(1-(1-x_i^{a})^{b}\right)^{2} - \frac{n\exp(\alpha)}{\exp(\alpha)-1} \tag{51}$$

$$\left(\frac{n}{2} + \sum_{i=1}^{n}\left(1-(1-x_i^{a})^{b}\right)^{2} - \frac{n\exp(\alpha)}{\exp(\alpha)-1}\right) = 0 \tag{52}$$

$$\frac{\partial\left(\log(\phi)\right)}{\partial a} = \frac{n}{a} + \sum_{i=1}^{n}\ln x_i - (b-1)\sum_{i=1}^{n}\frac{1}{1-x_i^{a}}x_i^{a}\ln a - (2b-1)\sum_{i=1}^{n}\frac{1}{1-x_i^{a}}\ln a + 2\alpha\sum_{i=1}^{n}\left(1-(1-x_i^{a})^{b}\right)b(1-x_i^{a})^{b-1}x_i^{a}\ln a$$

.... (53)

$$\left(\frac{n}{a} + \sum_{i=1}^{n}\ln x_i - (b-1)\sum_{i=1}^{n}\frac{1}{1-x_i^{a}}x_i^{a}\ln a - (2b-1)\sum_{i=1}^{n}\frac{1}{1-x_i^{a}}\ln a + 2\alpha\sum_{i=1}^{n}\left(1-(1-x_i^{a})^{b}\right)b(1-x_i^{a})^{b-1}x_i^{a}\ln a\right) = 0$$

.... (54)

$$\frac{\partial\left(\log(\phi)\right)}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\ln(1-x_i^a)+2\sum_{i=1}^{n}\ln(1-x_i^a)-2\alpha\sum_{i=1}^{n}\left(1-(1-x_i^a)^b\right)\left((1-x_i^a)^b\ln b\right) \quad (55)$$

$$\left(\frac{n}{b}+\sum_{i=1}^{n}\ln(1-x_i^a)+2\sum_{i=1}^{n}\ln(1-x_i^a)-2\alpha\sum_{i=1}^{n}\left(1-(1-x_i^a)^b\right)\left((1-x_i^a)^b\ln b\right)\right)=0 \quad (56)$$

Equations (52), (54) and (56) cannot be solved analytically, statistical software like R can be used to simultaneously solve them numerically using iterative methods. Solutions of these equations provides the maximum likelihood estimate $\hat{\phi}=(\hat{\alpha},\hat{a},\hat{b})$ of $\phi=(\alpha,a,b)^T$

5.2. Simulation Studies

The performance of the maximum likelihood estimates for the Z-Kum distribution parameters was evaluated using Monte Carlo simulation for a three parameter combinations. Different sample sizes (n = 50, 75 and 100) and some selected parameter values (α = 0.09, a = 0.08, b = 0.09) were used to perform the simulation. Result of the simulation is presented in the table below.

Table 1. Average MLEs, Variance and MSE of the MLEs of parameters of Z-Kum distribution with actual parameter values (α = 0.09, a = 0.08, b = 0.09)

N	Estimates			Variance			MSE		
	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$
50	0.0724	0.0880	0.0980	0.0012	0.0003	0.0003	0.0015	0.5071	0.0003
75	0.0735	0.0874	0.0975	0.0011	0.0002	0.0002	0.0014	0.5069	0.0003
100	0.0767	0.0860	0.0964	0.0009	0.0002	0.0002	0.0011	0.5061	0.0002

5.3 Model Comparison and Selection Criteria

To show how applicable and flexible our proposed model is, its performance is compared with other established models with reference to information lost. So, we tend to use information criteria techniques and goodness-of-fit statistics that correct model for complexity, to constrain the model from over fitting to assess the most effective model from a range different models which can have different number of parameters.

In this case, we will consider the generally well known criteria such as Akaike Information Critareion (AIC), the Bayesian Information Criterion (BIC), the Consistant Akaike Information Cretarion (CAIC) and Hannan-Quinn Information Criterion (HQIC) and illustrate the flexibility and applicability Z-Kum distribution, using three (3) real life data set.

Data set 1

This data was used and analyze by Musa et al. (2021)

0.68879, 0.50813, 0.66621, 0.74526, 0.86947, 0.88076, 0.84688, 0.91463, 0.75655, 0.55329, 0.79042, 0.82429, 0.92593, 0.80172, 0.79042, 0.83559, 0.68879, 0.74526, 0.80172, 0.93722, 0.85818, 0.98238, 0.29359, 0.99368, 0.67751, 0.80172, 0.93722, 0.63234, 0.64363, 0.73397, 0.89205, 0.64363, 0.77913, 0.41779, 0.58717, 0.88076, 0.91463, 0.80172, 0.68879, 0.72267, 0.90334, 0.76784, 0.93722, 0.21454, 0.38392

Data set 2

This data was used and analyze by Musa et al (2021)

0.42909, 0.83559, 0.85818, 0.79042, 0.67751, 0.99368, 0.88076, 0.88076, 0.93722, 0.74526, 0.76784, 0.82429, 0.77913, 0.68879, 0.98238, 0.71138, 0.76784, 0.51942, 0.77913, 0.70009, 0.54200, 0.75655, 0.86947, 0.99368, 0.76784, 0.92593, 0.80172, 0.46296, 0.76784, 0.76784, 0.48555, 0.89205, 0.36134, 0.65492, 0.79042, 0.84688, 0.80172, 0.64363, 0.42909, 0.74526, 0.80172, 0.48555, 0.67751, 0.75655, 0.47425, 0.94851, 0.92593, 0.63234, 0.93722, 0.73397, 0.71138, 0.90334, 0.72267, 0.99368, 0.63234, 0.45167, 0.65492, 0.92593, 0.41779, 0.72267, 0.75655, 0.47425, 0.94851, 0.48555, 0.63234, 0.54201, 0.89205, 0.80172, 0.65492, 0.46296, 0.75655, 0.84688, 0.47425, 0.65492, 0.51942, 0.39521, 0.91463, 0.37263, 0.66621, 0.49684, 0.86947, 0.82429, 0.63234, 0.41779, 0.74526, 0.80172, 0.12421, 0.16938, 0.15808, 0.09033, 0.88076, 0.37263, 0.66621, 0.18067, 0.85818, 0.83559, 0.64363, 0.49684, 0.76784, 0.77913, 0.89205, 0.35005, 0.99368, 0.60976, 0.75655, 0.77913, 0.65492, 0.39521, 0.74526, 0.82429, 0.92593, 0.97109, 0.68879, 0.94851, 0.7904, 0.99368, 0.71138, 0.49684, 0.06775, 0.91463, 0.97109, 0.91463, 0.86947, 0.76784, 0.86947, 0.79042, 0.79042, 0.41779, 0.77913, 0.99368, 0.51942, 0.67751, 0.84688, 0.80172, 0.90334, 0.80172, 0.90334, 0.71138, 0.63234, 0.74526, 0.54201, 0.39295, 0.76784, 0.71138, 0.67751, 0.63234, 0.77913, 0.85818, 0.63234, 0.99368, 0.55329, 0.75655, 0.82429, 0.37263, 0.56459, 0.15808, 0.45167, 0.64363, 0.67751, 0.99368, 0.92593, 0.67751, 0.84689, 0.68879, 0.76784, 0.50813, 0.68879, 0.82429, 0.67751, 0.28229, 0.49684, 0.62105, 0.66621, 0.62105, 0.86947, 0.89205, 0.68879, 0.50813, 0.66621, 0.74526, 0.86947, 0.88076, 0.84688, 0.91463, 0.75655, 0.55329, 0.79042, 0.82429, 0.92593, 0.80172, 0.79042, 0.83559, 0.68879, 0.74526, 0.80172, 0.93722, 0.85818, 0.98238, 0.29359, 0.99368, 0.67751, 0.80172, 0.93722, 0.63234, 0.64363, 0.73397, 0.89205, 0.64363, 0.77913, 0.41779, 0.58717, 0.88076, 0.91463, 0.80172, 0.68879, 0.72267, 0.90334, 0.76784, 0.93722, 0.21454, 0.38392

Data set 3

The following data was used and analyze by Saboor et al (2021). It consist 48 rock samples from petroleum reservoir obtained from the measurements on petroleum rock samples

0.0903296, 0.2036540, 0.2043140, 0.2808870, 0.1976530, 0.3286410, 0.1486220, 0.1623940, 0.2627270, 0.1794550, 0.3266350, 0.2300810, 0.1833120, 0.1509440, 0.2000710, 0.1918020, 0.1541920, 0.4641250, 0.1170630, 0.1481410, 0.1448100, 0.1330830, 0.2760160, 0.4204770, 0.1224170, 0.2285950, 0.1138520, 0.2252140, 0.1769690, 0.2007440, 0.1670450, 0.2316230, 0.2910290, 0.3412730, 0.4387120, 0.2626510, 0.1896510, 0.1725670, 0.2400770, 0.3116460, 0.1635860, 0.1824530, 0.1641270, 0.1534810, 0.1618650, 0.2760160, 0.2538320, 0.2004470

Table 2. Performance of Z-Kum distribution's goodness of fit using data set 1

	MODEL		
	Z-KUM	**LIB-KUM**	**KUM**
AIC	-101.1618	-97.37334	-79.19838
CAIC	-99.82845	-96.04001	-77.86504
BIC	-97.88865	-94.10021	-75.92525
HQIC	-100.3907	-96.60229	-78.42733
Rank	**1**	**2**	**3**

Table 3. Performance of Z-Kum distribution's goodness of fit using data set 2

	MODEL		
	Z-KUM	**LIB-KKUM**	**KUM**
AIC	-977.89	-670.72	-816.81
CAIC	-977.78	-670.61	-816.70
BIC	-967.70	-660.52	-806.61
HQIC	-973.78	-666.61	-812.69
Rank	**1**	**2**	**3**

Table 4. Performance of Z-Kum distribution's goodness of fit using data set 3

	MODEL		
	Z-KUM	**LIB-KUM**	**KUM**
AIC	-134.4426	-126.1913	-118.3165
CAIC	-134.3220	-126.0807	-118.2060
BIC	-124.2481	-115.9968	-108.1221
HQIC	-130.3263	-122.0749	-114.2002
Rank	**1**	**2**	**3**

It can be seen from Table **2, 3** and **4** that based on the values of the information criterion from the three different real life data sets, Z-Kum distribution having the less values performed better than the other two distributions in term of fitting/modeling real life data.

6. Discussion and Conclusion

In this paper, we developed new three parameter model called Zubair- Kumaraswamy (Z-Kum) distribution. The extension was done using Zubair G-Family (2018) of continuous probability distribution was used to extend well known Kumaraswamy distribution to make it more flexible in modeling and predicting real world phenomenon. Some basic structural properties of the new distributions like Quantile functions, moments, moment generating functions, characteristics functions and order statistics were obtained. Survival function, hazard function, reversed hazard rate function and a cumulative hazard rate function was also obtained. Behaviour of the hazard rate plot exhibit increase, decrease, Bathtub and inverted Bathtub shape. Maximum likelihood estimate was used to estimate the Z-Kum distribution parameters, Monte Carlo simulation also was carried out to evaluate the performance of MLE in estimating our distribution parameters. Result of the simulation studies revealed that as the sample size increases, the estimate values approaches actual parameter values, while the values of mean square errors approaches zero, this indicates that MLE is good for the estimation of our distribution parameters. To show how flexible and more efficient our proposed model is over some existing distributions, we compare the model with the other fitted existing models. Analytical measure of goodness of fit of some information criterion such as Akaike information criterion (AIC), consistent Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Hannan-Quinn information criterion (HQIC) was considered using three real life data sets. From the results obtained, it is evident that our proposed model give better fit than the other competing models and is therefore, more flexible in modeling and predicting real world phenomenon.

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