Oscillatory Solution of a Convolutional Volterra Integral Equation

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ABSTRACT

Oscillatory solutions are pivotal for understanding functional differential and integral equations. They provide insights into how the solutions of these equations behave, aiding in grasping their growth, stability, and convergence properties. This study formulates essential conditions for identifying oscillatory solutions within a convolutional Volterra integral equation, establishes and proves theorems concerning these solutions, and provides anexample. The proofs presented herein reveal that the convolutional Volterra integral equation can exhibit oscillatory or non-oscillatory behavior, depending on the characteristics of the function within the integral.

Keywords: Convolution, Volterra Integral Equation,Oscillatory, Infimum, Supremum

1. INTRODUCTION

In many functional differential and integral equations, the behaviour of oscillatory solutions plays an important role in their analysis and results interpretation. Oscillatory solutions help in understanding the behaviour of solutions to differential and integral equations, providing insights into their growth, stability, and convergence properties. This knowledge is essential for the analysis and design of systems in various fields, such as control theory, quantum chemistry, and mathematical physics. Oscillatory solutions are used in control theory to study the oscillation of solutions relative to a given hyper-plane of a system of equations. This helps in understanding the dynamic behaviour of the system and designing control strategies to stabilize or manipulate the oscillations. Oscillatory solutions are also relevant in quantum chemistry, where they can provide insights into the vibrational motion of molecules and the behaviour of electronic waves. Over the years, there have been extensive studies on the behaviour of the oscillatory solution for various forms of differential and integral equations.Numerous oscillation criteria are known and diversified approaches have proved fruitful. Among these is the use of the Volterra equations. An oscillatory behaviour of the solution of the Volterra integral equation of the form $x(t)=f(t)-\int_0^t a(t,s)\,g(s,x(s))ds$ and first-order functional differential equation was considered by [1]. [2] studied the necessary and sufficient conditions for the bounded and unbounded solutions of the Volterra integral equation considered in [1] to be oscillatory or non-oscillatory and also studied the asymptotic behaviour of such solutions. [3] studied that the oscillation solution of the Volterra integral equation of the form $x(t)=f(t)+\int_0^t k(t,s,x(s),xg(s))ds$ depends on the distance between consecutive zeros and established the theorem for localization of the zeros of the solution. Moreover, the oscillation solution of the Volterra integral equation and integro-differential equation with a high oscillatory kernel was considered by [4]. [5] employed an enhanced Levin quadrature technique to tackle a particular type of oscillatory integral equation, wherein the unknown function exhibits notably lower oscillations compared to the kernel function of these integrals. Nevertheless, in practical applications like electromagnetic scattering, integral equations commonly manifest in multi-dimensional structures. Hence,

extending this method from one dimension to multiple dimensions holds considerable importance.
Leveraging the asymptotic properties of non-oscillatory solutions, an exploration into the oscillatory characteristics of solutions for both the integro-dynamic and integral equations across a time scale was conducted by [6]. [7] tackled highly oscillatory Volterra integral equations using an exponential fitting collocation method. Their approach, developed after analysing the inherent oscillatory patterns, effectively captured the solutions' qualitative behaviour. The numerical examples showcased the method's superior efficiency and accuracy compared to conventional polynomial collocation methods. A Nyström-style method to solve highly oscillatory second-kind Volterra integral equations was introduced by [8]. This approach combines classical and dilation quadrature methods. The study extensively investigates convergence, stability, and accuracy using numerical assessments. Additionally, this method was applied to compute initial scattering data for the Korteweg-de Vries equation's initial value problem.
Although most of the studies considered the oscillatory solutions of various types of integral equations, one major type of integral equation that is yet to be exploited is the convolutional integral equation. The convolutional integral equations are specialintegral equations that are a subset of integral equations that involve the convolution operation [9].Convolution integral equations play a vital role in various scientific and engineering domains, offering a powerful tool to model and solve problems involving convolution operations. Their applications span across diverse fields, making them an essential aspect of mathematical analysis and problem-solving methodologies.Understanding the characteristics of convolution kernels and their influence on the solutions of convolution integral equations is crucial in various fields like signal processing, image processing, and system modelling [10].
According to [11], the properties of the kernel function directly impact the behaviour and amplitude of the resulting oscillations within the solution of the integral equation.
Analyzing the properties of convolution kernels that induce bounded oscillations helps in designing and selecting appropriate kernel functions for specific applications, ensuring stability and controlled oscillatory behaviour within desired bounds in various signal processing and system modelling scenarios.The significance of oscillatory solutions within convolutional integral equations lies in their ability to capture and elucidate behaviours characterized by periodic or alternating patterns. Understanding these solutions is crucial across diverse domains such as signal processing, control theory, and physics, offering insights into systems exhibiting oscillatory behaviour. They play a pivotal role in analylzing and characterizing signals with periodic components, aiding in stability assessments in control systems, and modelling wave-like phenomena in physical systems. Additionally, the study of oscillatory solutions contributes to advancing mathematical frameworks, providing tools for modelling biological rhythms, and offering deeper insights into complex systems governed by oscillatory dynamics, thus impacting various scientific, engineering, and mathematical disciplines.
Therefore, considering a convolutional integral equation of the form as shown in Equation (1)

$$x(t) = f(t) + \int_0^t \alpha(t-s) g\left(s, x(\varphi(s))\right) ds \ , t > 0 \qquad (1)$$

Where $x(t)$ represents the unknown function to be solved, $f(t)$ is a given function or the initial condition, $g\left(s, x(\varphi(s))\right)$ convolution-like operation or a more general nonlinear operation involving the unknown function $x(\varphi(s))$, and $\alpha(t-s)$ is the kernel function, the goal of this study is to investigate theoscillatory solutions of equation (1).

2. MATHEMATICAL PRELIMINARIES

This section provides fundamental definitions pertinent to the study.

2.1 INTEGRAL EQUATIONS

The standard form of an integral equation can be represented in the form:

$$u(t) = f(x) + \lambda \int_{g(x)}^{h(x)} k(x,t) u(t)\, dt \tag{2}$$

where $x(t)$ is the unknown function to be determined, $k(x,t)$ is the kernel of the integral equations and $f(x)$ is the known function sometimes called the forcing function, λ is a known parameter constant and $g(x)$ and $h(x)$ are the limits of the integral equation [12]. The limits of integration can sometimes be constant, variable or mixed type depending on the type of integral equation, and may as well as be one dimensional or more.The unknown function $u(x)$ in many cases appears inside and outside the integral sign.

2.1.1 Volterra Integral Equation

According to [13], the Volterra Integral equation is a type of integral equation that has one of its limits to be a variable. The standard form of a Volterra integral equation is given in the form:

$$\vartheta(t) u(t) = f(x) + \lambda \int_{0}^{x} k(x,t) u(t)\, dt \tag{3}$$

When the unknown function $u(x)$ appears inside and outside the integral sign and $\vartheta(x) = 1$ in Equation (3), the resulting integral equation is called a Volterra integral equation of the second kind and is represented by the form:

$$u(x) = f(x) + \lambda \int_{0}^{x} k(x,t) u(t)\, dt \tag{4}$$

Also, when $\vartheta(x) = 0$ and the unknown function appears only inside the integral sign, the resulting integral equation is called the Volterra integral equation of the first kind depicted in Equation (5) in the form:

$$0 = f(x) + \lambda \int_{0}^{x} k(x,t) u(t)\, dt \tag{5}$$

2.1.2 Linearity of Integral Equations

An integral equation is linear or nonlinear depending on the exponent of the unknown function inside the integral sign. If the exponent of the unknown function inside the integral sign has a value of one, then the integral equation is said to be linear otherwise it is nonlinear.It is also nonlinear when the unknown function $u(x)$ under the integral sign contains a nonlinear function [14].

2.1.3 Separable kernel

A kernel $k(x,t)$ of an integral equation is said to be separable or (degenerate) if it can be expressed in the form:

$$k(x,t) = \sum_{i=1}^{n} a_i(x) b_i(t) \tag{6}$$

where $a_i(x)$ and $b_i(t)$ are linearly independent [15].

2.1.4 Symmetric (or Hermitian kernel)

A complex-valued function $k(x,t)$ is called symmetric (or Hermitian) if the kernel

$$k(x,t) = k^*(x,t) \quad (7)$$

where $k^*(x,t)$ is the complex and $*$ is the complex conjugate.

If the kernel is real then

$$k(x,t) = k(t,x). \quad (8)$$

2.1.5 Convolution Integral Equation

Convolution Integral Equationis an equation thatinvolves an unknown function within the integral sign, representing a convolution transform. The distinctive characteristic of an integral equation of convolution type lies in its kernel, which relies on the disparity between the arguments [16].The kernel function $k(x,t)$ relies on the difference between x and t. For instance, in a linear shift-invariant system, the kernel might depend on $x-t$. A prime illustration of this is evident in the equation:

$$u(x) = f(x) + \lambda \int_{-\infty}^{\infty} k(x-t)u(t)\,dt, \; -\infty < t < \infty \quad (9)$$

2.2 SUPREMUM AND INFIMUM

Definition 2.1

A set $A \subset R$ of real numbers is bounded from above if there exists a real number $M \in R$, called an upper bound of A, such that $x \le M$ for every $x \in A$. Similarly, A is bounded from below if there exist $m \in R$, called a lower bound of A, such that $x \ge m$ for every $x \in A$. A set is bounded if it is bounded both from above and below [17].

The supremum of a set is its least upper bound and the infimum is its greatest upper bound.

Definition 2.2

Let $A \subset R$ is a set of real numbers. If $M \in R$ is an upper bound of A such that $M \le M'$ for every upper bound M' of A, then M is called the supremum of A, denoted $M = \sup A$. If $M \in R$ is a lower bound of A such that $m \ge m'$ for every lower bound m' of A, then m is called the or infimum of A, denoted $m = \inf A$. If A is not bounded from above, then $\sup A = \infty$, and if A is not bounded from below then $\inf A = -\infty$ [18].

In the case where $A = \phi$ is an empty set, then every real number can act as both an upper and a lower bound of A, and denoted by $\sup\phi = -\infty,\ \inf\phi = \infty$.The existence of supremum or infimum for a set can only be affirmed if it corresponds to a finite real number. For an indexed set $A = \{x_k : k \in J\}$, we often use the notation $\sup A = \sup_{k \in J} x_k,\ \inf A = \inf_{k \in J} x_k$ [18].

3. RESULTS AND DISCUSSION

[1] proposed that the oscillatory solutions of an equation in the form of equation (1) can be categorized into different types of oscillatory solutions, delineated as follows:

i. the solution is said to be oscillatory if each of the sets $\{t \ge 0 \mid x(t) > 0\}$ and $\{t \ge 0 \mid x(t) < 0\}$ is unbounded,

ii. it is said to be weakly oscillatory if the set $\{t \ge 0 \mid x(t) = 0\}$ is unbounded and;

iii. it is non-oscillatory if it is not weakly oscillatory

Therefore, to determine the oscillatory solution of equation (1), the following conditions are considered:

i. $f:[0,\infty] \to R$ and $g:[0,\infty]\times R \to R$ are continuous,
ii. $\alpha:[0,\infty]\times[0,\infty] \to R$
iii. $\alpha(t-s)=0$ if $s>t$ and $\alpha(t-s)\geq 0$, $0 \leq t \leq \infty$ and $0 \leq s \leq t$.
iv. $\alpha(t-s)$ be continuous for both $0 \leq t \leq \infty$ or $0 \leq s \leq t$.

This study focuses on exploring real solutions of equation (1) within the interval $[0,\infty)$. According to [1], solution $x(t)$ of equation (1) is considered oscillatory if $x(t)$ has zeros for arbitrarily large t ; otherwise, a solution $x(t)$ is said to be nonoscillatory.

The oscillatory conditions will then be deduced based on the following theorem.

Theorem 3.1 Suppose conditions (i)-(iv) are satisfied then all unbounded solutions of equation (1) are oscillatory if $f(t)$ is bounded and $g(t,x)>0$ if the function $h(t)=\int_0^t \alpha(t-s)ds,\ 0<t<\infty$ is bounded for every $t>0$.

Proof:
Suppose that the solution of equation (1) is not oscillatory to the extent that the solution $x(t)$ is unbounded on the interval $[0,\infty)$. This also implies that there exist a $T>0$ such that the solution $x(t)>0$ or $x(t)<0$ for $t \geq T$. Since $f(t)$ is bounded, let $K \in \square$ such that $|f(t)| \leq k$. Also, when a set is bounded then it is bounded above and below, therefore $-k \leq f(t) \leq k$.
Thus, the idea of contradiction is applied to establish the proof. Since it is assumed that the solution is unbounded, let $x(t)>0$, such that $t \geq T$ then equation (1) can be written in the form:

$$0<x(t)= f(t)+\int_0^t \alpha(t-s)g\big(s,x(\phi(s))\big)ds \quad (10)$$

Since $t \geq T$, equation (10)is expressed in the form:

$$0<x(t)= f(t)+\int_0^T \alpha(t-s)g\big(s,x(\phi(s))\big)ds +\int_T^t \alpha(t-s)g\big(s,x(\phi(s))\big)ds \quad (11)$$

Next, assuming the function is bounded above then $f(t) \leq k$, thus by equation (11);

$$0<x(t)\leq k+\int_0^T \alpha(t-s)g\big(s,x(\phi(s))\big)ds +\int_T^t \alpha(t-s)g\big(s,x(\phi(s))\big)ds \quad (12)$$

But $\int_T^t \alpha(t-s)g\big(s,x(\phi(s))\big)ds \geq 0$ if and only if $t \geq T$ therefore equation (12) is expressed as;.

$$0<x(t)\leq k+\int_0^T \alpha(t-s)g\big(s,x(\phi(s))\big)ds \quad (13)$$

Thus, taking the supremum of equation (13) results in equation (14);

$$0<x(t)\leq k+\sup_{t\in[0,T]} g\big(t,x(t)\big)\int_0^T \alpha(t-s)ds \text{ but let } S=\sup_{t\in[0,T]} g\big(t,x(t)\big) \quad (14)$$

$$0<x(t)\leq k+S\int_0^T \alpha(t-s)ds \quad (15)$$

This implies that $x(t)$ is bounded which is a contradiction from the earlier assertion that it is non-oscillatory and unbounded. Since for oscillatory solution the solution $x(t)>0$ or $x(t)<0$ for some $t>0$. The converse is also proved.
Therefore, conversely, suppose $x(t)<0$ for $t \geq T$ then from equation (1)

$0 > x(t) = f(t) + \int_0^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds$ (16)

$0 > x(t) = f(t) + \int_0^T \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds + \int_T^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds$ (17)

Since $t \geq T$

Let $f(t)$ be bounded below such that $f(t) \geq -k$ then from equation (17),

$0 > x(t) \geq -k + \int_0^T \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds + \int_T^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds$

(18)

which is expressed in the form;

$0 > x(t) \geq -k + \int_0^T \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds$ (19)

By assuming that $t \geq T$

$0 > x(t) \geq -k + S\int_0^T \alpha(t-s)\, ds$ (20)

This also implies that $x(t)$ is bounded and contradictory. Therefore equation (1) is said to be oscillatory. This completes the proof for theorem 3.1 and thus the conditions are valid for oscillatory solution.

Next, a condition for an oscillatory solution is considered.

Theorem 3.2

Suppose the conditions (i)-(iv) are satisfied then equation (1) has an oscillatory solution if $g(t,x) > 0$ for all $t \geq 0$ and $\int_0^\delta \alpha(t-s)\, ds$ is bounded for $\delta > 0$ such that $\lim_{t\to\infty} \text{Sup} f(t) = \infty$ and $\lim_{t\to\infty} \text{Inf} f(t) = -\infty$

Proof

The idea of proof by contraction will also be applied here. Suppose equation (1) has an oscillatory solution on $[0,\infty)$, then it implies that there exist a $T > 0$ such that $x(t) > 0$ or $x(t) < 0$ *for all* $t \geq T.$ The idea of validity of interval is applied here.

First suppose $x(t) > 0$ for $t \geq T$ then from equation (1)

$0 < x(t) = f(t) + \int_0^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds$ (21)

For $t \geq T,$ let

$0 < x(t) = f(t) + \int_0^T \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds + \int_T^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds$ (22)

Since $t \geq T,$ equation (22) can be expressed as:

$0 < x(t) \leq f(t) + \int_0^T \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds$ (23)

But for $t \geq T,$

$\int_T^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds \geq 0$ (24)

Thus,

$0 < x(t) \leq f(t) + \int_0^T \alpha(t-s)\, g\left(s, x\left(\phi(s)\right) ds\right)$ (25)

Taking the infimum of equation (25)

$0 < x(t) \leq \ln ff(t) + \text{Inf}\int_0^T \alpha(t-s)\, g\left(s, x\left(\phi(s)\right) ds\right)$ (26)

$$0 < x(t) \le \lim_{t\to\infty} \underset{t\in[0,T]}{\ln f}\, f(t) + \lim_{t\to\infty} \underset{t\in[0,T]}{\text{Inf}} \int_0^T \alpha(t-s)\, g\left(s, x\left(\phi(s)\right) ds\right) \quad (27)$$

But let $I = \underset{t\in[0,T]}{\text{Inf}} \left|g\left(t, x(t)\right)\right|$

$$0 < x(t) \le \lim_{t\to\infty} \text{Inf} f(t) + \lim_{t\to\infty} I \int_0^T \alpha(t-s)\, ds \quad (28)$$

But $\lim_{t\to\infty} \text{Inf} f(t) = -\infty$

Thus $0 < x(t) \le -\infty$ which is a contradiction on the validity of the interval and also to the assertion that $x(t) > 0$.

Conversely, suppose $x(t) < 0$ then

$$0 > x(t) = f(t) + \int_0^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds \quad (29)$$

$$0 > x(t) = f(t) + \int_0^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds + \int_T^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds \quad (30)$$

But $\int_T^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds \ge 0$ if and only if $t \ge T$, thus,

$$0 > x(t) \ge f(t) + \int_0^T \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds \quad (31)$$

Taking the supremum of the right-hand side of equation (31).

$$0 > x(t) \ge \underset{t\in[0,T]}{\text{Sup}} f(t) + \underset{t\in[0,\infty]}{\text{Sup}} \int_0^T \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds \quad (32)$$

$$0 > x(t) \ge \underset{t\in[0,T]}{\text{Sup}} f(t) + \underset{t\in[0,\infty]}{\text{Sup}} \left|g\left(t, x(t)\right)\right| \int_0^T \alpha(t-s)\, ds$$

Let $S = \underset{t\in[0,\infty]}{\text{Sup}} \left|g\left(t, x(t)\right)\right|$

$$0 > x(t) \ge \underset{t\in[0,T]}{\text{Sup}} f(t) + S \int_0^\infty \alpha(t-s)\, ds$$

But $\int_T^t \alpha(t-s)\, g\left(s, x\left(\phi(s)\right)\right) ds \le s \int_0^\infty \alpha(t-s)\, ds$ and also $\lim_{t\to\infty} \text{Sup} f(t) = \infty$

This implies that $0 > x(t) > \infty$ which is also a contradiction that the solution $x(t) < 0$.

Hence the proof. Thus, $x(t)$ has an oscillatory solution.

Example

Let consider the equation

$$x(t) = f(t) + \int_0^t \alpha(t-s)\, g\left(s, x\left(\varphi(s)\right)\right) ds \;, t > 0 \quad (33)$$

where $f(t) = (t\cos t + \sin t + 1)(t+2)^{-\frac{1}{4}} + (\cos t)^5$, $\alpha(t-s) = 0$ if $s > t$, $\alpha(t-s) = (t+1)^{\frac{1}{4}} s^{\frac{4}{5}}$ for $0 \le t < \infty$, $0 \le s \le t$ and $g\left(s, x(s)\right) = \left(s x(s)\right)^{\frac{1}{4}}$. Suppose all the conditions in theorem 3.2 are satisfied then it implies that equation (33) will have an oscillatory solution and one of suchsolutions will be $(\cos t)^5$. The other form of oscillatory solutions can be deduced numerically, as it would be challenging to do so analytically.

4. CONCLUSION

The study investigated oscillatory solutions of a convolutional Volterra integral equation. Conditions for identifying oscillatory solutions within such equations were specified, and theorems concerning these solutions were established, proven, and illustrated with an example. The proofs presented indicate that the convolutional Volterra integral equation can exhibit either oscillatory or non-oscillatory behavior, depending on the characteristics of the function within the integral.

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