On top local homology modules

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Abstract

Here, we can relate the theory of local cohomology, with respect to an ideal, to the theory of local homology modules. With the results of the article, we show the importance of local homology theory as a study tool within of the commutative algebra theory.

1 Introduction

Assume that $(R, \mathfrak{m})$ is a commutative Noetherian local ring, with non-zero identity, J is an ideal of R and A is an R-module.

In [3], we have the local homology module $\mathrm{H}_i^J(A)$,

$$\mathrm{H}_i^J(A) = \varprojlim_{n \in \mathbb{N}} \mathrm{Tor}_i^R(R/J^n, A).$$

About local cohomology modules, see [2]. For basic results about local homology we refer the reader to [3], and, [8].

We study the top local homology module $\mathrm{H}_s^{\mathfrak{m}}(A)$, where A is a non-zero Artinian R-module of Noetherian dimension s.

The module $\mathrm{H}_s^{\mathfrak{m}}(A)$ is called a top local homology module with respect to maximal ideal $\mathfrak{m}$, because $\max\{i : \mathrm{H}_i^{\mathfrak{m}}(A) \neq 0\} \leq s$, by [3, Proposition 4.8].

In the next section, we presented prerequisites of the theory.

In the Section 3, we put the results, and in the Section 4, some applications.

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As motivation, vanishing, finiteness and artinianness of local cohomology and local homology modules are the main problems in commutative algebra. In this paper, we put only the local homology modules. We finalize with a conclusion. We use properties of commutative algebra and homological algebra for the development of the results (see [1] and [11]).

2 Prerequisites

Definition 2.1 (see [5])**.** According to the previous context, A is called secondary if the multiplication function by any element is either surjective or nilpotent.

Definition 2.2. A secondary representation of A is an expression as a finite sum of secondary modules. If such a representation exists, we will say that A is representable.

Definition 2.3 (see [6])**.** A prime ideal $\mathfrak{q}$ of R is said to be an attached prime of A if $\mathfrak{q} = (N :_R A)$, for some submodule N of A. If A admits a reduced secondary representation

$$A = S_1 + S_2 + \ldots + S_n,$$

then the set of attached primes $\operatorname{Att}_R(A)$ is equal to

$$\left\{ \sqrt{(0 :_R S_i)},\ \text{for } i = 1, \ldots, n \right\}.$$

Now, we have Noetherian dimension $\operatorname{N}\dim_R(A)$.

For $A = 0$, we define $\operatorname{N}\dim_R(A) = -1$. Then by induction, for any integer $t \geq 0$, we define $\operatorname{N}\dim_R(A) = t$ when

(1) $\operatorname{N}\dim_R(A) < t$ is false, and

(2) for every ascending chain $A_1 \subseteq A_2 \subseteq \ldots$, of submodules of A there exists an integer m_0 such that

$$\operatorname{N}\dim_R(A_{m+1}/A_m) < t,$$

for all $m \geq m_0$.

Thus, A is non-zero and finitely generated if and only if $\mathrm{N}\dim_R(A) = 0$.

If A is Artinian R-module, then $\mathrm{N}\dim_R(A) < \infty$, according to [6].

In [4], is defined the cohomological dimension of A with respect to J as

$$\mathrm{cdi}(J, A) = \max\left\{i : \mathrm{H}^i_J(A) \neq 0\right\}.$$

By [2, Theorem 6.1.2] and by [2, Theorem 6.1.4], $\mathrm{cdi}(J, A) \leq \dim_R(A)$, and, $\mathrm{cdi}(\mathfrak{m}, A) = \dim_R(A)$.

We call, $\mathrm{hdi}(J, A) := \max\left\{i : \mathrm{H}^J_i(A) \neq 0\right\}$, the homological dimension of A with respect to J. From [3, Propositions 4.8 and 4.10], if A is an Artinian R-module, $\mathrm{hdi}(J, A) \leq \mathrm{N}\dim_R(A)$, and, $\mathrm{hdi}(\mathfrak{m}, A) = \mathrm{N}\dim_R(A)$.

Now, for an R-module A, $\mathrm{E}(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$ and $\mathrm{D}(\bullet)$ denotes the Matlis duality functor $\mathrm{Hom}_R(\bullet, \mathrm{E}(R/\mathfrak{m}))$.

We have that: $\dim_R(\mathrm{D}(A)) = \dim_R(A)$.

If A is an Artinian R-module, where R is complete ring, then $A \cong \mathrm{D}(\mathrm{D}(A))$, and $\mathrm{D}(A)$ is a Noetherian $\hat{R}$-module (see [2, Theorem 10.2.19], and, [10, Theorem 1.6(5)]).

If A is an Artinian R-module, $\mathrm{H}^J_i(A) \cong \mathrm{D}(\mathrm{H}^i_J(\mathrm{D}(A)))$, for all $i \geq 0$, according to [3, Proposition 3.3(ii)].

Therefore, $\mathrm{hdi}(J, A) = \mathrm{cdi}(J, \mathrm{D}(A))$.

So, $\mathrm{hdi}(J, A) \leq \dim_R(\mathrm{D}(A)) = \dim_R(A)$.

3 The results

Lemma 3.1. *Let*

$$0 \to L \to A \to N \to 0,$$

be an exact sequence of Artinian R-modules. So,

$$\mathrm{hdi}(J, A) = \max\left\{\mathrm{hdi}(J, L), \mathrm{hdi}(J, N)\right\}.$$

Proof. $\mathrm{D}(A)$ is Noetherian $\hat{R}$-module. So, by [4, Corollary 2.3(i)],

$$\mathrm{cdi}(J\hat{R}, \mathrm{D}(N)) \leq \mathrm{cdi}(J\hat{R}, \mathrm{D}(A)).$$

By [2, Theorem 4.2.1], $\mathrm{cdi}(J, \mathrm{D}(N)) \leq \mathrm{cdi}(J, \mathrm{D}(A))$.

So, $\mathrm{hdi}(J, N) \leq \mathrm{hdi}(J, A)$. And by

$$\mathrm{H}^J_{i+1}(L) \to \mathrm{H}^J_{i+1}(A) \to \mathrm{H}^J_{i+1}(N) \to \mathrm{H}^J_i(L) \to \mathrm{H}^J_i(A) \to \dots\ ,$$

we have $\mathrm{hdi}(J, L) \leq \mathrm{hdi}(J, A)$.

Hence, $\max\{\mathrm{hdi}(J, L), \mathrm{hdi}(J, N)\} \leq \mathrm{hdi}(J, A)$.

From the above long exact sequence,

$$\mathrm{hdi}(J, A) \leq \max\{\mathrm{hdi}(J, L), \mathrm{hdi}(J, N)\}.$$

□

Lemma 3.2. *Let $(R, \mathfrak{m})$ be a complete local ring, and let A as before. Then,*

$$\mathrm{cdi}(J, R/\mathfrak{q}) \leq \mathrm{hdi}(J, A),$$

for all $\mathfrak{q} \in \mathrm{Att}_R(A)$.

Proof. $\mathrm{D}(A)$ is a Noetherian R-module and

$$\mathrm{Supp}_R(R/\mathfrak{q}) \subseteq \mathrm{Supp}_R(\mathrm{D}(A)),$$

for all $\mathfrak{q} \in \mathrm{Ass}_R(\mathrm{D}(A))$. So, by [4, Theorem 2.2],

$$\mathrm{cdi}(J, R/\mathfrak{q}) \leq \mathrm{cdi}(J, \mathrm{D}(A)),$$

for all $\mathfrak{q} \in \mathrm{Ass}_R(\mathrm{D}(A))$.

$\mathrm{Att}_R(A) = \mathrm{Ass}_R(\mathrm{D}(A))$ and $\mathrm{cdi}(J, \mathrm{D}(A)) = \mathrm{hdi}(J, A)$. So,

$$\mathrm{cdi}(J, R/\mathfrak{q}) \leq \mathrm{hdi}(J, A),$$

for all $\mathfrak{q} \in \mathrm{Att}_R(A)$. □

Lemma 3.3. *We have,*

$$\mathrm{hdi}(J, A) \leq \mathrm{cdi}(J, R/\mathrm{Ann}_R(A)).$$

Proof. Let $R' = R/\mathrm{Ann}_R(A)$. By [12, Theorem 3.3], we have $\mathrm{H}_i^J(A) \cong \mathrm{H}_i^{JR'}(A)$, for all $i \geq 0$.

Thus, we have that $\mathrm{hdi}(J, A) = \mathrm{hdi}(JR', A)$. Since,

$$\mathrm{hdi}(JR', A) \leq \mathrm{cdi}(JR', R'),$$

according to [5, Corollary 3.2], and $\mathrm{cdi}(JR', R') = \mathrm{cdi}(J, R')$, according to [4, Lemma 2.1], $\mathrm{hdi}(J, A) \leq \mathrm{cdi}(J, R')$, and this finishes the proof. □

4 Application

Theorem 4.1. *Let* $(R, \mathfrak{m})$ *be a complete local ring, and let* A *be a non-zero Artinian* R*-module of finite Noetherian dimension* s*, with* $\operatorname{hdi}(J, A) = s$*. Thus,*

$$\sum = \left\{ N' : N' \text{ is a submodule of } A \text{ and } \operatorname{hdi}(J, A/N') < s \right\},$$

has a smallest element N*. The* R*-module* N *has the following properties:*

(1) $\operatorname{hdi}(J, N) = \dim(N) = s$.

(2) $\operatorname{Att}_R(N) = \{\mathfrak{q} \in \operatorname{Att}_R(A) : \operatorname{cdi}(J, R/\mathfrak{q}) = s\}$.

(3) $\mathrm{H}_s^J(N) \cong \mathrm{H}_s^J(A)$.

Proof. We have $A \in \sum$, and thus $\sum \neq \emptyset$. A is an Artinian R-module, and so the set $\sum$ has a minimal member N. By Lemma 3.1, if N_1, $N_2 \in \sum$, then $\operatorname{hdi}(J, A/(N_1 \cap N_2)) < n$. The intersection of any two members of $\sum$ is again in $\sum$. So N is contained in every member of $\sum$ implying that N is the smallest element of $\sum$.

(1) $\operatorname{hdi}(J, A/N) < s$, and so from the exact sequence

$$0 \to N \to A \to A/N \to 0,$$

and of the Lemma 3.1, $\operatorname{hdi}(J, N) = s$. From,

$$s = \operatorname{hdi}(J, N) \leq \dim(N) \leq \dim(A) = s,$$

$\dim(N) = s$.

(2) If $\mathfrak{q} \in \operatorname{Att}_R(N)$, then $\mathfrak{q} = \operatorname{Ann}_R(N/L)$, where L is a submodule of N. By item (2), we have that $\operatorname{hdi}(J, N/L) = s$. Hence,

$$s = \operatorname{hdi}(J, N/L) \leq \dim_R(R/\mathfrak{q}) \leq \dim_R(A) = s.$$

Thus, $\dim_R(R/\mathfrak{q}) = \dim_R(A)$. Since $\dim_R(A) = \dim_R(R/\operatorname{Ann}_R(A))$, $\mathfrak{q}$ is a minimal element of the set $V(\operatorname{Ann}_R(A))$. Thus, $\mathfrak{q} \in \operatorname{Att}_R(A)$. Now, using Lemma 3.3,

$$s = \operatorname{hdi}(J, N/L) \leq \operatorname{cdi}(J, R/\mathfrak{q}) \leq \dim_R(R/\mathfrak{q}) \leq \dim_R(A) = s.$$

Therefore, we have that $\mathrm{cdi}(J, R/\mathfrak{q}) = s$. Now, suppose that $\mathfrak{q} \in \mathrm{Att}_R(A)$ and $\mathrm{cdi}(J, R/\mathfrak{q}) = s$. Since, $\mathrm{hdi}(J, A/N) < s$, and, also $\mathrm{cdi}(J, R/\mathfrak{q}) = s$, Lemma 3.2, implies that $\mathfrak{q} \notin \mathrm{Att}_R(A/N)$. Therefore, we have $\mathfrak{q} \in \mathrm{Att}_R(N)$.

(3) The exact sequence $0 \to N \to A \to A/N \to 0$, induces the exact sequence

$$\ldots \to \mathrm{H}^J_{s+1}(A/N) \to \mathrm{H}^J_s(N) \to \mathrm{H}^J_s(A) \to \mathrm{H}^J_s(A/N) \to \ldots\ .$$

Since, $\mathrm{hdi}(J, A/N) < s$, we have $\mathrm{H}^J_{s+1}(A/N) = \mathrm{H}^J_s(A/N) = 0$. Therefore, $\mathrm{H}^J_s(N) \cong \mathrm{H}^J_s(A)$.

□

5 Conclusion

In this article, we can to relate the theory of local cohomology, with respect to an ideal, to the theory of local homology modules. With the results of the article, we show the importance of local homology theory as a study tool within of the commutative algebra theory. Moreover, by making this relationship, we get applications for the local homology module, with respect to an ideal, in a general theory of modules. See also the works, [14], [15], and [16].

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