On top local homology modules

Abstract

Let R be a local ring, and let A be a non-zero Artinian R-module, with Noetherian dimension equal to s. In this paper, we determine the associated primes of the top local homology module of A, with respect to unique maximal ideal of the ring R.

Keywords and phrases: local homology, Artinian modules, Noetherian ring, asso- ciated primes.

1 Introduction

Throughout this paper assume that $(R, \mathfrak{m})$ is a commutative Noetherian local ring, with non-zero identity, $\mathfrak{a}$ is an ideal of R and A is an R-module.

In [3], Cuong and Nam defined the local homology modules $\mathrm{H}_i^{\mathfrak{a}}(A)$, with respect to $\mathfrak{a}$ by

$$\mathrm{H}_i^{\mathfrak{a}}(A) = \varprojlim_{n\in\mathbb{N}} \mathrm{Tor}_i^R(R/\mathfrak{a}^n, A).$$

This definition is dual to Grothendieck's definition of local cohomology modules. For more details about local cohomology modules, see [2].

For basic results about local homology we refer the reader to [3], and, [8].

In this paper, we study the top local homology module $\mathrm{H}_s^{\mathfrak{m}}(A)$, where A is a non-zero Artinian R-module of Noetherian dimension s. The module $\mathrm{H}_s^{\mathfrak{m}}(A)$ is called a top local homology module with respect to maximal ideal $\mathfrak{m}$, because

$$\max\{i : \mathrm{H}_i^{\mathfrak{m}}(A) \neq 0\} \leq s,$$

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by [3, Proposition 4.8].

We use properties of commutative algebra and homological algebra for the development of the results (see [1] and [11]).

2 Prerequisites

Definition 2.1. A non-zero R-module A is called secondary if the multiplication map by any element a of R is either surjective or nilpotent. A secondary representation of the R-module A is an expression for A as a finite sum of secondary modules. If such a representation exists, we will say that A is representable.

Definition 2.2. A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of A if $\mathfrak{p} = (N :_R A)$, for some submodule N of A. If A admits a reduced secondary representation

$$A = S_1 + S_2 + \ldots + S_n,$$

then the set of attached primes $\mathrm{Att}_R(A)$ is equal to

$$\left\{ \sqrt{(0 :_R S_i)},\ \text{for } i = 1, \ldots, n \right\}.$$

It is well known that if N is a submodule of Artinian R-module A, then

$$\mathrm{Att}_R(A/N) \subseteq \mathrm{Att}_R(A) \subseteq \mathrm{Att}_R(N) \cup \mathrm{Att}_R(A/N) \quad \text{(see [7, Section 6])}.$$

We now recall the concept of Noetherian dimension $\mathrm{N}\dim_R(A)$ of an R-module A.

For $A = 0$, we define $\mathrm{N}\dim_R(A) = -1$. Then by induction, for any integer $t \geq 0$, we define $\mathrm{N}\dim_R(A) = t$ when

(1) $\mathrm{N}\dim_R(A) < t$ is false, and

(2) for every ascending chain $A_1 \subseteq A_2 \subseteq \ldots$, of submodules of A there exists an integer m_0 such that

$$\mathrm{N}\dim_R(A_{m+1}/A_m) < t,$$

for all $m \geq m_0$.

Thus, A is non-zero and finitely generated if and only if $\operatorname{N}\dim_R(A) = 0$.

If A is Artinian R-module, then $\operatorname{N}\dim_R(A) < \infty$, according to [6].

In [4], for any R-module A, we define the cohomological dimension of A with respect to $\mathfrak{a}$ as

$$\operatorname{cd}(\mathfrak{a}, A) = \max\left\{i : \operatorname{H}^i_{\mathfrak{a}}(A) \neq 0\right\}.$$

By [2, Theorem 6.1.2] and by [2, Theorem 6.1.4],

$$\operatorname{cd}(\mathfrak{a}, A) \leq \dim_R(A),$$

and,

$$\operatorname{cd}(\mathfrak{m}, A) = \dim_R(A).$$

We will call,

$$\operatorname{hd}(\mathfrak{a}, A) := \max\{i : \operatorname{H}_i^{\mathfrak{a}}(A) \neq 0\}$$

the homological dimension of A with respect to $\mathfrak{a}$. It follows from [3, Propositions 4.8 and 4.10] that if A is an Artinian R-module, then,

$$\operatorname{hd}(\mathfrak{a}, A) \leq \operatorname{N}\dim_R(A),$$

and,

$$\operatorname{hd}(\mathfrak{m}, A) = \operatorname{N}\dim_R(A).$$

Throughout the paper, for an R-module A, $\operatorname{E}(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$ and $\operatorname{D}(\bullet)$ denotes the Matlis duality functor

$$\operatorname{Hom}_R(\bullet, \operatorname{E}(R/\mathfrak{m})).$$

It is well known that:

$$\dim_R(\operatorname{D}(A)) = \dim_R(A).$$

Also, if A is an Artinian R-module, where R is complete ring, then

$$A \cong \operatorname{D}(\operatorname{D}(A)),$$

and $\operatorname{D}(A)$ is a Noetherian $\hat{R}$-module (see [2, Theorem 10.2.19], and, [10, Theorem 1.6(5)]).

Note that if A is an Artinian R-module, then

$$\operatorname{H}_i^{\mathfrak{a}}(A) \cong \operatorname{D}(\operatorname{H}^i_{\mathfrak{a}}(\operatorname{D}(A))),$$

for all $i \geq 0$, according to [3, Proposition 3.3(ii)].

Therefore,

$$\operatorname{hd}(\mathfrak{a}, A) = \operatorname{cd}(\mathfrak{a}, \operatorname{D}(A)).$$

Thus,

$$\operatorname{hd}(\mathfrak{a}, A) \leq \dim_R(\operatorname{D}(A)) = \dim_R(A).$$

3 The results

Lemma 3.1. *Let* $(R,\mathfrak{m})$ *be a local ring, and let*

$$0 \to L \to A \to N \to 0,$$

be an exact sequence of Artinian R*-modules. Then,*

$$\mathrm{hd}(\mathfrak{a}, A) = \max\left\{\mathrm{hd}(\mathfrak{a}, L), \mathrm{hd}(\mathfrak{a}, N)\right\}.$$

Proof. Since $\mathrm{D}(A)$ is Noetherian $\hat{R}$-module, by [4, Corollary 2.3(i)], we have that

$$\mathrm{cd}(\mathfrak{a}\hat{R}, \mathrm{D}(N)) \leq \mathrm{cd}(\mathfrak{a}\hat{R}, \mathrm{D}(A)).$$

Hence by the Independence Theorem, [2, Theorem 4.2.1], it follows that

$$\mathrm{cd}(\mathfrak{a}, \mathrm{D}(N)) \leq \mathrm{cd}(\mathfrak{a}, \mathrm{D}(A)).$$

Therefore, $\mathrm{hd}(\mathfrak{a}, N) \leq \mathrm{hd}(\mathfrak{a}, A)$. From the long exact sequence

$$\mathrm{H}^{\mathfrak{a}}_{i+1}(L) \to \mathrm{H}^{\mathfrak{a}}_{i+1}(A) \to \mathrm{H}^{\mathfrak{a}}_{i+1}(N) \to \mathrm{H}^{\mathfrak{a}}_{i}(L) \to \mathrm{H}^{\mathfrak{a}}_{i}(A) \to \dots\ ,$$

we deduce that $\mathrm{hd}(\mathfrak{a}, L) \leq \mathrm{hd}(\mathfrak{a}, A)$.

Hence,

$$\max\left\{\mathrm{hd}(\mathfrak{a}, L), \mathrm{hd}(\mathfrak{a}, N)\right\} \leq \mathrm{hd}(\mathfrak{a}, A).$$

From the above long exact sequence we also infer that

$$\mathrm{hd}(\mathfrak{a}, A) \leq \max\left\{\mathrm{hd}(\mathfrak{a}, L), \mathrm{hd}(\mathfrak{a}, N)\right\}.$$

Thus, the proof is complete. □

Lemma 3.2. *Let* $(R,\mathfrak{m})$ *be a complete local ring, and let* A *be a non-zero Artinian* R*-module. Then,*

$$\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \mathrm{hd}(\mathfrak{a}, A),$$

for all $\mathfrak{p} \in \mathrm{Att}_R(A)$.

Proof. Since $\mathrm{D}(A)$ is a Noetherian R-module and

$$\mathrm{Supp}_R(R/\mathfrak{p}) \subseteq \mathrm{Supp}_R(\mathrm{D}(A)),$$

for all $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{D}(A))$, by [4, Theorem 2.2] we infer that

$$\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \mathrm{cd}(\mathfrak{a}, \mathrm{D}(A)),$$

for all $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{D}(A))$.

Since $\mathrm{Att}_R(A) = \mathrm{Ass}_R(\mathrm{D}(A))$ and $\mathrm{cd}(\mathfrak{a}, \mathrm{D}(A)) = \mathrm{hd}(\mathfrak{a}, A)$, we obtain

$$\mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \mathrm{hd}(\mathfrak{a}, A),$$

for all $\mathfrak{p} \in \mathrm{Att}_R(A)$. □

Lemma 3.3. *Let $(R, \mathfrak{m})$ be a local ring, and let A be an Artinian R-module. Then,*

$$\mathrm{hd}(\mathfrak{a}, A) \leq \mathrm{cd}(\mathfrak{a}, R/\mathrm{Ann}_R(A)).$$

Proof. Let $R^{'} = R/\mathrm{Ann}_R(A)$. By [12, Theorem 3.3], we have

$$\mathrm{H}_i^{\mathfrak{a}}(A) \cong \mathrm{H}_i^{\mathfrak{a}R^{'}}(A),$$

for all $i \geq 0$.

Thus, we have that $\mathrm{hd}(\mathfrak{a}, A) = \mathrm{hd}(\mathfrak{a}R^{'}, A)$. Since,

$$\mathrm{hd}(\mathfrak{a}R^{'}, A) \leq \mathrm{cd}(\mathfrak{a}R^{'}, R^{'}),$$

according to [5, Corollary 3.2], and

$$\mathrm{cd}(\mathfrak{a}R^{'}, R^{'}) = \mathrm{cd}(\mathfrak{a}, R^{'}),$$

according to [4, Lemma 2.1], we conclude that

$$\mathrm{hd}(\mathfrak{a}, A) \leq \mathrm{cd}(\mathfrak{a}, R^{'}),$$

and this finishes the proof. □

Lemma 3.4. *Let $(R, \mathfrak{m})$ be a complete local ring, and let A be a non-zero Artinian R-module of finite Noetherian dimension s, with $\mathrm{hd}(\mathfrak{a}, A) = s$. Thus,*

$$\sum = \left\{ N^{'} : N^{'} \textit{ is a submodule of } A \textit{ and } \mathrm{hd}(\mathfrak{a}, A/N^{'}) < s \right\},$$

has a smallest element N. The R-module N has the following properties:

(1) $\mathrm{hd}(\mathfrak{a}, N) = \dim(N) = s$.

(2) *N has no proper submodule L such that* $\mathrm{hd}(\mathfrak{a}, N/L) < s$.

(3) $\mathrm{Att}_R(N) = \{\mathfrak{p} \in \mathrm{Att}_R(A) : \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) = s\}$.

(4) $\mathrm{H}^{\mathfrak{a}}_s(N) \cong \mathrm{H}^{\mathfrak{a}}_s(A)$.

Proof. It is clear that $A \in \sum$, and thus $\sum$ is not empty. Since A is an Artinian R-module, the set $\sum$ has a minimal member N. By Lemma 3.1, if $N_1, N_2 \in \sum$, then $\mathrm{hd}(\mathfrak{a}, A/(N_1 \cap N_2)) < n$. Since the intersection of any two members of $\sum$ is again in $\sum$, it follows that N is contained in every member of $\sum$ implying that N is the smallest element of $\sum$.

(1) Since $\mathrm{hd}(\mathfrak{a}, A/N) < s$, from the exact sequence

$$0 \to N \to A \to A/N \to 0,$$

and of the Lemma 3.1, we obtain that $\mathrm{hd}(\mathfrak{a}, N) = s$. From,

$$s = \mathrm{hd}(\mathfrak{a}, N) \leq \dim(N) \leq \dim(A) = s,$$

we derive $\dim(N) = s$.

(2) Suppose that L is a submodule of N such that $\mathrm{hd}(\mathfrak{a}, N/L) < s$. From the exact sequence

$$0 \to N/L \to A/L \to A/N \to 0,$$

and of the Lemma 3.1, we infer $\mathrm{hd}(\mathfrak{a}, A/L) < s$. Hence, $L \in \sum$, and then $L = N$.

(3) If $\mathfrak{p} \in \mathrm{Att}_R(N)$, then $\mathfrak{p} = \mathrm{Ann}_R(N/L)$, where L is a submodule of N. By item (2), we have that $\mathrm{hd}(\mathfrak{a}, N/L) = s$. Hence,

$$s = \mathrm{hd}(\mathfrak{a}, N/L) \leq \dim_R(R/\mathfrak{p}) \leq \dim_R(A) = s.$$

Thus,

$$\dim_R(R/\mathfrak{p}) = \dim_R(A).$$

Since $\dim_R(A) = \dim_R(R/\mathrm{Ann}_R(A))$, we conclude that $\mathfrak{p}$ is a minimal element of the set $V(\mathrm{Ann}_R(A))$. Thus, $\mathfrak{p} \in \mathrm{Att}_R(A)$.

On the other hand, using Lemma 3.3, we derive

$$s = \mathrm{hd}(\mathfrak{a}, N/L) \leq \mathrm{cd}(\mathfrak{a}, R/\mathfrak{p}) \leq \dim_R(R/\mathfrak{p}) \leq \dim_R(A) = s.$$

Therefore, we have that $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = s$.

Now, suppose that

$$\mathfrak{p} \in \operatorname{Att}_R(A)$$

and

$$\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = s.$$

Since, $\operatorname{hd}(\mathfrak{a}, A/N) < s$, and, also $\operatorname{cd}(\mathfrak{a}, R/\mathfrak{p}) = s$, Lemma 3.2, implies that $\mathfrak{p} \notin \operatorname{Att}_R(A/N)$.

Therefore, we have that $\mathfrak{p} \in \operatorname{Att}_R(N)$.

(4) The exact sequence

$$0 \to N \to A \to A/N \to 0,$$

induces the exact sequence

$$\ldots \to \mathrm{H}^{\mathfrak{a}}_{s+1}(A/N) \to \mathrm{H}^{\mathfrak{a}}_{s}(N) \to \mathrm{H}^{\mathfrak{a}}_{s}(A) \to \mathrm{H}^{\mathfrak{a}}_{s}(A/N) \to \ldots\ .$$

Since, $\operatorname{hd}(\mathfrak{a}, A/N) < s$, it follows that we have

$$\mathrm{H}^{\mathfrak{a}}_{s+1}(A/N) = \mathrm{H}^{\mathfrak{a}}_{s}(A/N) = 0.$$

Therefore, $\mathrm{H}^{\mathfrak{a}}_{s}(N) \cong \mathrm{H}^{\mathfrak{a}}_{s}(A)$.

□

4 Applications

Theorem 4.1. *Let $(R, \mathfrak{m})$ be a complete local ring, and let A be a non-zero Artinian R-module of Noetherian dimension s. Then,*

$$\operatorname{Ass}_R(\mathrm{H}^{\mathfrak{m}}_{s}(A)) = \{\mathfrak{p} \in \operatorname{Att}_R(A) : \operatorname{cd}(\mathfrak{m}, R/\mathfrak{p}) = s\}.$$

Proof. If $s = 0$, then A has a finite length and therefore $\mathfrak{m}^k A = 0$, for some $k \in \mathbb{N}$. Hence,

$$\operatorname{Ass}_R(\mathrm{H}^{\mathfrak{m}}_{s}(A)) = \operatorname{Ass}_R(A) = \{\mathfrak{m}\} = \operatorname{Att}_R(A) = \{\mathfrak{p} \in \operatorname{Att}_R(A) : \operatorname{cd}(\mathfrak{m}, R/\mathfrak{p}) = 0\}.$$

Thus, $s > 0$. If $\mathrm{H}_s^{\mathfrak{m}}(A) = 0$, then $\mathrm{hd}(\mathfrak{m}, A) < s$. By Lemma 3.2,

$$\mathrm{cd}(\mathfrak{m}, R/\mathfrak{p}) < s \text{ for all } \mathfrak{p} \in \mathrm{Att}_R(A).$$

This implies that,

$$\{\mathfrak{p} \in \mathrm{Att}_R(A) : \mathrm{cd}(\mathfrak{m}, R/\mathfrak{p}) = s\} = \emptyset = \mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(A)),$$

and the result has been proved in this case. Now assume that $s > 0$, and $\mathrm{H}_s^{\mathfrak{m}}(A) \neq 0$. Then,

$$\mathrm{hd}(\mathfrak{m}, A) = \dim_R(A) = s.$$

By Lemma 3.4, A has no proper submodule L with $\mathrm{hd}(\mathfrak{m}, A/L) < s$, and we must show that $\mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(A)) = \mathrm{Att}_R(A)$. If

$$r \notin \bigcup_{\mathfrak{p} \in \mathrm{Att}_R(A)} \mathfrak{p},$$

then the exact sequence

$$0 \to (0 :_A r) \to A \xrightarrow{r} A \to 0,$$

induces the exact sequence

$$\mathrm{H}_s^{\mathfrak{m}}((0 :_A r)) \to \mathrm{H}_s^{\mathfrak{m}}(A) \xrightarrow{r} \mathrm{H}_s^{\mathfrak{m}}(A).$$

Using [8, Lemma 4.7],

$$\mathrm{N}\dim_R((0 :_A r)) \leq s - 1,$$

and therefore $\mathrm{H}_s^{\mathfrak{m}}((0 :_A r)) = 0$. Since

$$0 \to \mathrm{H}_s^{\mathfrak{m}}(A) \xrightarrow{r} \mathrm{H}_s^{\mathfrak{m}}(A)$$

is exact, we infer that

$$r \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(A))} \mathfrak{p},$$

and

$$\bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(A))} \mathfrak{p} \subseteq \bigcup_{\mathfrak{p} \in \mathrm{Att}_R(A)} \mathfrak{p}.$$

Since $\mathrm{Att}_R(A)$ is a finite set, every

$$\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(A))$$

is included in some

$$\mathfrak{q} \in \mathrm{Att}_R(A).$$

For such $\mathfrak{q}$ there exists a submodule L of A satisfying $\mathfrak{q} = \mathrm{Ann}_R(A/L)$. Hence,

$$s = \mathrm{hd}(\mathfrak{m}, A/L) \leq \dim_R(A/L) \leq \dim_R(R/\mathfrak{q}) \leq \dim_R(R/\mathfrak{p}) \leq s.$$

This shows $\mathfrak{p} = \mathfrak{q}$, and, also,

$$\mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(A)) \subseteq \mathrm{Att}_R(A).$$

To prove the reverse inclusion, assume that $\mathfrak{p} \in \mathrm{Att}_R(A)$. There exists a submodule L of A such that $\mathrm{Att}_R(L) = \mathfrak{p}$. Since we have assumed that A has no proper submodule U with $\mathrm{hd}(\mathfrak{m}, A/U) < s$, Lemma 3.4 implies that $\mathrm{cd}(\mathfrak{m}, R/\mathfrak{p}) = s$. By Lemma 3.2, $\mathrm{hd}(\mathfrak{m}, L) = s$, and, $\mathrm{H}_s^{\mathfrak{m}}(L) \neq 0$. Since $\mathrm{cd}(\mathfrak{m}, R/\mathfrak{p}) = s$, and,

$$\mathrm{Att}_R(L/U) \subseteq \mathrm{Att}_R(L) = \{\mathfrak{p}\},$$

for all submodules U, Lemma 3.2 shows that L cannot have any proper submodule U such that $\mathrm{hd}(\mathfrak{m}, L/U) < s$. Analogously,

$$\mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(L)) \subseteq \mathrm{Att}_R(L) = \{\mathfrak{p}\}.$$

Since $\mathrm{H}_s^{\mathfrak{m}}(L) \neq 0$, we establish that $\mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(L)) = \{\mathfrak{p}\}$. However, from the exact sequence

$$0 \to \mathrm{H}_s^{\mathfrak{m}}(L) \to \mathrm{H}_s^{\mathfrak{m}}(A) \to \mathrm{H}_s^{\mathfrak{m}}(A/L),$$

we see that

$$\{\mathfrak{p}\} = \mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(L)) \subseteq \mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(A)).$$

Therefore,

$$\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{H}_s^{\mathfrak{m}}(A)),$$

that completes the proof. □

Theorem 4.2. *Let $(R,\mathfrak{m})$ be a complete local ring, and let A be a non-zero Artinian R-module with $\operatorname{N}\dim_R(A)=s$. Then,*

$$\operatorname{Ass}_R(\operatorname{H}_s^{\mathfrak{m}}(A))=\left\{\mathfrak{B}\cap R:\mathfrak{B}\in\operatorname{Att}_{\hat{R}}(A)\ \textit{and}\ \operatorname{cd}(\mathfrak{m}\hat{R},\hat{R}/\mathfrak{B})=s\right\}.$$

Proof. Since $\dim_{\hat{R}}(\operatorname{D}(A))=\dim_{\hat{R}}(A)=\operatorname{N}\dim_R(A)=s$, (for details consult [9]), by [2, Theorem 7.1.6],

$$\operatorname{H}^s_{\mathfrak{m}\hat{R}}(\operatorname{D}(A)),$$

is an Artinian local cohomology module and

$$\operatorname{D}(\operatorname{H}^s_{\mathfrak{m}\hat{R}}(\operatorname{D}(A)))\cong\operatorname{H}_s^{\mathfrak{m}\hat{R}}(A),$$

is a Noetherian $\hat{R}$-module. It is well known that

$$\operatorname{Ass}_R(L)=\{\mathfrak{B}\cap R:\mathfrak{B}\in\operatorname{Ass}_{\hat{R}}(L)\},$$

for each finitely generated $\hat{R}$-module L (see [7, Exercise 6.7]. Thus,

$$\operatorname{Ass}_R(\operatorname{H}_s^{\mathfrak{m}\hat{R}}(A))=\left\{\mathfrak{B}\cap R:\mathfrak{B}\in\operatorname{Ass}_{\hat{R}}(\operatorname{H}_s^{\mathfrak{m}\hat{R}}(A))\right\}.$$

Since by [13, Proposition 4.3],

$$\operatorname{H}_s^{\mathfrak{m}}(A)\cong\operatorname{H}_s^{\mathfrak{m}\hat{R}}(A),$$

as R-modules, we conclude that

$$\operatorname{Ass}_R(\operatorname{H}_s^{\mathfrak{m}}(A))=\left\{\mathfrak{B}\cap R:\mathfrak{B}\in\operatorname{Ass}_{\hat{R}}(\operatorname{H}_s^{\mathfrak{m}\hat{R}}(A))\right\}.$$

According to Theorem 4.1,

$$\operatorname{Ass}_{\hat{R}}(\operatorname{H}_s^{\mathfrak{m}\hat{R}}(A))=\left\{\mathfrak{B}:\mathfrak{B}\in\operatorname{Att}_{\hat{R}}(A)\text{ and }\operatorname{cd}(\mathfrak{m}\hat{R},\hat{R}/\mathfrak{B})=s\right\}.$$

Therefore,

$$\operatorname{Ass}_R(\operatorname{H}_s^{\mathfrak{m}}(A))=\left\{\mathfrak{B}\cap R:\mathfrak{B}\in\operatorname{Att}_{\hat{R}}(A)\text{ and }\operatorname{cd}(\mathfrak{m}\hat{R},\hat{R}/\mathfrak{B})=s\right\}.$$

This finishes the proof. □

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