On properties of zero points and poles of K-bianalytic functions

Abstract

In this paper, we first prove that the set of zero points of a nonzero K-bianalytic function $f(z) = \bar{z}(k)\phi_1(z(k))+\phi(z(k)),\ z \in D$, is not a region and the set of the second zero points has no accumulated point. Second, a sufficient and necessary condition is given for a K-bianalytic function to have a zero arc which has a parameter equation $\bar{z}(k) = \gamma(z(k))$ where γ is an analytic function in a region $D(k)$. Finally, the traits of a K-bianalytic function which has a zero arc, even straight, one of whose ends is a $(c_1, c_2)-$th pole at $z = 0$, are discussed. Some examples are also shown for our topic.

1 Introduction

Zhang [1] gave the definition of K-analytic functions. Zhang and his coauthors extended some properties of analytic functions to K-analytic functions in the theories of integral, Laurent series expansion, residual theorem, fractional linear transformation, conformal mapping, and so on. See [2]-[8]. Sang and Li [9] studied the mean value theorems of K-analytic functions. Lin and Xu [10] investigated Riemann problem of (λ, k) bi-analytic functions.

There are many works to study the properties of bianalytic functions. For example, Zhu, Huang, Liu and Zhu [11] gave the distribution of nonisolated zero points, properties of mapping, classification of isolated singular points of bianalytic functions. Fu [12] summarized some properties of analytic and bianalytic functions in the dissertation for master's degree. Wang, Huang and Liu [13] researched the properties of bianalytic functions with zero arc at a pole.

Li and Liu [14] naturally put forward the concept of K-bianalytic functions and investigated Cauchy theorem, Cauchy integral formula, power series expansion, Fourier series expansion of K-bianalytic functions. Hitherto, a lot of properties of K-bianalytic functions have not yet been studied and thus it is necessary to continue to look into the attributes of K-bianalytic functions.

In this paper, we mainly explore the properties of zero points and poles in K-bianalytic functions, which generalize the corresponding results of [11] and [13] in bianalytic functions.

Definition 1.1. [1] The forms of complex number as $x + iky$ ($k \in \mathbb{R},\ k \neq 0$) are called K-complex number of $x + iy$, denoted by $z(k)$.

Definition 1.2. [1] Let the function $f(z)$ be defined in a neighborhood of z_0. If

$$\lim_{\Delta z(k)\to 0} \frac{\Delta f}{\Delta z(k)} = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z(k) - z_0(k)}$$

exists, then we call that $f(z)$ is K-differential at z_0, the limit is the K-derivative of $f(z)$ at z_0, denoted by $f'_{(k)}(z_0)$ or $\frac{df(z)}{z(k)}|_{z=z_0}$, i.e.,

$$f'_{(k)}(z_0) = \frac{df(z)}{dz(k)}\Big|_{z=z_0} = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z(k) - z_0(k)}.$$

If $f(z)$ is K-differential at each $z \in D$, then the second K-derivative of $f(z)$ at $z_0 \in D$ is defined as

$$f''_{(k)}(z_0) = \frac{df'_k(z)}{dz(k)}\Big|_{z=z_0} = \lim_{z \to z_0} \frac{f'_k(z) - f'_k(z_0)}{z(k) - z_0(k)}.$$

Similarly, the nth K-derivative $f^{(n)}_{(k)}(z)$ can be defined as the same way.

Definition 1.3. [1] If $f(z)$ is K-differential in a region D, we say that $f(z)$ is analytic in D; If $f(z)$ is K-analytic in a neighborhood of z_0, then we say that $f(z)$ is K-analytic at z_0.

Definition 1.4. [14] Let the function $f(z)$ have the second partial derivative $\frac{\partial^2 f(z)}{\partial \bar{z}(k)^2}$ in a region D. If $\frac{\partial^2 f(z)}{\partial \bar{z}(k)^2} = 0$ for any $z \in D$, then $f(z)$ is called a K- bianalytic function in D.

Denote $D(k) = \{\xi(k) | \xi \in D\}$ if D is a set of $\mathbb{C}$.

Lemma 1.1. [14, Theorem 1] *If function $f(z)$ is a bianalytic function in a region D, then the following is established*

$$f(z) = \bar{z}(k)\phi_1(z(k)) + \phi(z(k)), \ z \in D$$

where $\phi_1(z)$ and $\phi_2(z)$ are arbitrary analytic functions in $D(k)$.

Similar to the definition of the nth zero point of bianalytic function in [11], the nth zero point of a K-bianalytic function $f(z)$ is defined as follows.

Definition 1.5. Let $f(z)$ be a K-bianalytic function in a region D, $n \geq 1$, $z_0 \in D$. If $f(z_0) = 0$ and $\frac{\partial^{i+j}}{\partial^i z(k) \partial^j \bar{z}(k)} f(z) = 0$ for any $0 < i + j \leq n - 1, i, j \in \mathbb{N}$ and there exists $s, t \in \mathbb{N}$ such that $s + t = n$ and $\frac{\partial^n}{\partial^s z(k) \partial^t \bar{z}(k)} f(z) \neq 0$, then z_0 is called a nth zero point of $f(z)$.

2 The main results

The zero points of K-bianalytic function are not definitely isolated. For example, the zero points of

$$w(z) = z(k)\bar{z}(k) - 1, \quad w(z) = z(k) - \bar{z}(k)$$

are the ellipse $x^2 + k^2y^2 = 1$ and the real axis, respectively, but they are not zero functions. Although zero points of K-bianalytic functions are not non-isolated, the distribution of zero points is not very wide.

Theorem 2.1. *Let $w(z)$ be a nonzero K-bianalytic function in a region D. Then the set of zero points is not a region.*

Proof. Let $w(z) = 0$, $z \in \sigma$, where σ is a subregion of D. By Lemma 1,

$$w(z) = \bar{z}(k)\phi_1(z(k)) + \phi_2(z(k)),$$

where $\phi_1(z)$ and $\phi_2(z)$ are arbitrary analytic functions in $D(k)$. If $\phi_1(k(z)) = 0$ for $z \in \sigma$, by the isolation of zero points of $K-$analytic functions [3], we know that $\phi_1(k(z)) = 0$ for $z \in D$ and thus $\phi_2(k(z)) = 0$ for $z \in D$. This contradicts the condition of the theorem. If $\phi_1(k(z)) \not\equiv 0$ for $z \in \sigma$, then there exists $z_0 \in \sigma$ such that $\phi_1(k(z_0)) \neq 0$ and thus there is a neighborhood of z_0, $U(z_0) \in \sigma$, such that $\phi_1(k(z)) \neq 0$ for $z \in U(z_0)$. Since

$$w(z) = \bar{z}(k)\phi_1(z(k)) + \phi_2(z(k)) = 0, \quad z \in \sigma,$$

it follows that

$$\bar{z}(k) = -\frac{\phi_2(z(k))}{\phi_1(z(k))}, \quad z \in \sigma,$$

which is wrong obviously. ∎

Theorem 2.2. *The second zero points of a K-bianalytic function has no accumulated point.*

Proof. Suppose that

$$w(z) = \bar{z}(k)\phi_1(z(k)) + \phi_2(z(k)),$$

where $\phi_1(z)$ and $\phi_2(z)$ are arbitrary analytic functions in $D(k)$. If the set of the second zero points z_n, $n = 1, 2, ...$, has a accumulated point $z_0 \in D$. In the light of

$$w(z_n(k)) = \phi_1(z_n(k)) = 0, \quad n = 1, 2, ...,$$

we have that

$$\phi_2(z_n(k)) = 0, \quad n = 1, 2...$$

Uniqueness of $K-$analytic function [3] gives that

$$\phi_1(z(k)) = \phi_2(z(k)) = 0, \quad z \in D,$$

which contradicts the fact that w is a nonzero function in D. □

Definition 2.1. If the points of an arc γ are zero points or the accumulated points of zero points of a $K-$bianalytic function $w(z)$, then γ is called a zero arc of the $K-$bianalytic function $w(z)$. If $w(z) = C$ for $z \in \gamma$ where C is a constant, then γ is called a constant arc of the $K-$bianalytic function $w(z)$.

Theorem 2.3. *Let the curve $\widehat{\gamma}$ have a parameter equation $\bar{z}(k) = \gamma(z(k))$ where γ is an analytic function in a region $D(k)$. Then $\widehat{\gamma}$ is a zero arc of a $K-$bianalytic function in D $w(z) = \bar{z}(k)\phi_1(z(k)) + \phi_2(z(k))$ if and only if*

$$\phi_2(z(k)) = -\gamma(z(k))\phi_1(z(k)), \quad z \in D. \tag{2.1}$$

Proof. Necessity. If

$$\phi_1(z(k)) = 0, \quad w(z) = 0, \quad z \in \widehat{\gamma},$$

then

$$\phi_2(z(k)) = 0, \quad z \in \widehat{\gamma}.$$

Therefore by uniqueness of $K-$analytic function [3] we know that

$$w(z) = \phi_1(z(k)) = \phi_2(z(k)) = 0, \quad z \in D.$$

If there exists $z_0 \in \widehat{\gamma}$ such that $\phi_1(z_0(k)) \neq 0,\ z_0 \in \widehat{\gamma}$, then there is a neighborhood of z_0, $U(z_0) \in \sigma$, such that $\phi_1(k(z)) \neq 0$ for $z \in U(z_0)$. If $z \in \gamma \cap U(z_0)$, then

$$\begin{aligned} w(z) &= \bar{z}(k)\phi_1(z(k)) + \phi_2(z(k)) \\ &= \left(\bar{z}(k) + \frac{\phi_2(z(k))}{\phi_1(z(k))}\right)\phi_1(z(k)) \\ &= \left(\gamma(z(k)) + \frac{\phi_2(z(k))}{\phi_1(z(k))}\right)\phi_1(z(k)). \end{aligned}$$

Thus

$$\gamma(z(k)) + \frac{\phi_2(z(k))}{\phi_1(z(k))} = 0,\ z_0 \in \widehat{\gamma} \cap U(z_0).$$

Uniqueness of $K-$analytic function [3] yields (2.1).

Sufficiency. Since $\phi_2(z(k)) = -\gamma(z(k))\phi_1(z(k))$, $\bar{z}(k) = \gamma(z(k))$, $z \in \widehat{\gamma}$, we have

$$\bar{z}(k)\phi_1(z(k)) + \phi_2(z(k)) = 0,\ z \in \widehat{\gamma},$$

and thus

$$w(z) = \bar{z}(k)\phi_1(z(k)) + \phi_2(z(k)) = 0,\ z \in \widehat{\gamma},$$

i.e., γ is a zero arc of w. □

Corollary 2.1. *Under the assumptions of Theorem 2.3,* $\widehat{\gamma}$ *is a constant arc of a* $K-$*bianalytic function* $w(z) = \bar{z}(k)\phi_1(z(k)) + \phi_2(z(k))$ *if and only if there exists a constant* C *such that*

$$\phi_2(z(k)) - C = -\gamma(z(k))\phi_1(z(k)), \quad z \in D.$$

Example 2.1. *Note that* $\bar{z}(k) = z(k)$ *is the parameter equation of the real axis. Thus by Theorem 2.3 a* $K-$*bianalytic function* $w(z)$ *which takes a subarc of the real axis to be zero arc must be*

$$w(z) = (\bar{z}(k) - z(k))\phi_1(z(k)),$$

where $\phi_1(z)$ *is analytic on the real axis.*

Definition 2.2. Let $w(z) = \bar{z}(k)\phi_1(z(k)) + \phi_2(z(k))$ be a $K-$bianalytic function in a region D, where $\phi_1,\ \phi_2$ are analytic functions in $D(k)$ and $z = a$ is the c_i-th pole of ϕ_i, $i = 1, 2$, respectively (if $z = a$ is a removable singular point of ϕ_i, then $z = a$ is called a 0-th pole of ϕ_i). If $0 \le c_i < \infty$ $(i = 1, 2)$ and $c_1^2 + c_2^2 \neq 0$, then $z = a$ is called (c_1, c_2)-th pole of $w(z)$.

Without loss of generality, we only need to discuss the behavior near $z = 0$ of $w(z)$. If $a \neq 0$, under the transformation $z(\zeta) = \zeta + a$, we can similarly investigate the behavior near $\zeta = 0$ of the function

$$\begin{aligned} w(z(\zeta)) &= \overline{\zeta + a}(k)\phi_1(z(\zeta)(k)) + \phi_2((z(\zeta))(k)) \\ &= \bar{\zeta}(k)\phi_1(z(\zeta)(k)) + [\bar{a}(k)\phi_1(z(\zeta)(k)) + \phi_2((z(\zeta))(k))]. \end{aligned}$$

Obviously, if ϕ_i, $i = 1, 2$, have c_i-th poles of $\phi(z)$ at $z = 0$, respectively, then ϕ_i, $i = 1, 2$, can be expressed by Laurent expansions as follows:

$$\phi_1(z) = \frac{a_{-c_1}}{z^{c_1}} + \frac{a_{-c_1+1}}{z^{c_1-1}} + ... + a_0 + a_1 z + ... = \frac{1}{z^{c_1}}\psi_1(z); \tag{2.2}$$

$$\phi_2(z) = \frac{b_{-c_2}}{z^{c_2}} + \frac{b_{-c_2+1}}{z^{c_2-1}} + ... + b_0 + b_1 z + ... = \frac{1}{z^{c_2}}\psi_2(z), \tag{2.3}$$

respectively. If $c_1 \ge 1$, then $a_{-c_1} \neq 0$ as well as c_2 does. The notations of (2.2) and (2.3) are used in the remaining part.

Theorem 2.4. *Let a* $K-$*bianalytic function*

$$w(z) = \bar{z}(k)\phi_1(z(k)) + \phi_2(z(k))$$

have $(c_1, c_2)-$*th pole at* $z = 0$*. If there is an arc* γ *with an end* $z = 0$ *such that*

$$w(z) = 0, \quad z \in \gamma\backslash\{0\}, \tag{2.4}$$

then

$$c_1 = c_2 + 1, \quad |a_{-c_1}| = |b_{-c_2}|.$$

Proof. By contradiction. If $c_1 \le c_2$, by (2.2) and (2.3) we have

$$w(z) = \frac{\bar{z}(k)z(k)^{c_2-c_1}\psi_1(z(k)) + \psi_2(z(k))}{z(k)^{c_2}}, \quad z \in \gamma\backslash\{0\}.$$

By (2.4), we have

$$\bar{z}(k)z(k)^{c_2-c_1}\psi_1(z(k)) + \psi_2(z(k)) = 0, \quad z \in \gamma\backslash\{0\}.$$

But

$$\lim_{z \to 0} \left(\bar{z}(k)z(k)^{c_2-c_1}\psi_1(z(k)) + \psi_2(z(k))\right) = b_{-c_2} \neq 0,$$

which is a contradiction. The similar method is suitable for explaining incorrectness of the case $c_2 < c_1 - 1$. Thus $c_1 = c_2 + 1$. In this case we obtain

$$w(z) = \frac{\bar{z}(k)\psi_1(z(k)) + z(k)\psi_2(z(k))}{z(k)^{c_1}}, \quad z \in \gamma\backslash\{0\},$$

which yields that

$$\bar{z}(k)\psi_1(z(k)) + z(k)\psi_2(z(k)) = 0, \quad z \in \gamma\backslash\{0\}. \tag{2.5}$$

So

$$\lim_{z\to 0, z\in\gamma}\left|\frac{\bar{z}(k)}{z(k)}\right| = \lim_{z\to 0, z\in\gamma}\left|-\frac{\psi_2(z(k))}{\psi_1(z(k))}\right| = \left|\frac{b_{-c_2}}{a_{-c_1}}\right| = 1.$$

□

Theorem 2.5. *Let a $K-$bianalytic function*

$$w(z) = \bar{z}(k)\phi_1(z(k)) + \phi_2(z(k))$$

have a $(c_1, c_2)-$th pole at $z = 0$. Then there exists a line segment η with an end $z = 0$, such that

$$w(z) = 0, \quad z \in \eta\backslash\{0\}, \tag{2.6}$$

if and only if there exists a neighborhood $U(0)$ of $z = 0$ such that

$$\frac{\phi_2(z(k))}{\phi_1(z(k))} = e^{i\theta_0}z(k), \quad z \in U(0),$$

where $\theta_0 = arg\frac{b_{-c_2}}{a_{-c_1}}$.

Proof. Sufficiency. If $\frac{\phi_2(z(k))}{\phi_1(z(k))} = e^{i\theta_0}z(k)$, then there exists a line segment η with an end $z = 0$, such that (2.7) holds, where the line segment η satisfies the equation $y = \frac{1}{k}\left(\cot\frac{\theta_0}{2}\right)x$ for $x + iy \in \eta$.

Necessity. By assumptions and the proof of Theorem 2.4, we get $c_1 = c_2 + 1$ and (2.5) holds with γ replaced by η. Let the inclination of the line segment $\{\eta(k) = z(k) : z \in \eta\}$ is α. Hence there exists a deleted neighborhood $U^0(0)$ of $= 0$ such that

$$\frac{\psi_2(z(k))}{\psi_1(z(k))} = -e^{-2i\alpha}, \quad z \in \eta \cap U^0(0).$$

The fact that $\frac{\psi_2(z(k))}{\psi_1(z(k))}$ is $K-$analytic, uniqueness of $K-$analytic function and (2.5) implies that there exists a neighborhood $U(0)$ of $z = 0$ such that

$$\frac{\phi_2(z(k))}{\phi_1(z(k))} = \frac{\psi_2(z(k))}{\psi_1(z(k))}z(k) = e^{i\theta_0}z(k), \quad z \in U(0).$$

□

Example 2.2. *Let $K-$bianalytic function be given by $w(z) = \frac{\bar{z}(k)+a_1z(k)+a_2z(k)^2+...+a_jz(k)^j+...}{z(k)^n}$, where $n \in \mathbb{N}^*$ and $a_i \in \mathbb{C}$. Then by Theorem 2.5 we know that there exists a line segment η with an end $z = 0$, such that*

$$w(z) = 0, \quad z \in \eta\backslash\{0\} \tag{2.7}$$

if and only if $a_1 = e^{i\theta}$, $\theta \in \mathbb{R}$ and $a_i = 0$, $i \geq 2$.

3 Conclusion

In the present paper, we extend some properties of zero points, zero arcs and poles of bianalytic functions to K-bianalytic functions.

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The authors did the same contributions in writing this paper.

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Not applicable.

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